5. The Time Arrow of Spacetime Geometry

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In the framework of general relativity, gravity is a consequence of spacetime curvature. Its dynamical laws are still symmetric under time reversal. However, if the actual global spacetime structure defines an arrow of time (as in an expanding universe), this must be reflected by the dynamics of all matter. While this had been well known, it came as a surprise during the early seventies that the most strongly gravitating systems possess properties that have to be described in thermodynamical terms, thus indicating an intimate connection between these apparently quite different fields of physics.

Gravitating systems are thermodynamically peculiar even in Newton's theory, since they possess negative heat capacities as a consequence of the universal attractivity of this force. In particular, forces depending homogeneously on the minus second power of distance, such as gravity and Coulomb forces, lead according to the virial theorem to the relation

$$\overline{E_{\rm kin}} = -\tfrac{1}{2}\overline{E_{\rm pot}} = -E \tag{5.1}$$

between the mean values of kinetic and potential energies, and therefore between them and the total energy. This virial theorem is usually valid in a statistical (ergodic) sense only. For bound systems it applies to mean values over a (quasi-) period of the motion, for quasi-bound systems it holds approximately for mean values over sufficiently large times. In quantum theory it applies to *expectation values* of proper (normalizable) energy eigenstates, as can be shown by using Fock's *ansatz* $\psi(\lambda\boldsymbol{r}_1, \ldots, \lambda\boldsymbol{r}_N)$ and the homogeneity of T and V in a variational procedure $\delta(\langle\psi|T+V|\psi\rangle/\langle\psi|\psi\rangle) = 0$ with respect to λ. The theorem must then also hold for mixtures which are diagonal in energy eigenstates, or (approximately) if non-diagonal terms can be neglected because of random phases. (For relativistic generalizations of the virial theorem see Gourgoulkon and Bonazzola 1994).

The anti-intuitive negative sign relating kinetic and total energy in (5.1) means, for example, that satellites are *accelerated* by weak friction in the earth's atmosphere, and that stars *heat up* by radiating energy away. (This second example is valid only as far as the quantum mechanical zero-point energy does not dominate $\overline{E_{\rm kin}} = \mathrm{Trace}\{\varrho T\}$ — as it does in white dwarf stars or solid bodies. Early astrophysicists believed that stars always cool down after exhausting their fuel.) It also means that the heat flow from hot to cold objects that are governed by gravity causes a thermal inhomogeneity to *grow*.

As an example, consider two monatomic ideal gases with entropies given according to (3.14) by

$$S_i = kN_i \left(\tfrac{3}{2} \ln T_i - \ln \varrho_i + C_i\right) \tag{5.2}$$

(with $i = 1, 2$, and constants C_i). Since the internal energy, $U = \overline{E_{\text{kin}}}$, is here given by $U = (3/2)NkT$, the net change of entropy resulting from an exchange of energy, $\delta U_1 = -\delta U_2$, and of particles, $\delta N_1 = -\delta N_2$, becomes for fixed volumes V_i or for fixed densities $\varrho_i = N_i/V_i$

$$\delta S_{\text{total}} = \delta S_1 + \delta S_2 = \left(\frac{1}{T_1} - \frac{1}{T_2}\right) \delta U_1 + k \left(\frac{3}{2} \ln \frac{T_1}{T_2} - \ln \frac{\varrho_1}{\varrho_2}\right) \delta N_1 \quad . \tag{5.3}$$

This describes entropy changes δS_1 and δS_2 with opposite sign, but cancelling only in thermodynamical equilibrium ($T_1 = T_2$ and $\varrho_1 = \varrho_2$). An entropy increase in accordance with the Second Law is in this normal situation achieved by a *reduction* of thermal and density inhomogeneities (aside from the transient *thermo-mechanical effect*, that is, a thermally induced pressure difference due to the temperature dependence of the second term).

However, the density of a star is not a free variable that can be kept fixed. Since a normal star, which will here for simplicity be assumed to be in thermal equilibrium, may in very good approximation be described as an ideal gas, its volume is related to the potential energy, and by means of the virial theorem then also to the temperature, according to

$$NT \propto U = \overline{E_{\text{kin}}} \propto -\overline{E_{\text{pot}}} \propto \frac{N^2}{R} \propto \frac{N^2}{V^{1/3}} \quad , \tag{5.4}$$

that is, $V \propto N^3/T^3$. The entropy (5.2) of a star is therefore

$$\begin{aligned} S_{\text{star}} &= kN \left(\tfrac{3}{2} \ln T - \ln N + \ln V + C\right) \\ &= kN \left(-\tfrac{3}{2} \ln T + 2 \ln N + C'\right) \quad . \end{aligned} \tag{5.5}$$

In the second line the signs of $\ln T$ and $\ln N$ are reversed. The total entropy change, $\delta S_{\text{star}} + \delta S_{\text{gas}}$, of a star embedded in an interstellar gas becomes (using the virial theorem $E_{\text{star}} = -U_{\text{star}}$ again)

$$\delta S_{\text{total}} = \left(\frac{1}{T_{\text{star}}} - \frac{1}{T_{\text{gas}}}\right) \delta E_{\text{star}} + k \ln \left(\frac{C'' N_{\text{star}}^2 \varrho_{\text{gas}}}{(T_{\text{star}} T_{\text{gas}})^{3/2}}\right) \delta N_{\text{star}} \quad . \tag{5.3'}$$

While heat still flows from the hot star into the cold gas according to the Second Law, this leads now to further increase of the star's temperature, and to accretion of matter provided N_{star} is large enough. Thermal and density inhomogeneities thus grow in the generic astrophysical situation, although 'pathological' (non-ergodic) solutions do exist, such as gravitationally collapsing spherical matter shells or pressure-free dust spheres, where the virial theorem does not hold.

This demonstrates that the gravitational contraction of normal stars is dynamically controlled by thermodynamics, but *not* by gravity itself. If the thermodynamical arrow of time did change direction in a recontracting universe (as first suggested by Gold 1962 — see Sect. 5.3), stars and similar gravitating objects would have to re-expand during that epoch.

A homogeneous universe thus describes a state of very low entropy (improbable but 'simple' in the sense discused at the end of Chap. 3). This leads to the question whether its potential inhomogeneous contraction under gravitational forces represents an entropy capacity that is sufficient to explain the observed thermodynamical arrow of time. The *Kaltgeburt* could then be replaced with a *homogeneous birth*, while inhomogeneous contraction leads to the required thermodynamical non-equilibrium.

In order to estimate the improbability (negentropy) of a homogeneous universe, one has to know the maximum entropy that can be gained from gravitational contraction. Possible limits of the negative heat capacity are:

a) *Quantum degeneracy* (primarily of electrons). It is essential for the stability of solid gravitating bodies and of white dwarfs. By emitting heat these objects can cool down 'normally' rather than heating up.
b) *Repulsive short range forces.* They may be important in neutron stars, while they have similar consequences as a degenerate electron gas.
c) *Gravitation* itself. Even in Newton's theory, any radiation with bounded propagation velocity that is affected by gravity cannot escape from the surface of a sufficiently massive object. If this speed limit is as universal as gravity (as it is according to the theory of relativity), any further collapse of the mass distribution remains *irrelevant* to an external observer. The maximum radius from which light may escape for a given mass defines an *event horizon.* Matter disappearing behind the horizon — whatever will happen to it — cannot participate any more in the thermodynamics of the universe, while only its external gravitational field remains observable.

Such *non-relativistic black holes* were discussed as early as 1795 by Laplace, and even before him by J. Mitchel. In general relativity, black holes are described by specific spacetime structures. This theory leads to the further consequence that neither of the first two bounds to gravitational contraction may prevent an object of sufficiently large mass (that may always be formed by accretion of matter) from collapsing into a black hole. In case (b), repulsive forces give rise to a positive potential energy, which must eventually dominate as a source of gravity, while for (a) the increasing zero point pressure of the fermions forces them to combine into effective bosons, with all of them being able to occupy the same spatial state.

Therefore, black holes define an upper limit for the entropy production by gravitational contraction of matter from the point of view of an external observer. But what *is* the entropy of a black hole? This question cannot be answered by the investigation of relativistic stars, that is, of equilibrium systems, since the essential stages of the collapse proceed irreversibly. However, a unique and finite answer is obtained from a quantum aspect of black holes, *viz.* their Hawking radiation (Sect. 5.1).

Since the curvature of space represents the gravitational degrees of freedom in general relativity, it may carry entropy by itself. The dynamics of spatial curvature (regarded as the time-dependent state — see Sect. 5.4) is described by Einstein's equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \quad , \tag{5.6}$$

(in units of $G = c = 1$), where $T_{\mu\nu}$ is the energy-momentum tensor of matter. These equations define an initial (or final) value problem again, since they are generically of hyperbolic type (cf. Sect. 2.1). The Einstein tensor $G_{\mu\nu}$ is a linear combination of the components of the Ricci tensor $R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu}$, that is, a trace of the Riemann curvature tensor. Forming this trace is analogous to forming the d'Alembertian in the wave equation (2.1) for the electromagnetic potential from the tensor of its second derivatives $\partial_\nu\partial_\lambda A^\mu$. Aside from the nonlinearities responsible for the self-interaction of gravity, the Riemann curvature tensor is similarly defined by the second derivatives of the metric $g_{\mu\nu}$, which thus assumes the role of the gravitational potential (analogous to A^μ in electrodynamics). In both cases, the trace of the tensor of derivatives is determined locally by the sources, while its trace-free parts represent the local field variables, which can be chosen freely as initial conditions.

Penrose (1969, 1981) used this freedom to conjecture that the trace-free part of the curvature tensor (the *Weyl tensor*) was zero when the universe began. This situation describes a 'vacuum state of gravity', that is, a state of minimum gravitational entropy, and a space as flat as is compatible with the sources. It is analogous to the cosmic initial condition $A^\mu_{\text{in}} = 0$ for the electromagnetic field discussed in Sect. 2.2 (with Gauß' law as a similar constraint). Gravity would then represent an exactly retarded field, requiring 'causes' in the form of advanced sources. Since Penrose intends to explain the thermodynamical arrow from this initial condition (see Sect. 5.3), his conjecture revives Ritz's position in his controversy with Einstein (mentioned in Chap. 2) by applying it to gravity rather than electrodynamics.

In the big bang scenario, the beginning of the universe is characterized by a time-like curvature singularity (where time itself begins). Penrose used this fact to postulate his Weyl tensor hypothesis to past singularities in general, since this would allow only *one* (homogenous) past singularity (the big bang itself). However, in the absence of an *absolute* direction of time the past would then be distinguished from the future precisely and solely by this boundary condition. If the Weyl tensor condition could be derived from other assumptions (that did not *ad hoc* specify an asymmetry in time), it would have to be valid for future singularities as well.

5.1 Thermodynamics of Black Holes

In order to discuss the spacetime geometry of black holes, it is convenient to consider the static and spherically symmetric vacuum solution discovered by Schwarzschild, and originally expected to represent a point mass. In terms of spherical spatial coordinates this solution is described by the metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \tag{5.7}$$

Here, r measures the size of a sphere (but not the distance from $r = 0$). This metric form is singular at $r = 0$ and $r = 2M$, but the second singularity, at the *Schwarzschild radius* $r = 2M$, is merely the result of an inappropriate choice of these coordinates. The condition $r = 2M$ describes a surface with area $A = 4\pi(2M)^2$ (using Planck units $G = c = \hbar = k_{\mathrm{B}} = 1$ from now on). In its interior (that is, for $r < 2M$) one has $g_{tt} = 2M/r - 1 > 0$ and $g_{rr} = (1 - 2M/r)^{-1} < 0$. Therefore, r and t interchange their physical meaning as spatial and temporal coordinates. The internal solution is *not* static, while the genuine singularity at $r = 0$ represents a time-like boundary rather than the space point expected by Schwarzschild.

Physical (time-like or light-like) world lines, that is, curves with $ds^2 \le 0$, hence with $(dr/dt)^2 \le (1 - 2M/r)^2 \to 0$ for $r \to 2M$, can only approach the Schwarzschild radius parallel to the t-axis (see Fig. 5.1). Therefore, the interior region $r < 2M$ is physically accessible only via $t \to +\infty$ or $t \to -\infty$, albeit within finite proper time. These world lines can be extended regularly into the interior when t goes beyond $\pm\infty$. Their proper times continue into the finite future (for $t > +\infty$) or past (for $t < -\infty$) with the new time coordinate $r \to 0$. There are hence *two* internal regions (II and IV in the figure), with their own singularities at $r = 0$ (at a finite distance in proper times). These internal regions must in turn each have access to a new external region, also in their past or future, respectively, via different Schwarzschild surfaces at $r = 2M$, but with opposite signs of $t = \pm\infty$. There, proper times have to decrease with growing t. These two new external regions may then be identified with one another in the simplest possible topology (region III in the figure).

This complete Schwarzschild geometry may be regularly described by means of the *Kruskal-Szekeres coordinates* u and v, which eliminate the coordinate singularity at $r = 2M$. In the external region they are related to the Schwarzschild coordinates r and t by

$$u = \sqrt{\frac{r}{2M} - 1}\; \mathrm{e}^{r/4M} \cosh\left(\frac{t}{4M}\right) \tag{5.8a}$$

$$v = \sqrt{\frac{r}{2M} - 1}\; \mathrm{e}^{r/4M} \sinh\left(\frac{t}{4M}\right) \quad . \tag{5.8b}$$

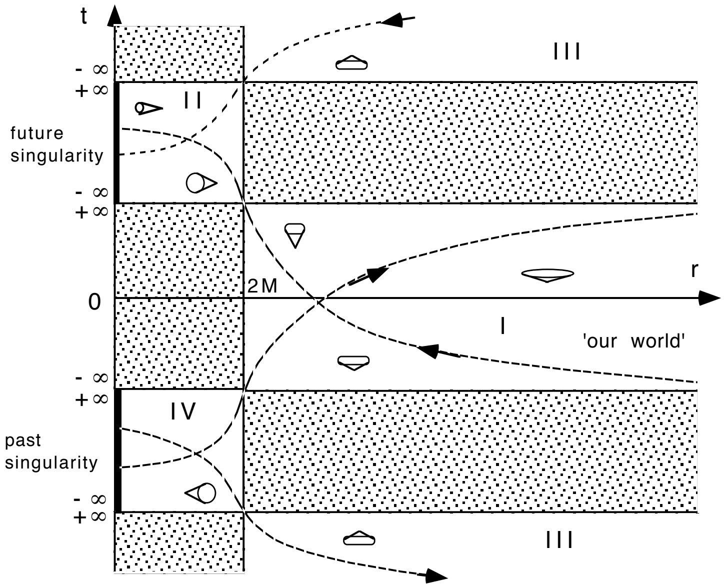

Fig. 5.1. Extension of the Schwarzschild solution from 'our world' beyond the *two* coordinate singularities at $r = 2M$, $t = \pm\infty$. Each point in the diagram represents a 2-sphere of size $4\pi r^2$. A consistent orientation of forward light cones (required from the continuation of physical orbits, such as those represented by dashed lines) is indicated in the different regions. There are also *two* genuine curvature singularities with coordinate values $r = 0$

The Schwarzschild metric in terms of these new coordinates reads

$$ds^2 = \frac{32M^2}{r} \mathrm{e}^{-r/2M}(-dv^2 + du^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad , \tag{5.9}$$

with $r = r(u, v)$. It is evidently regular for $r \to 2M$ and $t \to \pm\infty$, where u and v may remain finite. The Kruskal coordinates are chosen in such a way that future light cones everywhere form an angle of 45° around the $+v$-direction (see Fig. 5.2). Sector I is again the external region outside the Schwarzschild radius ('our world'). One also recognizes the two distinct internal regions II and IV (connected only through the 'instantaneous sphere' that is represented by the origin, $u = v = 0$) with their two separate singularities $r = 0$. Both Schwarzschild surfaces are light-like, and thus represent one-way passages for physical orbits. Their interpretation as past and future horizons is now evident. Sector III represents the second asymptotically flat 'universe'. (It is *not* connected with the original one by a rotation in space, since u is not limited to positive values like a radial coordinate.)

This vacuum solution of the Einstein equations is clearly T-symmetric, that is, symmetric under reflection at the hyperplane $v = 0$ (or any other

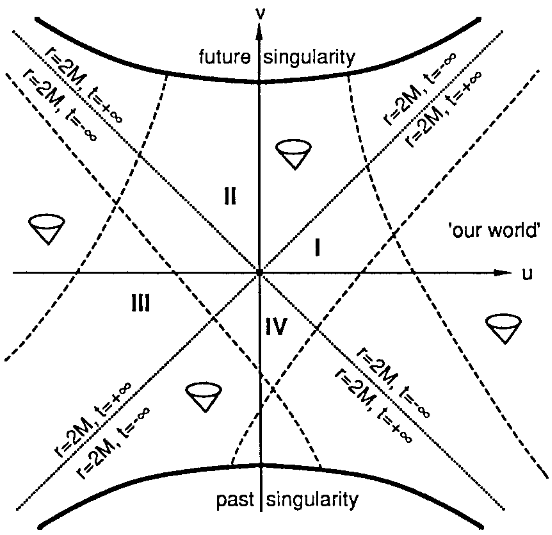

Fig. 5.2. Completed Schwarzschild solution represented in terms of Kruskal coordinates. Forward light cones appear now everywhere with a 45° opening angle around the $+v$-direction. Horizons are indicated by dense-dotted lines, possible orbits as dashed lines. Although the future horizon, say, moves in the outward direction with the speed of light from an inertial point of view, it does not increase in size. The center of the Kruskal diagram defines an 'instantaneous sphere' as a symmetry center, even though it does not specify a specific *external* time t_0

hyperplane t = constant). Therefore, it does neither represent a black hole, nor would it be compatible with the Weyl tensor hypothesis. In the absence of gravitational sources, the Ricci tensor must vanish according to the Einstein equations (5.6), while a singular curvature can only be due to the rest of the Riemann tensor (the Weyl tensor).

A black hole is instead defined as an asymmetric spacetime structure that *arises* dynamically by the gravitational collapse of matter. For example, if the infalling geodetic sphere indicated by the dashed line passing through Sectors I and II of Fig. 5.2 represents the collapsing surface of a spherically symmetric star, the vacuum solution is valid only outside it. Neither a past horizon with its past singularity, nor a second asymptotically flat spacetime exist in this case. They are indeed excluded by regular initial conditions. The coordinates u and v can be extended into the interior only with different interpretation (see Fig. 5.3a, where $u = 0$ is chosen as the center of the collapsing star). This black hole is drastically asymmetric under time reversal, as it contains only a future horizon and a future singularity.

Because of the symmetry of the Einstein equations, a time-reversed black hole — not very appropriately called a *white hole* (Fig. 5.3b) — must also represent a solution. However, its existence in nature would be excluded by the Weyl tensor hypothesis. If it were the precise mirror image of a black

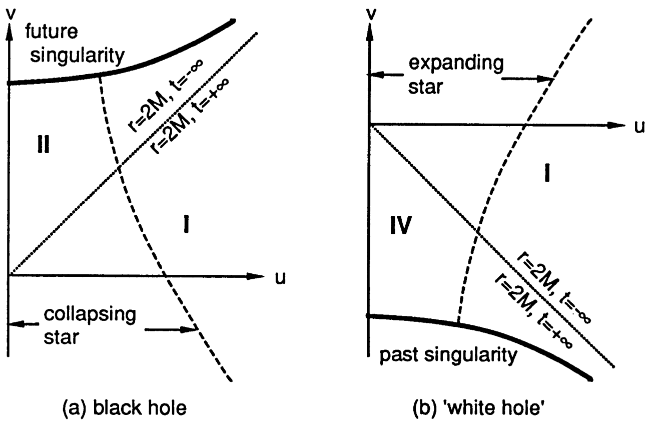

Fig. 5.3a,b. Geometry of a Schwarzschild *black hole* (a) which forms by the gravitational collapse of a spherically symmetric mass, and its time-reverse (b) — usually called a *white hole*

hole, the white hole could describe a star (perhaps with planets carrying life) emerging from a past horizon. This would be inconsistent with an arrow of time that is valid everywhere in the external region. If a white hole were allowed to exist, we could receive light from its singularity, although this light would be able to carry retarded *information* about the vicinity of the singularity only if our arrow of time extended into this region. This seems to be required for thermodynamical consistency, but may be in conflict with an initial singularity (see Sect. 5.3).

Similar to past singularities, also space-like singularities — so-called *naked singularities* — could be 'visible' to us. They, too, were assumed to be absent by Penrose. However, this *Cosmic Censorship* assumption cannot generally be imposed directly as an initial condition. Rather, it has to be understood as a conjecture about the nature of singularities which may *form dynamically* during a collapse from generic initial value data which comply with the Weyl tensor hypothesis. Although counterexamples (in which naked singularities form during a gravitational collapse from appropriate initial conditions) have been explicitly constructed, they seem either to form sets of measure zero (which could be selected by imposing exact symmetries that would be thermodynamically unstable in the presence of quantum matter fields), or to remain hidden behind black hole horizons (see Wald 1997, Brady, Moss and Myers 1998). In the first case, they may be compared with pathological mechanical systems that have occasionally been considered as counterexamples to ergodic behavior. This analogy may readily indicate a relationship between these aspects of general relativity and statistical thermodynamics.

The Schwarzschild-Kruskal metric may be generalized to become the *Kerr-Newman metric*, which describes axially symmetric black holes with angular momentum J and charge Q. This solution is of fundamental importance, since its external part characterizes the final stage of *any* gravitationally collapsing object. For $t \to +\infty$ (although very soon in excellent approximation during a collapse) every black hole may be completely described by the three parameters M, J and Q (except for Lorentz transformations and translations). This result is known as the *no-hair theorem*, as it means that black holes cannot maintain any external structure ('no hair'). It requires that the collapsing star radiates away all higher multipoles of matter and charge (in accordance with a Sommerfeld radiation condition), while conserved quantities connected with short-range forces, such as the lepton or baryon number, disappear from observability. A *white hole* would therefore require coherently incoming (advanced) radiation in order to 'grow hair'. For this reason, *white holes seem to be incompatible with the radiation arrow of our world.* (For their fate — if they had ever come into existence — see Eardley 1974, or Barrabès, Brady and Poisson 1993).

If the internal structure of a black hole is regarded as *irrelevant* for our future in the sense of statistical thermodynamics (in accordance with the no-hair theorem), the gravitational collapse *appears* to violate baryon and lepton number conservation. Even the entropy carried by collapsing matter would disappear from this point of view, in violation of the Second Law. A 'real' violation of any of these conservation (or non-conservation) laws would occur at the singularity that according to a *singularity theorem* must always arise behind a future horizon in the presence of 'normal' matter (see Hawking and Ellis 1973).

Spacetime singularities have particularly dramatic consequences in quantum theory because of the latter's kinematical nonlocality (cf. Sect. 4.2). Consider a global quantum state, propagating on space-like hypersurfaces ('simultaneities'), which define an arbitrary foliation of spacetime and thereby a time coordinate t. If these hypersurfaces somewhere met a singularity, not only the state on this singularity, but also its entanglement with the rest of the universe would cease to exist. In classical description, correlations can exist only as a consequence of incomplete information. Quantum mechanically, definite states would remain left for the non-singular rest of the universe only if the complete state approached the factorizing form $\psi = \psi_{\text{singularity}}\psi_{\text{elsewhere}}$ whenever a singularity forms. Unless all correlations with arising singularities vanished conspiratorially in this way (thus representing a strong final condition), the non-singular part of the universe would *for objective reasons* have to be described by a density matrix ϱ rather than a pure state. Therefore, several authors have argued that quantum gravity must violate unitarity and CPT invariance, and this idea has created a popular speculation ground for introducing an objective dynamical collapse of the wave function induced by gravity (Wald 1980, Penrose 1986, Károlyházy, Frenkel and Lukácz 1986, Diósi 1987, Ellis, Mohanty and Nanopoulos 1989, Percival 1997, Hawking and Ross 1997). Future horizons could then well explain the first step of (4.51).

This indeterminism of a global state vector would not only be inconsistent with canonical quantum gravity (Sect. 6.2), it may also be avoided in quantum field theory on a *classical* spacetime if the foliation defining a time coordinate is always chosen never to encounter a singularity. For example, the spherical Schwarzschild-Kruskal metric could be foliated according to Schwarzschild time t in the external region, and according to the new time coordinate $r < 2M$ (rather than the Kruskal time coordinate v) in its interior. This choice, which would always leave the whole black hole interior in *our* infinite future, has been advocated by 't Hooft (1990). A general singularity-free foliation is given by *York time*, which is defined by hypersurfaces of constant extrinsic spatial curvature scalar K (that describes the embedding into spacetime — see Qadir and Wheeler 1985). Such a foliation appears reasonable, in particular since consequences of an elusive unified field theory may become relevant close to curvature singularities.

This salvation of *global* unitarity is irrelevant to (local) observers who remain outside the horizon, since the reality accessible to them can be completely described by a density matrix ϱ_{ext} in the sense of a Zwanzig projection $\hat{P}_{\text{sub}}$ (cf. (4.26)) — regardless of the choice of a global foliation. The non-unitary dynamics of these density matrices has the same origin as it did in the quantum mechanical subsystems of Sect. 3.3: entanglement. The horizon appears only as a natural objectivization of the boundary, valid for all external observers. One may therefore appropriately describe the phenomenological properties of black holes (including their Hawking radiation — see below) regardless of their singularities. (The latter may even signal the need for a *new* theory). In a universe that is compatible with the Weyl tensor condition, this 'effective non-unitarity' of black holes mimics an indeterminism that would represent a quantum mechanical arrow of time, although this consequence does not *require* black holes and horizons (cf. Chap. 4 and Sect. 6.2.3).

From the point of view of an external observer, the information about matter collapsing under the influence of gravity becomes irreversibly irrelevant, except for the conserved observable quantities M, J and Q. However, the mass of a Kerr-Newman black hole is not completely lost (even if Hawking radiation is neglected). Its rotational and electromagnetic contributions can be recovered by means of a process discovered by Penrose (1969) — see Fig. 5.4. It requires boosting a rocket in the *ergosphere*, the region between the Kerr-Newman horizon, $r_+ := M + \sqrt{M^2 - Q^2 - (J/M)^2}$, and the *static limit*, $r_0(\theta) := M + \sqrt{M^2 - Q^2 - (J/M)^2 \cos^2\theta}$. In this ergosphere, the cyclic coordinate ϕ becomes time-like ($g_{\phi\phi} < 0$) as a consequence of extreme relativistic frame dragging. Because of the properties of this metric, ejecta from the booster which fall into the horizon may possess negative energy with respect to an asymptotic frame (even though this energy is locally positive). Similar arguments hold if the ejecta carry electric charge with a sign opposite to the black hole charge Q. In both cases the mass of the black hole may be reduced by reducing its charge or angular momentum.

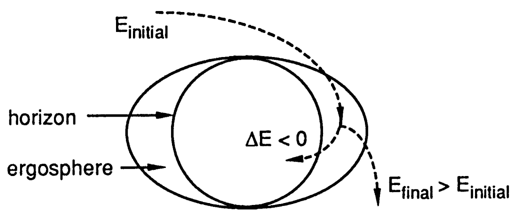

Fig. 5.4. Extraction of rotational energy from a black hole by means of the *Penrose mechanism*, using a booster in the *ergosphere* close to the horizon

The efficiency of this process of drawing energy from a black hole is limited — precisely as for a heat engine. According to a geometro-dynamical theorem (Hawking and Ellis 1973), the area A of a future horizon (or the sum of several such horizon areas) may never decrease. For all known processes which involve black holes, this can be written in analogy to thermodynamics as

$$dM = dM_{\text{irrev}} + \Omega\, dJ + \Phi\, dQ \tag{5.10}$$

(Christodoulou 1970), where the 'irreversible mass change' $dM_{\text{irrev}} \geq 0$ is given by the change of total area of future horizons, $dM_{\text{irrev}} = \frac{\kappa}{8\pi}\, dA$. Here, κ is the *surface gravity*, which turns out to be constant on the horizon. Φ is the electrostatic potential at the horizon, and Ω the angular velocity defined by the dragging of inertial frames at the horizon. The last two terms in this equation describe work done reversibly at the black hole by adding angular momentum or charge, while the first one is analogous to $T\, dS$ because of the inequality $dA/dt \geq 0$. All quantities are defined not with respect to a local frame (where they may diverge), but rather with respect to an asymptotic rest frame of the black hole (where they remain regular because of their diverging red-shift). For the Schwarzschild metric, the surface gravity is $\kappa = 1/4M$. The quantities Φ and Ω are also constant on the horizon, in analogy to other thermodynamical equilibrium parameters, such as pressure and chemical potential, which appear in the expression for the work done on a thermodynamical system in the form $\mu dN - pdV$.

These analogies led to the proposal of the following *Laws of Black Hole Dynamics*, which form a complete analogy to the Laws of Thermodynamics (cf. Bekenstein 1973, Bardeen, Carter and Hawking 1973, Israel 1986):

0. The surface gravity of a black hole must approach an equilibrium value $\kappa(M, Q, J)$ everywhere on the horizon for $t \to \infty$.
1. The total energy of black holes and external matter, measured from asymptotically flat infinity, is constant in time.
2. The sum of the surface areas of all horizons, $A := \sum_i A(M_i, Q_i, J_i)$, never decreases:

$$\frac{dA}{dt} \geq 0 \quad . \tag{5.11}$$

3. It is impossible to reduce the surface gravity to zero by a finite number of physical operations.

The analogy between this version of the third law and its thermodynamical counterpart may have to be modified for other versions because of the negative heat capacity of a black hole. In particular, the surface area A does not disappear with vanishing surface gravity in a similar way as the entropy does with vanishing temperature.

Bekenstein conjectured that these analogies are not just formal, but indicate genuine thermodynamical properties of black holes. He proposed not only a complete *equivalence of thermodynamical and spacetime-geometrical laws and concepts*, but even their *unification*. In particular, in order to 'legalize' the transformation of thermodynamical entropy into black hole entropy A (when dropping hot matter into a black hole), he required that instead of the two separate Second Laws, $dS/dt \geq 0$ and $dA/dt \geq 0$, there be only one *Unified Second Law*

$$\frac{d(S + \alpha A)}{dt} \geq 0 \quad , \tag{5.12}$$

with an appropriate constant α. Its value remains undetermined from the analogy, since the term $\frac{\kappa}{8\pi} dA$, equivalent to $T\,dS$, may as well be written as $\frac{\kappa}{8\pi\alpha} d(\alpha A)$. The *black hole temperature* $T_{\text{bh}} := \frac{\kappa}{8\pi\alpha}$ must classically be expected to vanish, since the black hole would otherwise have to emit heat radiation proportional to AT_{bh}^4 according to Stefan and Boltzmann's law. The constant α should therefore be infinite, and so should the *black hole entropy* $S_{\text{bh}} := \alpha A$. This would require the absence of processes, otherwise allowed by the Unified Second Law, wherein black hole entropy is transformed into thermodynamical entropy (this would violate the area theorem).

Nonetheless, Bekenstein suggested a finite value for α (of the order of unity in Planck units). This was confirmed by means of quantum field theory by Hawking's (1975) prediction of *black hole radiation*. His calculation revealed that black holes must emit heat radiation according to the value $\alpha = 1/4$. This process may be described by means of virtual particles with negative energy tunnelling from a virtual ergosphere into the singularity (York 1983), while their correlated partners with positive energy may then propagate towards infinity. (Again, all energy values refer to an asymptotic frame of reference). The probabilities for these processes lead precisely to a black body radiation with temperature

$$T_{\text{bh}} = \frac{\kappa}{2\pi} \quad , \tag{5.13}$$

and therefore to the black hole entropy [1]

[1] It is important to realize that this is a general result — independent of the precise nature of contributing fields. Therefore, it cannot be used to support any specific theory (such as M-theory). Physical theories can only be confirmed by comparison with *empirical* data.

$$S_{\mathrm{bh}} = \frac{A}{4} \quad . \tag{5.14}$$

The mean wave length of the emitted radiation is of the order $A^{1/2}$.

A black hole not coupled to any quantum fields ($\alpha = \infty$) would possess the temperature $T = 0$ and infinite entropy, corresponding to an ideal absorber in the sense of Sect. 2.2. This is indeed a general property of *classical black body radiation*, that is, of classical electromagnetic waves in thermal equilibrium (Gould 1987). According to (5.13), a black hole of solar mass would possess a temperature $T_{\mathrm{bh}} \approx 10^{-6}\,\mathrm{K}$. In the presence of the 2.7 K background radiation, it would absorb far more energy than it emits (even in the absence of dust or any other matter). Only a black hole of less than about 3×10^{-7} solar masses is sufficiently small, thus hot enough to *lose* mass at the present temperature of the universe (Hawking 1976). There also exists a solution that describes T-symmetric 'holes' embedded in an isotropic heat bath of its own temperature (Zurek and Page 1984). Instead of a future horizon and a future singularity it describes a *spatial* singularity at $r = 0$, representing negative mass. It seems to be unstable against quantum fluctuations of the metric in the region of the Schwarzschild radius, signalling the need for quantum gravity.

A small black hole may completely decay into thermal radiation. The resulting entropy has been estimated to be slightly larger than the black hole entropy if the radiation is emitted into empty space (Zurek 1982b). Since the future horizon and the singularity would thereby also disappear, one seems now confronted with a *genuine global indeterminism* — regardless of any choice of foliations. This problem is known as the *information loss paradox*.

Several possible resolutions of this paradox have been discussed in the literature. Conventional quantum theory would require that the entropy of the radiation is merely a result of its statistical description, which regards photon correlations as irrelevant. The photons in this black body radiation may be entangled to form a pure total state (as they would according to a Schrödinger equation that was formulated on a foliation with slices that never hit the singularity). A similar pure global state of radiation would arise according to the unitary description if a *macroscopic* body decayed from a high energy *pure* state into its ground state by emitting many photons (Page, 1980). This state of the radiation field can for all practical purposes (local or independent photon measurements) be assumed to be a thermal mixture. However, quantum correlations between photons of the incoming (advanced) radiation of a white hole would be dynamically relevant for it to 'grow hair'. A general correlation between the time arrows of horizons and radiation has been derived in the form of a 'consistency condition' for certain de Sitter type universes by Gott and Li (1997). Their model is remarkable in possessing *different* arrows of time in different spacetime regions (separated by an event horizon).

In spite of the absence of any explicit mechanism, this proposal of a unitary description of thermal radiation arising from black hole evaporation remains conventional (except for the possible violation of baryon and lepton number conservation). While not even the *Stoßzahlansatz* is in conflict with microscopic determinism, non-zero physical entropy has been shown to be compatible with a pure state in Sect. 4.2. However, the complete information representing the pure state can only be contained in the whole superposition of all Everett branches. It would be *absolutely* lost in a genuine collapse.

Therefore, the description of black holes by a probabilistic super-scattering matrix $ (Hawking 1976) could have similar roots as the apparent collapse of the wave function or the *Stoßzahlansatz*. An S- (or $-) matrix would in any case not represent a meaningful concept for describing black holes, which — like all macroscopic objects — never become 'asymptotically free' because of their never ending decoherence by their environment (Demers and Kiefer 1996). *Microscopic* 'holes', which could be described as quantum objects by means of an S-matrix, cannot possess the *classical* properties 'black' or 'white', that are analogous to the chirality of sugar molecules — cf. Sect. 4.3.2). They can only be expected to exist as T eigenstates in their symmetric or anti-symmetric superpositions of black and white.

According to other proposals, a *black hole remnant* that conserves all relevant properties (including lepton and baryon number and the complete entanglement with the environment) must always be left behind when a black hole is transformed into radiation. A third possibility will be discussed in Sect. 6.2.3.

General literature: Bekenstein 1980, Unruh and Wald 1982

5.2 Thermodynamics of Acceleration

While the time arrow of black holes is defined by their classical spacetime structure, Hawking radiation is a consequence of *quantum* fields on their spacetime. The existence of this radiation requires the presence of an event horizon, which in turn depends on the world line of an 'observer of reference' (or a family of observers, such as all inertial ones in a flat asymptotic spacetime). For example, the future horizon at $r = 2M$ would not exist for an inertial detector freely falling into the black hole, in contrast to one being held at a certain distance $r > 2M$. The latter would feel a gravitational force unless it were in appropriate rotational motion.

Homogeneous gravitational forces are 'equivalent' to uniformly accelerated frames of reference.[2] Must accelerated detectors then be expected to register heat radiation in an inertial vacuum — similar to the classical

[2] Homogeneous gravitational fields do *not* imply spacetime curvature. The popular error that they do has led to the folklore that accelerated systems have to be described

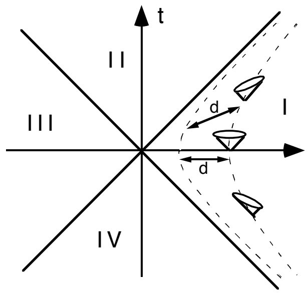

Fig. 5.5. Horizons appearing in Minkowski spacetime to uniformly accelerated observers (characterized by hyperbolic orbits $\varrho := (x^2 - t^2)/4 = \text{constant}$), with proper accelerations given by $a = (2\sqrt{\varrho})^{-1}$. The distance d between two parallel observers (who share co-moving rest frames and asymptotes) remains constant

electromagnetic radiation that they would measure in the Coulomb field of an inertial charge (Sect. 2.3) and equivalent to (5.13)? Horizons may indeed exist in flat spacetime for non-inertial observers. For example, there would be a past and a future horizon for a uniformly accelerated observer in flat Minkowski spacetime, given by the asymptotes of his hyperbolic relativistic orbit (see Fig. 5.5). This observer shares his horizons with a whole family of 'parallelly accelerated' ones (who require different accelerations in order to remain on parallel hyperbolae — similar to two observers at different fixed distances from a black hole). These observers also share comoving rest frames, and thus define accelerated rigid frames with fixed distances d in spite (or rather because) of their different acceleration.

In the x, t-plane this situation appears analogous to the Schwarzschild-Kruskal spacetime (Fig. 5.2), although it is singularity-free, since each point in the diagram now represents a flat R^2 rather than a 2-sphere. If the acceleration *began* at a certain finite time, no past horizon would exist (in analogy to a T-asymmetric black hole). The orbits of this family of observers can be used to define a new spatial coordinate $\varrho(x, t)$ that is constant along each orbit and may be conveniently scaled by $\varrho(x, 0) = x^2/4$. Together with a new time coordinate $\phi(x, t)$ that is related to proper times τ along the orbits by $d\tau = \sqrt{\varrho}\, d\phi$, and the flat coordinates y and z, it defines the *Rindler coordinates* in flat spacetime. In region I of Fig. 5.5 they are related to the Minkowski coordinates by

$$x = 2\sqrt{\varrho}\cosh\frac{\phi}{2} \quad \text{and} \quad t = 2\sqrt{\varrho}\sinh\frac{\phi}{2} \quad . \tag{5.15}$$

---

by *general* relativity. However, all that is required is curved spacetime *coordinates* and the identification of inertial frames with freely falling (apparently accelerated) ones.

The proper accelerations a along $\varrho =$ constant are given by $a(\varrho) = (2\sqrt{\varrho})^{-1}$, while the resulting non-Minkowskian representation of the Lorentz metric,

$$ds^2 = -\varrho\, d\phi^2 + \varrho^{-1}\, d\varrho^2 + dy^2 + dz^2 \quad , \tag{5.16}$$

describes a coordinate singularity at $\varrho = 0$ that is analogous to $r - 2M = 0$ for the Schwarzschild solution. The Minkowski coordinates can therefore be compared with the Kruskal coordinates u and v of Fig. 5.2, while the Rindler coordinates are analogous to the Schwarzschild coordinates.

The Rindler coordinates are also useful for describing the uniformly accelerated point charge of Sect. 2.3 and its relation to a co-accelerated detector, (even though this situation does not possess the full cylinder symmetry of the coordinates). The radiation propagating along the forward light cone of an event on the accelerated world line of the charge must somewhere hit the latter's future horizon, and asymptotically disappear completely into region II from the point of view of an inertial observer. However, it would require *infinite* time for the radiation to reach the horizon if described in terms of co-accelerated (Lorentz-rotated) simultaneities $\phi = constant$, which must all intersect the horizon at the origin. These simultaneities also characterize co-accelerated detectors (those at a fixed distance $\Delta\varrho$ from the charge).

This may explain why, from the point of view of an inertial observer, but not for a co-accelerated one, the accelerated charge radiates (Boulware 1980). While Dirac's radiation reaction (2.24) vanishes for uniform acceleration, the distinction between near-fields and far-fields by their powers of distance according to (2.13), and therefore the definition of radiation, depends on the acceleration of the reference frame. Time reversal symmetry is expressed by the fact that in region I the total retarded field of the charge is identical with its advanced field (except on the horizons), while one has either only retarded outgoing fields in region II, or only advanced incoming fields in region IV (or any superposition of these cases). Therefore, even though global inertial frames are absolutely defined in special relativity, only *relative* acceleration between source and detector is relevant if uniform.

Unruh (1976) was indeed able to show — similar to Mould (1964) for classical radiation — that an accelerated 'particle' detector in the inertial vacuum of a quantum field must register isotropic thermal radiation with an *Unruh temperature*

$$T_{\mathrm{U}} := \frac{a}{2\pi} = \frac{a\hbar}{2\pi c k_{\mathrm{B}}} \quad . \tag{5.17}$$

This is precisely what had to be expected in analogy to (5.13) according to the principle of equivalence. However, the response of a detector cannot be a matter of perspective or definition (as was the distinction between radiation and near-field).

Unruh's result can also be derived by representing the inertial *Minkowski vacuum* $|0_{\mathrm{M}}\rangle$ in terms of 'Rindler particle states', which are defined as harmonic wave modes factorizing in the Rindler coordinates (with frequency

Ω with respect to the time coordinate ϕ).[3] If plane wave modes $e^{i(kx-\omega t)}$ (Minkowski modes) are expanded in terms of these Rindler modes, this expansion induces a *Bogoljubow transformation*, $a_k^+ \to b_{\Omega s}^+ := \sum_k (\alpha_{\Omega s,k} a_k^+ + \beta_{\Omega s,k} a_k)$, for the corresponding 'particle' creation operators. In this relation, the index $s = \text{I}$ or III specifies two Rindler modes (both with time dependence $e^{-i\Omega\phi}$) which vanish in the regions III or I of Fig. 5.5, respectively. On flat simultaneities through the origin they are thus complete on half-spaces with $x > 0$ or $x < 0$. These Bogoljubow transformations linearly combine creation and annihilation operators, since the non-linear coordinate transformations do not preserve the sign of the frequency (ω or Ω). These signs distinguish particle and anti-particle modes in the usual interpretation, and the two terms appearing in the Fourier representation of the field operator, $\Phi(\boldsymbol{r}, t) \propto \int \{\exp[i(kx + \omega t)]a_k + \exp[i(kx - \omega t)]a_k^+\}\, dk$, do not transform separately in this general case.

In terms of the Rindler particle representation, the Minkowski vacuum assumes the entangled form of a BCS ground state of superconductivity,

$$|0_M\rangle = \prod_\Omega \left[\sqrt{1 - e^{-4\pi\Omega}} \sum_n e^{-2\pi\Omega n} \, |n\rangle_{\Omega,\text{I}} \, |n\rangle_{\Omega,\text{III}} \right] \tag{5.18}$$

(Bardeen, Cooper and Schrieffer 1957), where $|n\rangle_{\Omega,s} = (n!)^{-1/2}(b_{\Omega s}^+)^n|0_R\rangle$ are the Rindler particle occupation number eigenstates. The *Rindler vacuum* $|0_R\rangle$, defined by $b_{\Omega s}|0_R\rangle = 0$ for all Ω and s, is therefore different from the Minkowski vacuum. In terms of Minkowski particles it still forms a *pure* state — not a thermal mixture. (The Hawking radiation remaining after the disappearance of a black hole may be expected to form a similar coherent superposition.) The global concepts of quantum particles and their vacua are thus not invariant under non-Lorentzian transformations. In contrast, the actual *quantum state* is invariant and physically defined ('real'), while its interpretation in terms of particle numbers depends on a 'reference basis'in Hilbert space that represents the kinematical status of the corresponding observational instruments. In particular, the Rindler basis (or any 'observable' defined in terms of it) characterizes measurements by means of correspondingly accelerated devices, while a specific 'vacuum' represents an actual state (defined or prepared by means of physical, usually cosmological, boundary conditions). This difference would be obscured in the Heisenberg picture.

Equation (5.18) is the *Schmidt canonical representation* (4.25) of nonlocal quantum correlations between the two sectors I and III (which together are spatially complete for hyperplanes intersecting the origin $x = t = 0$). It illustrates the kinematical nonlocality of a relativistic Minkowski vacuum (see

[3] According to the discussion in Sects. 4.3 and 4.6, the term 'particles' here means field modes with their occupation numbers (oscillator quantum numbers). The concept of *classical* particles should be avoided on a fundamental level of quantum theory. For example, Feynman diagrams merely form an intuitive scheme for calculating the propagation of *quantum states* (wave functionals) $\Psi(t)$.

also Gerlach 1988). The diagonal elements of the reduced density matrix that describes the corresponding 'mixed state' on sectors I or III in the sense of $\hat{P}_{\text{sep}}$ are proportional to $\exp(-4\pi n\Omega)$ for each frequency Ω. They represent a canonical distribution with dimensionless temperature $1/4\pi$ (compatible with the dimensionless time coordinate ϕ). Since proper times along the world lines $\varrho, y, z =$ constant are given by $d\tau = \sqrt{\varrho}\, d\phi = (2a)^{-1}\, d\phi$, the physical energies are $2an\Omega$. The (ϱ-dependent) temperature is therefore $T = a/2\pi$ — in accordance with (5.17). Disregarding quantum correlations with the other half-space thus leads to the *apparent ensemble* of states representing a heat bath. As the detector requires measurement times Δt of the order of $(a\Omega)^{-1}$ in order to resolve a frequency Ω, the acceleration has to remain approximately uniform during this interval of time in order to mimic the presence of an event horizon for this mode.

Although the result (5.17) might have been expected according to the principle of equivalence, the latter is in general only locally applicable. Non-inertial rigid systems are an exception valid for uniform acceleration in flat spacetime (cf. Sect. 2.3). Therefore, Unruh radiation cannot be *globally* compared with Hawking radiation. While the whole future light cone of an event on the world line of a uniformly accelerated object will asymptotically intersect the latter's horizon, only *part* of the future light cone of an event in the external region of a black hole will ever enter its internal region. Similarly, only part of the (past) celestial sphere of an observer may have come close to a black hole horizon, where Hawking radiation originates. (The horizon tends to cover the *whole* celestial sphere of an observer approaching a black hole as a consequence of spacetime curvature. He would have to speed towards the remaining 'hole in the sky' in order not to be swallowed.) Such geometric aspects also determine the generalized Sommerfeld radiation condition that characterizes a specific 'vacuum' (Unruh 1976). Only in the immediate neighbourhood of the horizon can the freely falling observer be equivalent to the inertial one in flat spacetime, and therefore experience a vacuum. While the Unruh radiation is isotropic and T-symmetric, the Hawking radiation specifies a direction in space as well as in time by its non-vanishing energy flux.

A real and observable Rindler vacuum of QED could be produced by a uniformly accelerated ideal mirror (Davies and Fulling 1977). A mirror, representing a plane boundary condition to the field, leads to the removal of an infinite number of field modes (those not matching the boundary condition) with their infinite zero-point energy. For an inertial mirror this leads to an infinite renormalization of energy, defining a 'dressed mirror'. The dressing would not be additive — though still meaningful — for *several* parallel mirrors set at a fixed distance in their rest frame, while the adiabatic variation of their distances would give rise to the finite and observable *Casimir effect* (a force between two closely-spaced mirrors). An *accelerated* mirror, acting as an accelerated boundary, produces a quantum state that would be experienced as a vacuum by a co-accelerated detector, but as a thermal bath by an inertial one. A uniformly accelerated mirror would completely determine this QED state on the concave side of its spacetime hyperbola in Fig. 5.5, while the convex side offers the freedom of additional boundary conditions in regions II or IV (similar to the classical field of a uniformly accelerated charge). According to the equivalence principle, the mirror would even have to 'drag' inertial frames if it were an ideal 'graviton mirror'. (A *virtual* accelerated graviton mirror can be mimicked by a massive plane, since the mass causes relative uniform acceleration between the innertial frames of the two half-spaces.)

The thermodynamical effects of acceleration or curvature are too small to be confirmed with presently available techniques. (For this reason, Bekenstein and Hawking could not yet receive the Nobel prize.) However, they are the consequence of combining two well established theories, and they are all required for the consistency of Bekenstein's generalized thermodynamics. This has been demonstrated by means of beautiful *gedanken* experiments with cyclic processes in the vicinity of black holes (Unruh and Wald 1982).

Although the boosted detector in flat spacetime is widely 'equivalent' to one at a fixed distance from a black hole, its excitation energy is provided by a quite different source. In the case of acceleration, the energy must be drawn from the booster, while for a black hole it would have to reduce the latter's mass, and ultimately lead to its disappearance. However, since the presence of Hawking radiation depends on the inertial state of the observer, the black hole's mass loss must then also be observer-dependent (Hawking's observer-dependent *back-reaction of the metric*). To quote Hawking (1976):

"*If spacetime is quantized, one has to abandon the idea of a metric which is independent of the observer. … The reason is that to determine where one is in space-time one has to 'measure' the metric, and this act of measurement places one in one of the different branches of the wave function in the Everett-Wheeler interpretation.*"

This requires taking into account quantum gravity (Sect. 6.2), while similar arguments would apply to the fuel consumption of the booster according to quantum mechanics on classical (such as flat) spacetime.

General literature: Birrell and Davies 1983

5.3 Expansion of the Universe

Modern cosmology is based on the global spacetime structure, which is described by the same theory (Einstein's general relativity) that was used to describe black holes in Sect. 5.1. Since Hubble's discovery of 1923 we know that the universe is expanding. This is often regarded as a confirmation of general relativity (by an effect that Einstein missed predicting by introducing his cosmological constant, while it was present in Friedmann's solutions of 1922). However, a dynamical universe could as well have been discussed in terms of Newton's theory. In the spherically symmetric case, one would even have obtained the same expansion law (although this model would then have specified an inertial center). Apparently, applying the laws of mechanics and gravity to the whole universe met almost as much reservation (until the beginning of this century) as had their application to the celestial objects a few hundred years earlier. (Kepler and his contemporaries were surprised that planets 'fly like the birds'.) In particular, a *static* universe would require a new and fundamental repulsive force in order to compensate gravity — precisely what Einstein's cosmological constant was supposed to do before the discovery of the Hubble flow. In an open universe these consequences could at most be *obscured*, but not avoided. In Newton's theory, too, an expanding universe would require a big bang that provided the ordered initial kinetic energy.

In Einstein's theory, a homogeneous and isotropic universe is described by the Friedmann-Robertson-Walker (FRW) metric,

$$ds^2 = -dt^2 + a(t)^2 \left\{ d\chi^2 + \Sigma^2(\chi) \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right] \right\} \quad , \tag{5.19}$$

with $\Sigma(\chi) = \sin\chi, \sinh\chi$, or χ, depending on the sign of the spatial curvature, $k = +1, -1$ or 0, respectively. The Friedmann time coordinate t in (5.19) represents the proper time of objects which are at rest in these coordinates ('comoving clocks'). This metric may remain valid at the big bang (for $a = 0$) in accordance with the Weyl tensor hypothesis. Its Weyl tensor does indeed *always* vanish, although this model can be generalized by means of a multipole expansion on the Friedmann sphere (see Halliwell and Hawking 1985, and Sect. 5.4).

The FRW metric depends only on the expansion parameter $a(t)$. Its dynamics, derived from the Einstein equations (5.6) with an added cosmological constant, then assumes the form of an 'energy integral' (with vanishing value of the energy) for $a(t)$, or for some function of it. For example, one obtains for its logarithm, $\alpha = \ln a$,

$$\frac{1}{2}\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{1}{2}\left(\frac{d\alpha}{dt}\right)^2 = -V(\alpha) \quad . \tag{5.20}$$

This logarithmic measure of spatial extension, α, which sends the big bang to minus infinity, will prove convenient in Sect. 6.2. The Friedmann potential $V(\alpha)$ is given by the energy density of matter, $\varrho(a)$, the cosmological constant Λ, and the spatial curvature k/a^2, as

$$V(\alpha) = -\frac{4\pi\varrho(e^\alpha)}{3} - \frac{\Lambda}{3} + ke^{-2\alpha} \quad . \tag{5.21}$$

Essentially the same equation (without curvature term and cosmological constant, but with variable energy) is obtained from Newton's dynamics for the radius of a gravitating homogeneous sphere of matter (Bondi 1961).

The energy density ϱ may depend on a in various ways. In the matter-dominated epoch it is proportional to a^{-3}, while during the radiation era — at less than 10^{-3} times the present age of the universe — it changed with with a^{-4}, since all wave lengths expand with a. Much earlier (for extremely high matter density or curvature), novel and yet speculative theories may have been important. Several proposed quantum field theories imply that the vacuum state of matter fields went through phase transitions in this era (see Sect. 6.1). They may be characterized by an energy density that is independent of a (as during a condensation process). In this era, the matter term in the potential V would *simulate* a cosmological constant. Other contributions, due to a *dilaton field*, have been postulated in string theory as being relevant in the still earlier *Planck era* (of the order $a \lesssim 1$).

The Friedmann potential $V(\alpha)$ is indicated in Fig. 5.6. Just as in classical mechanics, allowed regions require positive kinetic energy $E - V = -V > 0$, while $V = 0$ defines turning points of the cosmic expansion. An upper turning point describes a recollapsing universe, while a lower turning point represents a 'bouncing' universe (without big bang or big crunch singularities).

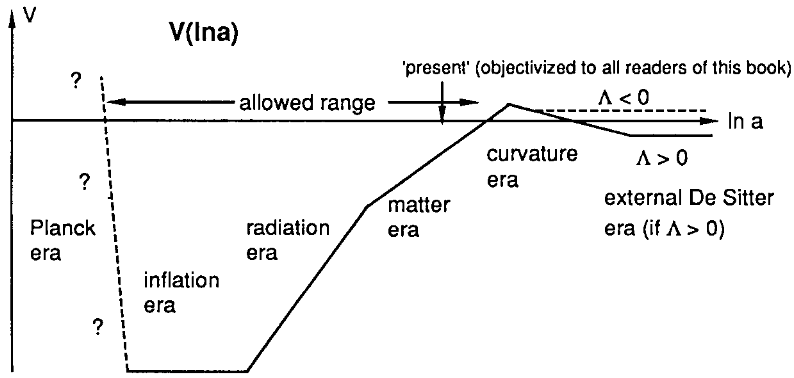

Fig. 5.6. Schematic behavior of the 'potential energy' for the dynamics of $\ln a$ (assuming positive spatial curvature). Turning points of the motion are defined by $V = 0$. Recent observations of distant supernovae indicate that the universe has already passed the maximum of the potential V (which would then have to lie below the axis). Therefore, V will always remain negative, while the dynamical effect of curvature remains negligible

Different eras, described by different equations of state $\varrho(a)$, may possess different analytical solutions $a(t)$. For a dominating (fundamental or simulated) cosmological constant, $a(t) = c\mathrm{e}^{\pm Ht}$, with a 'Hubble' constant $H = \dot{a}/a = \dot{\alpha}$. This situation is called a *de-Sitter era.* For a matter or radiation dominated universe, $a(t) = c't^{2/3}$ or $a(t) = c''t^{1/2}$, respectively, while for very large values of a (low matter densities), and in the absence of a fundamental cosmological constant, the curvature term would dominate. For positive curvature it would then require reversal of the expansion at some time. (See also the model (5.34) and Fig. 5.7 in the subsequent section!)

The finite age of an expanding universe with an initial singularity (a big bang) leads to the consequence that the backward light cones of two events might nowhere overlap. These events would then not possess any (partial) common cause. A sphere formed by the light front originating in a point-like event at the big bang (where $a(0) = 0$) is called a *causality horizon*. Its radius $s(t)$ at Friedmann time t is given by

$$s(t) = \int_0^t \frac{a(t)}{a(t')}\, dt' \quad . \tag{5.24}$$

For matter- or radiation-dominated universes this integral converges even though $a(t) \to 0$ for $t \to 0$, and for vanishing cosmological constant, s remains smaller than the semi-circumference of a closed Friedmann universe. Only (growing) *regions* of the universe may then be causally connected. Quantum entanglement may also be expected to require local *causes*, and therefore to be limited to distances within the causality horizon. This would explain the initial condition (4.56), while it leaves the homogeneity of the universe (that is, its simultaneous beginning) unexplained.

This latter *horizon problem* was the main motivation for postulating a phase transition of the vacuum that leads to an early de-Sitter phase (an exponentially expanding universe). The big bang singularity can then in principle be shifted arbitrarily far into the past, depending on the duration of this phase transition. However, in an extremely short time span (of the order of 10^{-33} sec) the universe, and with it all causality horizons, would *inflate* by a huge factor that was sufficient for the whole now observable 2.7 K background radiation to be causally connected (Linde 1979).

While the Friedmann model is an exact solution of the Einstein equations, and apparently an approximation to the large scale behavior of the real universe, it is not thermodynamically stable against density fluctuations (as discussed at the beginning of this chapter and in Sect. 5.1). This instability is expected to explain the formation of galaxies, galaxy clusters, and possibly larger structures in the present universe. Breaking translational symmetry may either require small inhomogeneities as a seed, or result from 'quantum fluctuations' that become 'real' by means of decoherence (see Calzetta and Hu 1995, Kiefer, Polarski and Starobinsky 1998, and Sect. 6.1). The second mechanism is also known to lead to a quantum limit for the retardation (hysteresis) of phase transitions. The onset of these structures is now believed being observed in the cosmic background radiation.

The arrow of time characterizing these irreversible processes is thus based again on an improbable cosmic initial condition: homogeneity. Boltzmann (1896) already discussed the Second Law in a cosmological context. Under the assumption of an infinite universe in space and time, he concluded that we, here and now, happen to live in the aftermath of a gigantic cosmic fluctuation.

Its maximum (that is, the state of lowest entropy) must have occurred in the distant past in order to explain the existence of fossils and other documents in terms of causal history and evolution. This *improbable* assumption (cf. the remarks at the end of Chap. 3) is an application of the *weak anthropic principle* (Carr and Rees 1979, Barrow and Tipler 1986): we could not exist at another place and time according to this mechanistic world model.

How improbable is the novel initial condition of homogeneity that Boltzmann did not even recognize as an essential assumption? We may calculate just how improbable from the entropy by means of Einstein's relation (3.52). The conventional physical entropy of a homogeneous state in local equilibrium is proportional to the number of particles, N. Gravitational entropy is negligible in the present state of our universe. However, the number of photons in the 2.7 K background radiation exceeds that of massive particles by a factor 10^8. (The mean entropy of a baryon is approximately the same as that of a photon under present conditions — both are of order k_B.) A standard model of a closed universe, with 10^{80} baryons (now often regarded merely as a causally connected 'bubble' in an infinite world), would therefore possess an entropy of order 10^{88} (the number of photons).

In Planck units, the area of a horizon of a neutral spherical black hole of mass M is given by $A = 4\pi(2M)^2$. Its entropy thus grows with the second power of its mass,

$$S_{\mathrm{bh}} = 4\pi M^2 \quad . \tag{5.25}$$

Merging black holes will therefore produce a considerable amount of entropy. If the standard universe of 10^{80} baryons consisted of 10^{23} solar mass black holes (since $M_{\mathrm{sun}} \approx 10^{57} m_{\mathrm{baryon}}$), it would possess a total entropy of order 10^{100}, that is, 10^{12} times its present value. If most of the matter eventually formed a single black hole, this value would increase by another factor 10^{23}. The probability for the present, almost homogeneous universe of $2.7K$ is therefore a mere

$$p_{\mathrm{hom}} \approx \frac{\exp(10^{88})}{\exp(10^{123})} = \exp(10^{88} - 10^{123}) \approx \exp(-10^{123}) \quad , \tag{5.26}$$

when compared with its most probable state in this logarithmic approximation (Penrose 1981). Gravitational contraction offers an enormous entropy capacity for structure and complexity to evolve out of homogeneity in accordance with the Second Law (even though the Second Law may not represent the complete story — cf. Figs. 3.5 and 4.3).

This improbable initial condition of homogeneity as the origin of the Second Law is different from Gold's (1962) proposal to derive the thermodynamical time arrow from expansion *in a causal way* (cf. Price 1996 and Schulman 1997 for critical discussions). While it is true that any non-adiabatic expansion of an inhomogeneous equilibrium system (such as droplets in a saturated gas) leads to a retarded non-equilibrium in our causal world, this would as well be true for non-adiabatic *contraction*. The growing space (and thus phase space, representing increasing entropy capacity) has often been proposed as the master arrow of time, although it is clearly insufficient to explain causality (such as non-equilibrium being *caused* by expansion or contraction). Non-adiabatic *compression* of a vessel would lead to retarded pressure waves emitted from the walls, but not to a reversal of the thermodynamical arrow. The quantitative arguments demonstrate that the phase space created by gravitational contraction (ultimately into black holes) is far more important than that due to the cosmic expansion.

There are other examples of *using* causality in thermodynamical arguments rather than *deriving* it in this cosmic scenario. Gal-Or (1974) discussed retarded equilibration caused by the slow nuclear reactions in stars. Even though nuclear fusion controls the time scale and energy production during early stages of stellar contraction, its entropy production is marginal compared to black hole formation. Another example of causal reasoning is provided by a positive cosmological constant that may counteract gravity and induce *accelerated* expansion. Can it then also reverse the gravitational trend towards inhomogeneity, or even causally explain the observed homogeneity?

This has indeed been suggested on the basis of the *cosmological no-hair conjecture* (Hawking and Moss 1982), which states that the universe must lose all structure and become homogeneous during inflation. However, this conjecture is questionable, since the long range forces described by a cosmological constant cannot force local gravitating systems, in particular black holes, to expand fast and enter a state of homogeneity. (Expanding white holes would require acausally incoming advanced radiation, as explained in Sect. 5.1.) The expansion of diluted parts of the universe would thus accelerate faster under the influence of a cosmological constant than that of its denser parts (unless this difference were compensated by a conspiratorial inhomogeneous initial expansion velocity), thus causing inhomogeneities to grow. 'Proofs' of this no-hair conjecture had therefore to exclude positive spatial curvature.

However, a cosmological constant simulated by a phase transition of the vacuum during the early stages of the universe could overcompensate the effect of gravity until strong inhomogeneities begin to form. Therefore, Davies (1983, 1984) suggested that the homogeneity of our universe was caused by its inflation (see also Page 1984, Goldwirth and Piran 1991). This would mean that the Weyl tensor 'cooled down' as a consequence of this driven spatial expansion — similar to the later occurring red-shifting of the electromagnetic radiation. While these *direct* implications of the expansion define reversible processes, equilibration (in the radiation era or during the phase transition) would be irreversible in the statistico-thermodynamical sense (based on microscopic causality).

This explanation of homogeneity would still have to exclude strong initial inhomogeneities (initial black holes, in particular). In order to represent a causal explanation, it would require the state that precedes inflation to be even less probable than the homogenous state after inflation. What would then happen backwards in time to those many *inhomogeneous* states which are 'now' conceivable?

These questions may be appropriately discussed in terms of a recollapsing universe. Its consistent analysis is meaningful regardless of what will ever happen to our own universe. Would the thermodynamical arrow have to reverse direction when the universe starts recollapsing towards the big crunch after having reached maximum extension? The answer would have to be 'yes' if the cosmic expansion represents the master arrow, but it is often 'no' on the basis of causal arguments continued into this region (as it proved sufficient to counter Gold's original argument). Several authors argued that the background radiation would first have to heat up through wave lenghth contraction (blue shifting), while the temperature gradient between interstellar space and the fixed stars would then be inverted only long after the universe has reached its maximum extension. However, this argument presupposes the overall validity of the 'retarded causality' in question, that is, the absence of conspiratorial correlations in the contraction phase. It would be justified if the relevant improbable initial condition, such as Penrose's Weyl tensor hypothesis, held at only one 'end' of this cosmic history. The absence or negligibility of irreproducible conspiratorial events in our present epoch seeems to indicate either that the universe is globally asymmetric in time, or that it is still 'improbably young' when related to its total duration.

Davies (1984) argued similarly that there can be no reversed inflation at the big crunch, since correlations required for an inverse phase transitions can be excluded as improbable. Instead of a homogeneous big crunch one would either obtain 'de-Sitter bubbles', which would reverse the cosmic contraction *locally* and lead to an inhomogeneous 'bounce', or inhomogenous singularities at variance with a reversed Weyl tensor condition, or both. This argument fails, however, if the required correlations are *caused in the backward direction of time* by a final condition that is thermodynamically symmetric in time to the initial one (cf. also Sect. 6.2.3). Similarly, if the big bang were replaced with a non-singular *homogeneous bounce* in some way, entropy must have decreased previous to the bounce. In all cases, an observer complying with the Second Law would *experience* an expanding universe.

On the other hand, a low entropy big crunch together with a low entropy big bang would constrain the possible histories of the universe far more than the low entropy condition at *one* end. The general boundary value problem (cf. Sect. 2.1) leaves only one complete (initial or final) condition free. Although a condition of low entropy does not define the initial state completely, statistically independent two-time boundary conditions would square the probability (5.26), that is,

$$p_{\mathrm{two-time}} = p^2_{\mathrm{hom}} \approx [\exp(-10^{123})]^2 \approx \exp(-10^{123.301}) \quad . \tag{5.27}$$

The RHS may appear as a small correction to (5.26) in this exponential form, but an element of phase space as small as described by (5.27) would be much smaller than a Planck cell (that represents a pure quantum state). A two-time boundary condition of homogeneity may thus be inconsistent with 'ergodic' quantum cosmology.

The consistency of two-time boundary conditions has been investigated for simple deterministic (dynamically closed) systems (see Cocke 1967 for mechanical, and Schulman 1997 for wave mechanical systems). For example, an open Friedmann universe with vanishing cosmological constant would not even allow two boundaries with $a = 0$, but rather expand forever (in one time direction) in a drastically asymmetric way. Davies and Twamley (1993) discussed the situation of classical electromagnetic radiation in an expanding and recollapsing universe. According to their estimates, our universe will remain essentially transparent all the way between the two opposite radiation eras (in spite of the reversible frequency shift over many orders of magnitude in between). Following an argument by Gell-Mann and Hartle they then concluded that the light emitted causally by all stars before the 'turning of the tide' must propagate according to the Maxwell equations, and remain dominant until the reversed radiation era is reached. Therefore, our universe should possess a macroscopically asymmetric history even if it were bound to recontract.

Craig (1996) argued on this basis, but by *assuming* cosmic history to be essentially symmetric, that the brightness of our night sky at optical frequencies would have to contain a homogeneous component (that is, a non-Planckian high frequency tail in the isotropic background radiation) with intensity at least equalling the intensity of the light now observed from all stars and galaxies. This conclusion assumes tacitly that the retarded radiation emitted by all galaxies before the turning of the tide is *different* from the advanced radiation from (thermodynamically reversed) galaxies during the contraction phase. If this advanced radiation were identical with the retarded one (cf. Sect. 2.1), this would require the light from the expansion era to 'conspire' in order to focus on sources during the formal contraction era. Even though the star light would considerably spread in space during the life cycle of a long-lasting recollapsing universe, it would classically preserve all information about its origin, and thus not be compatible with local reversed sources in our distant future. Therefore, Craig assumed the star light from one side to continue into the radiation era on the temporally opposite side (so that the total intensity is doubled).

However, these conclusions are invalidated when the photon aspect of electromagnetic radiation is taken into account. The restricted information content of photons, emphasized already by Brillouin (1962), was essential also for establishing limits to Borel's argument of Sect. 3.1.2 (cf. Sect. 4.3.3). Each photon, even if emitted into intergalactic space as a spherical wave, disappears from the whole quasi-classical universe as soon as it is absorbed *somewhere*. There is an asymmetry between emission and absorption in the conventional quantum description (cf. Fig. 4.4). If the absorber is itself described quantum mechanically, the localization of the absorbed photon is a consequence of decoherence — reflecting the fact that wave functions live in configuration space rather than in space. In quantum cosmology (without taking into account quantum gravity yet), a reversal of this process would require many Everett components to conspire in order to recohere, while the corresponding final condition *for the total Everett wave function* would not be in any obvious conflict with the initial condition. If the Schrödinger dynamics were instead modified by a collapse of the wave function, this new dynamical law would also have to be reversed in time for Craig's considerations to remain valid.

This problem of consistent cosmic two-time boundary conditions will assume a conceptually novel form in the context of quantum gravity, since any concept of time then disappears from cosmology (Sect. 6.2).

5.4 Geometrodynamics and Intrinsic Time

In general relativity, the 'block universe picture' is traditionally preferred to a dynamical description, as its unified spacetime concept is then manifest. (A historically remarkable exception, aside from cosmology, is the Oppenheimer-Snyder model of gravitational collapse.) Only during the second half of the last century has the dynamical (Hamiltonian) approach to general relativity been appropriately explored, particularly by Arnowitt, Deser and Misner (1962). This has not always been welcomed, as it seems to destroy the beauty of relativistic spacetime symmetry by re-introducing a 3+1 (space and time) representation. However, only in this *form* can the dynamical content of general relativity be fully appreciated. A similarly asymmetric form in spite of relativistic invariance is known from the dynamical description of the electromagnetic field in terms of the vector potential **A** (cf. Chap. 21 of Misner, Thorne and Wheeler 1973).

This dynamical reformulation of the theory requires the separation of unphysical gauge degrees of freedom (which in general relativity simply represent the choice of coordinates), and the skilful handling of boundary terms. The result of this technically demanding procedure turns out to have a simple interpretation. It describes the *dynamics of spatial three-geometry*, ${}^{(3)}G(t)$, that is, a propagation of the intrinsic curvature of space-like hypersurfaces with respect to a time coordinate t that labels a foliation of the dynamically arising spacetime. This foliation has to be *chosen* during the construction of the solution. The extrinsic curvature, describing the embedding of the three-geometries into spacetime, is represented by the corresponding canonical momenta. The configuration space of the three-geometries ${}^{(3)}G$ has been dubbed *superspace* by Wheeler. Trajectories in this superspace define four-dimensional spacetime geometries ${}^{(4)}G$.

This 3+1 description may appear ugly not only as it hides Einstein's beautiful spacetime concept, but also since the required foliation by means of space-like hypersurfaces (on which ${}^{(3)}G$ is defined) is arbitrary. Hence, many trajectories ${}^{(3)}G(t)$ represent the same spacetime ${}^{(4)}G$, which is the physically meaningful object. Only in special situations (such as the FRW metric (5.19)) may there be a 'preferred choice' of coordinates, which reflects their exceptional symmetry. The time coordinate t of a given foliation is just one of the four arbitrary (physically meaningless) spacetime coordinates. As a parameter labelling trajectories it could as well be eliminated and replaced with one of the dynamical variables (cf. Chap. 1), such as the size of an expanding universe. The resulting four-geometry then defines all spacetime distances — including *all* proper times of real or imagined local objects ('clocks'). Classically, spacetime may always be assumed to be filled with a 'dust of test clocks' of negligible mass (cf. Brown and Kuchař 1995). However, they are not required to *define* proper times (except operationally); in general relativity, time as a metric property is itself a dynamical variable (see below). According to the principle of relativity, proper time then assumes the role of Newton's absolute time as a controller of motion for all material clocks.

Einstein's equations (5.6) are second order differential equations, similar to the wave equation (2.1). They may therefore be expected to determine the metric $g_{\mu\nu}(x, y, z, t)$ by means of two boundary conditions for $g_{\mu\nu}$ — at t_0 and t_1, say. (For $t_1 \to t_0$ this would correspond to $g_{\mu\nu}$ and its 'velocity' at t_0. This pair may then also determine the extrinsic curvature.) Since the time coordinate is physically meaningless, its value on the boundaries is irrelevant; two metric functions on three-space, $g^{(0)}_{\mu\nu}(x, y, z)$ and $g^{(1)}_{\mu\nu}(x, y, z)$, are sufficient. Not even their order is geometrically essential, since there is no *absolute* direction of light cones. Similarly, the t-derivative of $g_{\mu\nu}$, resulting in the limit $t_1 \to t_0$, is required only up to a numerical factor (that specifies a meaningless initial 'speed of three-geometry' in superspace).

If one similarly eliminates all *spatial* coordinates from the metric $g_{\mu\nu}$, it describes precisely the coordinate-independent three-geometry ${}^{(3)}G$. In the 'normal' situation, the coordinate-independent content of the Einstein equations should then determine the complete four-dimensional spacetime geometry in between, and in general also beyond, two spatial geometries, ${}^{(3)}G^{(0)}$ and ${}^{(3)}G^{(1)}$. Although the general existence and uniqueness of a solution to this boundary value problem remains an open mathematical problem (Bartnik and Fodor 1993, Giulini 1998), this does not seem to be physically relevant.

The procedure is made transparent by writing the metric with respect to a chosen foliation as

$$\begin{pmatrix} g_{00} & g_{0l} \\ g_{k0} & g_{kl} \end{pmatrix} = \begin{pmatrix} N^i N_i - N^2 & N_l \\ N_k & g_{kl} \end{pmatrix} \quad . \tag{5.28}$$

The submatrix $g_{kl}(x, y, z, t)$ (with $k, l = 1, 2, 3$) on a hypersurface $t =$ constant is now its spatial metric there, while the *lapse function* $N(x, y, z, t)$ and the three *shift functions* $N_i(x, y, z, t)$ define arbitrary increments of time and space coordinates, respectively, for a normal transition to a neighboring space-like hypersurface. These four *gauge functions* have to be *chosen* for convenience when solving an initial value problem.

The six functions forming the remaining symmetric matrix $g_{kl}(x, y, z, t)$ still contain three gauge functions representing the spatial coordinates. Their initial choice is specified by the initial matrix $g_{kl}^{(0)}(x, y, z)$, while the free shift functions determine their change with time. The three remaining, geometrically meaningful functions may be physically understood as representing the two (spin) components of gravitational waves and the 'many-fingered' (local) physical time of relativity, that is, a function that defines the increase of proper times along all time-like world lines connecting points on two neighboring space-like hypersurfaces. These three degrees of freedom need not always be practically separable; all three are gauge-free dynamical variables. In contrast, the lapse function $N(x, y, z, t)$, here in conjunction with the shift functions, determines merely how a specific time coordinate is related to this many-fingered physical time.

Therefore, the three-geometry ${}^{(3)}G$, representing the dynamical *state* of general relativity, is itself the 'carrier of information on physical time' (Baierlein, Sharp and Wheeler 1962): it *contains* physical time rather than *depending* on it. By means of the Einstein equations, ${}^{(3)}G$ determines a *continuum of physical clocks*, that is, all time-like distances from an 'initial' ${}^{(3)}G_0$ (provided a solution of the corresponding boundary value problem does exist). Given yesterday's geometry, today's geometry could not be tomorrow's — an absolutely non-trivial statement, since ${}^{(3)}G_0$ by itself is not a complete initial condition that would determine the solution of (5.6) up to a gauge. A mechanical clock can meaningfully go 'wrong', while for a rotating star one would have to know the initial orientation *and* the initial rotation velocity in order to read time from motion. However, a speed of three-geometry (in contrast to its *direction* in superspace) would be as tautological as a 'speed of time'. Time cannot be time-dependent in any nontrivial way.

In this sense, Mach's principle (not only with respect to time) [4] is anchored in general relativity: time must be realized by physical (dynamical) objects. Dynamical laws (as required by Mach) that do *not* specify (or depend on) an absolute time are characterized by their *reparametrization invariance*, that is, invariance under monotonic transformations, $t \rightarrow t' = f(t)$. In general relativity, the time parameter t labels trajectories in superspace by the values of a time coordinate. No specific choice may 'simplify' the laws according to Poincaré (cf. Chap. 1), and no distinction between active and passive reparametrizations remains meaningful (see Norton 1989).

Newton's equations are, of course, *not* invariant under such a reparametrization. His time t is not merely an arbitrary parameter, but represents a dynamically preferred (absolute) time. Its reparametrization would be characterized by 'Kretzschmann invariance', that is, the trivial invari-

[4] See Barbour and Pfister (1995) for various interpretations of Mach's principle.

ance of the theory under a rewriting of the dynamical laws (for example, by adding a pseudo-force). Nonetheless, Newton's equations can be brought into a reparametrization-invariant *form* by parametrizing the time variable t itself, $t(\lambda)$, and treating it as an additional dynamical variable with respect to λ. If $L(q, \dot{q})$ is the original Lagrangean, this leads to the new variational principle

$$\delta \int \tilde{L}\left(q, \frac{dq}{d\lambda}, \frac{dt}{d\lambda}\right) d\lambda := \delta \int L\left(q, \frac{dq}{d\lambda}\frac{d\lambda}{dt}\right) \frac{dt}{d\lambda} d\lambda = 0 \quad , \tag{5.29}$$

where $t(\lambda)$ has to be varied as a dynamical variable, too. This procedure describes the true meaning of the 'Δ-variation' that often appears somewhat mysteriously in analytical mechanics. Evidently, (5.29) is invariant under the reparametrization $\lambda \rightarrow \lambda' = f(\lambda)$.

Eliminating the formal variable t from (5.29) leads to *Jacobi's principle* (see below), that was partially motivated by the pragmatic requirements of astronomers who did not have better clocks than the objects they were describing dynamically. Their best available clock time was *ephemeris time*, defined by stellar positions that were taken from *tables of ephemeris* which were produced by their colleagues. Since celestial motions are usually strongly 'perturbed' by other celestial objects, they do not offer any obvious possibility to define Newton's time operationally. Jacobi's principle allowed astronomers to solve the equations of motion without explicitly using Newton's time. The dynamical system (5.29) is now often used as a toy model for reparametrization invariant theories. Einstein's equations of general relativity, on the other hand, are invariant under reparametrization of their original time coordinate, $t \rightarrow t' = f(t)$, without any further parametrization $t(\lambda)$. There is no time beyond the many-fingered dynamical variable ${}^{(3)}G$ any more!

In (5.29), $dt/d\lambda =: N(\lambda)$ may be regarded as a Newtonian lapse function. For a time-independent Lagrangean L, t appears as a cyclic variable in this formalism. Its canonical momentum, $p_t := \partial\tilde{L}/\partial N = L - \sum p_i \dot{q}_i = -H$, which is conserved in this case, is remarkable only as its quantization leads to the time-dependent Schrödinger equation. However, the 'super-Hamiltonian' $\tilde{H}$ that describes the thus extended system is trivial:

$$\tilde{H} := \sum p_i \frac{dq_i}{d\lambda} + p_t \frac{dt}{d\lambda} - \tilde{L} = N\left(\sum p_i \frac{dq_i}{dt} - H - L\right) \equiv 0 \quad . \tag{5.30}$$

More dynamical content can be extracted from Dirac's procedure of treating $N(\lambda)$ rather than $t(\lambda)$ as a new variable. The corresponding momentum, $p_N := \partial\tilde{L}/\partial(dN/d\lambda) \equiv 0$, has to be regarded as a constraint, while the *new* super-Hamiltonian is

$$\tilde{H} := \sum p_i \frac{dq_i}{d\lambda} + p_N \frac{dN}{d\lambda} - \tilde{L} = NH \quad . \tag{5.30'}$$

Although $dN/d\lambda$ can here not be eliminated in the normal way by inverting the definition of canonical momentum, it drops out everywhere except in the derivative $\partial\tilde{H}/\partial p_N$, as it occurs only multiplied with the factor p_N which vanishes. The two new resulting Hamiltonian equations are (1) the identically fulfilled $dN/d\lambda = \partial\tilde{H}/\partial p_N = dN/d\lambda$, and (2) $dp_N/d\lambda = -\partial\tilde{H}/\partial N = -H$. Because of $p_N \equiv 0$ one obtains the (secondary) *Hamiltonian constraint* $H = 0$ (but not $\equiv 0$), characteristic for reparametrization invariant theories. In general relativity, there are also three *momentum constraints*, characterizing invariance under spatial coordinate transformations which are 'dynamically' related by the shift functions.

Because of (5.30), with $p_t = -H$, Hamilton's principle (5.29) can be written as $\delta\int(\sum p_i\dot{q}_i - H)dt = 0$. For fixed energy, $H = E$, the second term, H, can be omitted under the integral in the variation. For the usual quadratic form of the kinetic energy, $2T = \sum a_{ij}\dot{q}_i\dot{q}_j = \sum p_i\dot{q}_i = 2(E - V)$, the integrand under the variation can then be written in a form that is homogeneously linear in $dq_i/d\lambda$,

$$\delta\int\sqrt{2(E-V)\sum a_{ij}\frac{dq_i}{d\lambda}\frac{dq_j}{d\lambda}}d\lambda = 0 \quad . \tag{5.31}$$

This is Jacobi's principle (cf. Lanczos 1970), useful for given energy. It is manifestly invariant under reparametrization of λ, as it describes only timeless orbits. Even though the equations of motion could be simplified by using Newton's time, (5.31) does evidently not depend on the choice of λ.

Since the energy E depends on absolute velocities in Newton's theory, Jacobi's principle would be 'Machian' only if the fixed energy represented a universal constraint. Therefore, Barbour and Bertotti (1982) suggested an illuminating nonrelativistic model of Machian mechanics. Their action principle in analogy to (5.31),

$$\delta\int\sqrt{-VT}dt = 0 \quad , \tag{5.32}$$

is universally invariant under reparametrizations of t (just as general relativity). Nothing new can then be discovered from parametrizing t in order to vary $t(\lambda)$ as in (5.29). Barbour and Bertotti also eliminated absolute rotations from their configuration space. While this has other important consequences, it is irrelevant for the problem of time. In general relativity, this 'Leibniz group', consisting of time reparametrizations and spatial rotations, has to be generalized to the whole group of *diffeomorphisms* (general coordinate transformations). In order to eliminate any absolute meaning of a time *coordinate* on spacetime, the Hamiltonian constraint has to be a function on space.

This situation has been carefully analyzed by Barbour (1994a,b). He refers to the empty physical content of a parametrization of t in the form $t(\lambda)$, in contrast to (5.29) in Newton's theory, as *timelessness*. This terminology may be misleading — not only because there are various concepts of time (see also Rovelli 1995). I will argue in Sect. 6.2 that the *rigorous timelessness of the Wheeler-DeWitt equation* (*viz.*, the absence of any time *parameter*) is a specific *quantum* aspect, as it reflects the absence of classical trajectories, or the time-energy uncertainty relation in the case of a Hamiltonian constraint. In classical general relativity, even a constrained Hamiltonian would define a time parameter for a trajectory that describes cosmic history, but not any meaningful concept of a 'speed of cosmic history', which would have to include a 'speed of physical time'.

In a general sense, *global time* may thus be defined by the succession of global physical states (a trajectory through a universal configuration space). Even the topology of this succession (its continuity and order up to a reversal) is determined by *intrinsic* properties of these states. No external parameter is required for this purpose. In general relativity, the global states are ambiguous (they depend on the chosen spacetime foliation), while the resulting spacetime geometry is invariantly defined. The latter determines many-fingered time (that is, invariant *proper times*) for all world lines of local objects (such as 'test clocks' or observers). In contrast to Newtonian action-at-a-distance gravity, proper time as a controller of motion is defined as a *local* concept. However, the dynamical laws define *relative* motion, since the metric is itself a dynamical object. A preferred global time parameter that 'simplified' these dynamical laws would be regarded as *absolute time*.

In the Friedmann model (5.19), where $N \equiv 1$, the increment of the time coordinate t is identical (up to a sign) with the increment of proper times τ of 'comoving' matter (being at rest in the chosen coordinates with $N_i = 0$). In general, the time coordinate t corresponds to the physically meaningless parameter λ of (5.29), while τ is a local version of Newton's time t as the controller of motion. Since homogeneity is presumed on all chosen simultaneities in this simple model, the many fingers of time form one closed 'fist'.

Elimination of the global time parameter t would here merely reproduce the equation of state, $\varrho(a)$ as the corresponding 'trajectory', since ϱ is not a dynamical degree of freedom. There is evidently no intrinsic distinction between expansion and contraction. The single variable a would determine proper times τ for comoving matter up to this degeneracy, since $\dot{a}^2$ is given as a function of a by the energy constraint (5.20). (Because of similar ambiguities, the boundary condition by means of two three-geometries is usually formulated as a '*thin* sandwich problem'.)

Matter can also be described dynamically by means of a homogeneous scalar field $\Phi(t)$, for example with an energy density

$$\varrho = \frac{1}{2}(\dot{\Phi}^2 + m^2\Phi^2) \quad , \tag{5.33}$$

while keeping the symmetries of the Friedmann model intact. The Hamiltonian of this model (without cosmological constant) with respect to the variables $\alpha = \ln a$ and Φ reads

$$H = \frac{e^{-3\alpha}}{2}\left(p_\alpha^2 - p_\Phi^2 + ke^{4\alpha} - m^2\Phi^2 e^{6\alpha}\right) \quad , \tag{5.34}$$

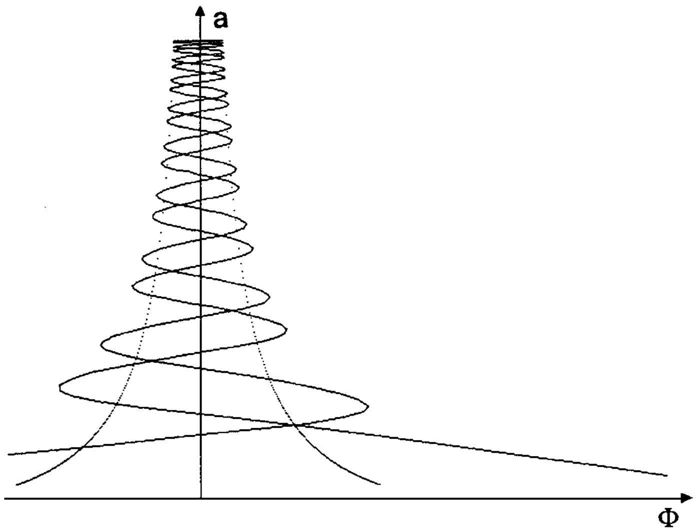

Fig. 5.7. Returning classical orbit representing the dynamics of a closed Friedmann universe, described by its expansion parameter a and a homogeneous massive scalar field Φ. The dotted curve represents vanishing Friedmann potential, $V = 0$. For slightly larger initial values $\Phi(a_0)$ than chosen in the example, the 'inflation era' (that is, the rising non-oscillating lower right part of the orbit) would extend over many orders of magnitude in a before the orbit entered the 'matter-dominated era', where it would then perform a huge number of oscillations before arriving at its turning point of the expansion, a_{max}. (After Hawking and Wu 1985)

where k is again the sign of the spatial curvature, while the canonical momenta are $p_\alpha = e^{3\alpha}\dot{\alpha}/N$ and $p_\Phi = -e^{3\alpha}\dot{\Phi}/N$. A timeless trajectory for a closed universe ($k = 1$) in this model is depicted in Fig. 5.7.

The symmetry of this model can be relaxed by means of a multipole expansion on the Friedmann sphere,

$$\Phi(\chi, \theta, \phi) = \sum a_{nlm} Q^n_{lm}(\chi, \theta, \phi) \quad , \tag{5.35}$$

where $Q^n_{lm}(\chi, \theta, \phi)$ are three-dimensional spherical harmonics (Halliwell and Hawking 1985). The variable Φ in (5.33) represents the monopole component, $\Phi = a_{000}$, since $Q^0_{00} = 1$. A similar expansion of the metric tensor field g_{kl} requires vector and tensor harmonics in addition to the scalar harmonics Q^n_{lm}. Only the tensor harmonics turn out to represent physical (geometrical) properties, while the scalar and vector harmonics describe gauge degrees of freedom. In particular, the time parameter t does not exactly represent proper time of comoving matter any more in this 'perturbed Friedmann model'.

If there is a Hamiltonian constraint, $H(p, q) = 0$ in its general form, multiplication of the Hamiltonian with a function $f(p, q)$, $H \to H' = fH = 0$, would induce an orbit-dependent reparametrization $t \to t'(t)$, with $dt'/dt =$

$f(p(t), q(t))$, as can be seen by writing down the new Hamiltonian equations. For example, the choice $f \equiv -1$ would induce an inversion of the time parameter for all trajectories. Therefore, the factor $e^{-3\alpha}$ in (5.34) is irrelevant for the time-less trajectory.

In (5.34) and its generalizations, the kinetic energy of matter occurs with a negative sign (that is, with negative dynamical mass), since it entered the Hamiltonian as a source of gravity (representing negative potential energy). In Friedmann type models, all gauge-free geometric degrees of freedom but the global expansion parameter a (or its logarithm) share this property (Giulini and Kiefer 1994, Giulini 1995). The kinetic energy is thus not positive definite in cosmology, while the metric that it defines in infinite-dimensional superspace is *super-Lorentzian* (with signature $- + + + + + \ldots$) in this case.

In the familiar case of mechanics, vanishing kinetic energy, $E - V = 0$, describes turning points of the motion. However, since there are no forbidden regions for indefinite kinetic energy, the boundary $V = V - E = 0$ need here not force the trajectories to come to a halt and reverse direction. In general, this condition now describes a smooth transition between 'subluminal' and 'superluminal' directions in superspace (not in space!) — see Fig. 5.7. A trajectory would be reflected from an *infinite* potential 'barrier' only if this were either negative at a time-like boundary, or positive at a space-like one. Reversal of the cosmic expansion at $a_{\max}$ requires the vanishing of an appropriate $V_{\text{eff}}(\alpha)$ that includes the actual kinetic energy of the other degrees of freedom (similar to the effective radial potential in the Kepler problem).

In the Friedmann model, a point on the trajectory in configuration space determines Friedmann time t (that could be read from comoving test clocks) — except where the curve intersects itself. In a mini-superspace with more than two degrees of freedom (with a material clock, for example), physical time on a trajectory is generically determined *uniquely* by the dynamical state. This demonstrates that the essential requirement for the state to represent a carrier of information about time is reparametrization invariance of the dynamical laws — not its spacetime geometrical interpretation.

A drastic abuse of a time *parameter* is entertained in Veneziano's (1991) string model, based on a dilaton field Φ (with dynamics different from (5.34)). Its equations of motion lead to a time dependence of the form $f(t - t_0)$, with an integration constant t_0 that determines the value of the time parameter at the big bang (where $\alpha = -\infty$). A translation $t_0 \to t_0 + T$ would be meaningless (as pointed out already by Leibniz). The solution for $t < t_0$, where expansion *accelerates* exponentially in this model, has been interpreted as 'pre-big-bang', while the absence of a smooth connection between pre- and post-big-bang has then been regarded as a 'graceful exit problem' (Brustein and Veneziano 1994). However, this (in any case speculative) model has simply two different solutions, which could possibly be related through an infinite parameter time, $t = \pm\infty$ — similar to Schwarzschild time at a horizon. Coordinate times $t < t_0$ would then represent physical times *later* than $t > t_0$, while a continuation through t_0 is merely formal (Dabrowski and Kiefer 1997).

The shift functions N_i of (5.28) can be *chosen* to vanish even when symmetries are absent. The secondary *momentum constraints* $H_i := \partial\tilde{H}/\partial N_i = 0$ (which conserve vanishing canonical momenta p_{N_i}, and which are fulfilled automatically for the Friedmann solution because of its symmetry) have then to be solved explicitly. The lapse function $N(x, y, z, t)$ now determines truly *many-fingered* time (as a spatial *field* on the dynamically evolving hypersurface) with respect to the coordinate t. If, nonetheless, N is chosen as a function of t only, the foliation proceeds everywhere according to physical time (*normal* to the hypersurface) with fixed 'comoving' coordinates.

This is not always a convenient choice. For example, observers passing very close by a black hole horizon (without entering it) would experience an extreme time dilation (or observe the stars moving very fast through a little hole in the sky). In a recollapsing universe they could enter the expansion era within relatively short proper time. (This would render the immediate vicinity of horizons very sensitive to any conceivable cosmic *final* condition — see Zeh 1983 and Sect. 6.2.3.) In this case, a foliation according to *York time*, mentioned in Sect. 5.1, may be appropriate, since it arrives 'simultaneously' at all final singularities. Note, however, that the external curvature scalar K, which defines York time, is *not* a function of state, $f({}^{(3)}G)$.

Among the simplest inhomogeneous models are the spherically symmetric ones, with a metric

$$ds^2 = -N(\chi, t)^2 dt^2 + L(\chi, t)^2 d\chi^2 + R(\chi, t)^2 \left[d\theta^2 + \sin^2\theta d\phi^2\right] \quad . \tag{5.36}$$

They contain one remaining spatial gauge function, that has to be eliminated by means of the momentum constraint $H_\chi = 0$. It is analogous to Gauß' law in electrodynamics, as it refers to the radial coordinate.

Qadir (1988) proposed an illustrative toy model for such an inhomogeneous universe (Fig. 5.8). It forms a generalization of the Oppenheimer-Snyder model for the gravitational collapse of a homogeneous spherical dust cloud (see Misner, Thorne and Wheeler 1973, Chap. 32). The latter model pastes a spherical spatial boundary of a contracting, positive curvature Friedmann universe smoothly and consistently onto the spatial boundary of a Schwarzschild-Kruskal solution that extends to spatial infinity. Qadir connects the Schwarzschild solution in turn with another (much larger) partial Friedmann universe with much smaller mass density. This pasting at two spatial boundaries, with Friedmann coordinate values χ_1 and χ_2, say, is consistent if the masses of the two partial Friedmann universes are identical, and can thus be identified with the Schwarzschild mass M. The interpolating vacuum region is a strip from Fig. 5.2, spatially bounded by two non-intersecting geodesics which lead from the past to the future Kruskal singularity.

In order to comply with the Weyl tensor hypothesis as much as possible, Qadir assumed the two partial Friedmann universes to touch at the big bang. (In this deterministic model, an initial inhomogeneity is required as a seed.) Since the denser part of this toy universe feels stronger gravitational attraction, it contracts (or its expansion decelerates) faster. An empty

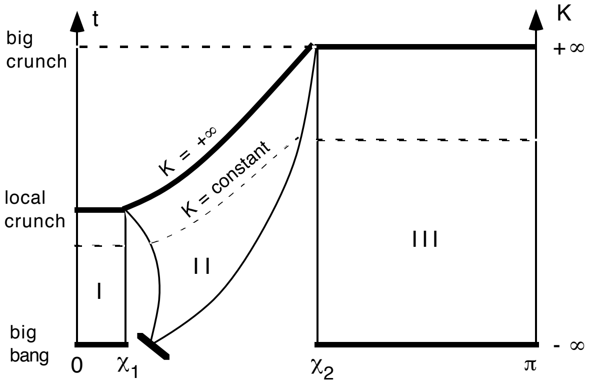

Fig. 5.8. Qadir's 'suture model' of a collapsing homogeneous dust cloud, I, in an expanding and recollapsing Friedmann universe, III. The Friedmann surfaces at χ_1 (left) and χ_2 (right) are initially identified. The Weyl tensor, representing the gravitational degrees of freedom, is chosen to vanish initially (except at the spatial boundary between the two regions), but will evolve and grow with the emerging interpolating 'Schwarzschild-Kruskal corridor', II, (a strip from Fig. 5.2). This corridor, with vanishing stress-energy tensor, must build up between the two regions with different positive spatial curvatures and energy densities. The spatial boundaries of the three spacetime regions have to be identified (including proper times on them, all chosen to start at the big bang). In a picture due to Penrose, the local crunch inside the black hole (region I) together with its attached Kruskal singularity (in region II) appears as a 'stalactite' hanging from the ceiling (representing the big crunch singularity — in region III). In contrast, there is only one (piecewise homogeneous) big bang singularity (a flat floor in Penrose's picture) at $K = -\infty$, that has been chosen as the first slice of the foliation (with $t = 0$)

'Schwarzschild corridor' must then form at the density discontinuity, and grow in size with increasing temporal distance from the big bang. As the energy-momentum tensor vanishes in the Schwarzschild-Kruskal region, its curvature is entirely due to the Weyl tensor, while the latter vanishes inside the two partial Friedmann universes. The time arrow of this process of 'gravitational monopole radiation' (the formation of the corridor with its non-zero gravitational degrees of freedom) is once more a consequence of the special initial condition.

This model is certainly interesting as an illustration of the Weyl tensor hypothesis, but it does not describe statistical (entropic) aspects. For this purpose, many degrees of freedom (such as transversal gravitational waves) would have to be taken into account. Qadir's cosmic evolution process simply describes an example of motion away from the chosen initial state; motion similar to that normally found in unbound mechanical systems (regardless of any statistical considerations). The term 'initial' refers here to the starting point of the computation, but not necessarily to a physical direction of time.

General literature: Chap. 21 of Misner, Thorne and Wheeler 1973