Matter X waves

Claudio Conti* and Stefano Trillo†
National Institute for the Physics of Matter, INFM- Roma Tre, Via della Vasca Navale 84, 00146 Rome - Italy
(Dated: July 8, 2004)

We predict that an ultra-cold Bose gas in an optical lattice can give rise to a new form of condensation, namely matter X waves. These are non-spreading 3D wave-packets which reflect the symmetry of the Laplacian with a negative effective mass along the lattice direction, and are allowed to exist in the absence of any trapping potential even in the limit of non-interacting atoms. This result has also strong implications for optical propagation in periodic structures.

PACS numbers:

Matter waves are a natural manifestation of large scale coherence of an ensemble of atoms populating a fundamental quantum state. The observation of Bose-Einstein condensates (BECs) in dilute ultra-cold alkalis [1] has initiated the exploration of many intriguing properties of matter waves, whose mascroscopic behavior can be successfully described via mean-field approach in terms of a single complex wave-function with well defined phase across the atom cloud [2]. Large scale coherence effects are usually observed by means of 3D magnetic or optical confining potentials (also 1D cigar-like or 2D disk-shaped BEC are possible [3]) in which BECs are described by their ground-state wave-function However, trapping can also occur in free space (i.e. without a trap) through the mutual compensation of the leading-order (two-body) interaction potential and kinetic energy, leading to bright (dark) solitons for negative (positive) scattering lengths. This phenomenon has been observed only in 1D (i.e., soliton waveguides, a trap confining the BEC transversally to the soliton) [4]. In 2D and 3D, free-space localization cannot occur due to collapse instability of solitons, and even in a trap collapse usually prevents stable formation of BEC [5, 6], needing stabilizing mechanisms [7].

A lot of attention was also devoted to periodic potentials due to optical *lattices* [8], where the behavior of atom mimic those of electrons in crystals or photons in periodic media [9], and exhibit effects which stem from genuine atom coherence [10]. In 1D (elongated) lattices bright (gap) solitons can form also in the presence of repulsive interactions [11]. In this letter we predict that a novel trapping phenomenon occurs when the full 3D dynamics is retained in a 1D lattice. Specifically, under the conditions for which the Bloch state associated with the lattice has a negative effective mass, the natural state of the BEC is a localized *matter X wave* characterized by a peculiar bi-conical shape. The atoms are organized in this way in the absence of any trap, solely as the result of the strong anisotropy between the 1D modulation and the free-motion in the 2D transverse plane. Furthermore, due to axial symmetry, the atoms can experience a collective motion with given velocity along the lattice, resulting into wave-packets traveling undistorted. Remarkably, these matter X waves are not only observable individually, but allow also to describe physically realizable BECs as their superposition, each component maintaining a constant (in time) number of atoms.

The very first step of our derivation is similar to that of matter gap solitons. Starting from the (mean-field) Gross-Pitaevskii (GP) equation [2] with an optical standing wave potential and no additional trap (we set $\eta \equiv \frac{\hbar^2}{2m}$)

$$i\hbar\partial_t\psi = -\eta\nabla^2\psi + 4\Gamma\sin^2(kz/2)\psi + a|\psi|^2\psi = 0, \quad (1)$$

we assume axial symmetry around z and decompose the wave-function $\psi = \psi(r, z, t)$ ($r^2 \equiv x^2 + y^2$) into its forward and backward components as

$$\psi = [\psi_f(r, z, t)e^{ikz/2} + \psi_b(r, z, t)e^{-ikz/2}]e^{i\frac{k^2-8\Gamma}{4\hbar}t}, \quad (2)$$

which, for $\psi_{f,b}$ slowly varying in z, and dropping rapidly rotating terms, allows us to reduce the GP equation (1) to the coupled equations

$$\begin{aligned}\mathcal{L}_+\psi_f + \Gamma\psi_b - a(|\psi_f|^2 + 2|\psi_b|^2)\psi_f = 0,\\ \mathcal{L}_-\psi_b + \Gamma\psi_f - a(|\psi_b|^2 + 2|\psi_f|^2)\psi_b = 0,\end{aligned} \quad (3)$$

where $\mathcal{L}_\pm \equiv i\hbar\partial_t \pm i\eta k\partial_z + \eta\nabla^2_\perp$, and $\nabla^2_\perp \equiv \partial^2_r + r^{-1}\partial_r$. In the linear limit $a = 0$, the plane-wave [$\exp(i\kappa z - iEt/\hbar)$] linear dispersion relation associated with Eqs. (3) has two branches $E = E_\pm(\kappa) = \pm\Gamma\sqrt{1 + p^2}$ (we set $p \equiv \kappa k\eta/\Gamma$), exhibiting an energy gap of width 2Γ. The coupling between ψ_f and ψ_b causes the structure to be strongly dispersive near band-edge and the linear dynamics of atoms to be governed by strong Bragg reflection. Nevertheless, in the 1D limit ($\eta = 0$), where Eqs. (3) were obtained previously [11], the nonlinearity (both attractive $a < 0$ and repulsive $a > 0$) induces self-transparency mediated by a two-parameter family of moving bright gap solitons, so-called because they exist in the gap seen in the soliton moving frame [12]. In the attractive case, one might think that the nonlinearity can balance also the kinetic transverse term $\nabla_\perp$ leading to bell-shaped 3D atom wave-packets [13]. We show in the following that, contrary to this expectation, close to the lower band edge $E = E_-$, the atomic wave-function takes a completely different form. To this end we apply a standard envelope function (or effective mass [15]) approximation [14], searching for spinor solutions $\overrightarrow{\psi} = [\psi_f \ \psi_b]^T$ of the form

$$\overrightarrow{\psi} = \epsilon\phi(\epsilon r, \epsilon z, \epsilon t)\overrightarrow{\psi}_- \exp(i\kappa z - itE_-/\hbar) + O(\epsilon^2) \quad (4)$$

where ϵ is a small expansion parameter, ϕ is slowly modulating the Bloch state with amplitude $\overrightarrow{\psi}_- = [\psi_{f-} \ \psi_{b-}]^T$ (eigenvector of Eqs. (3) with $a = 0$ corresponding to the eigenvalue E_-). At the leading order we find that ϕ obeys the following asymptotic equation

$$i\hbar\partial_t\phi + iE'_-\partial_z\phi + E''_-\partial_z^2\phi + \eta\nabla_\perp^2\phi - \chi|\phi|^2\phi = 0 \quad (5)$$

where $\chi = \frac{a}{2}\frac{3+2p^2}{1+p^2}$, and $E'_- \equiv \frac{dE_-}{d\kappa}$, $E''_- \equiv \frac{d^2E_-}{d\kappa^2}$ account for dispersion. For sake of simplicity we deal henceforth with the strict band-edge case $\kappa = 0$, which can be prepared by acting on the wave-number and on the potential parameter [16]. In this case Eq. (5) reads explicitly as

$$i\hbar\partial_t\phi + \frac{\hbar^2}{2m}\left(\nabla_\perp^2 - \frac{m}{m_e}\partial_z^2\right)\phi - \frac{3a}{2}|\phi|^2\phi = 0, \quad (6)$$

where $-m_e \equiv -m\Gamma/(\eta k^2)$ is the negative effective mass associated with the lattice, in turn determining the *hyperbolic* character of (GP) Eq. (6). The scaling transformation $z, r, t, \phi \to z_0 z, r_0 r, t_0 t, c_0\phi$ with $r_0^2 = z_0^2 m_e/m$, $t_0 = 2m_e z_0^2/\hbar$, $c_0^2 = 2\hbar/(3|a|t_0)$, z_0 being a length scale, allows us to use dimensionless variables. In the attractive case ($a < 0$), Eq. (6) admits matter X waves solutions of the kind $\phi = \varphi(r, z)\exp(i\mu t)$, where $\varphi(r, z)$ is indeed an X-shaped invariant envelope [17]. Notice that X waves, well known in optics [18], acoustics [19], or microwaves [20], as non-spreading (in space and time) solutions of the *linear* Helmotz wave equation [21], have been only recently discovered for Schödinger-type models of the form of Eq. (6), by analyzing the so-called paraxial (linear [22] or nonlinear [17]) propagation in dispersive media. However, we emphasize that, while an optical or acoustic field retains a directly observable spatio-temporal X-shape [17, 18], in the case of a Bose gas, the local density of atoms $|\psi|^2$ has an X-shaped spatial envelope $|\phi|^2$ modulating a term $\cos^2(kz/2)$ (periodic with lattice period), due to the form of Eq. (2).

To assess further the regimes of observability of matter X waves, we discuss two crucial issues. First, by extending the analysis of Ref. [17], we are able to show that X wave solutions of Eq. (6) exist also in the (far more common) case of repulsive nonlinearity ($a > 0$). As an example, we display in Fig. 1 the atom density corresponding to the stationary solution with eigenvalue $\mu = 0$, which shows a dense core accompanied by bi-conically shaped regions of lower density. As shown in the insets, the signature of the X-shape is a single peak on-axis ($r = 0$) and a double peak off-axis ($r \neq 0$). The envelope $|\phi|^2$ exhibits also slow oscillations and modulate the fast sinusoidal variation of the density $|\psi|^2$.

Second, our aim is to show that non-spreading atom

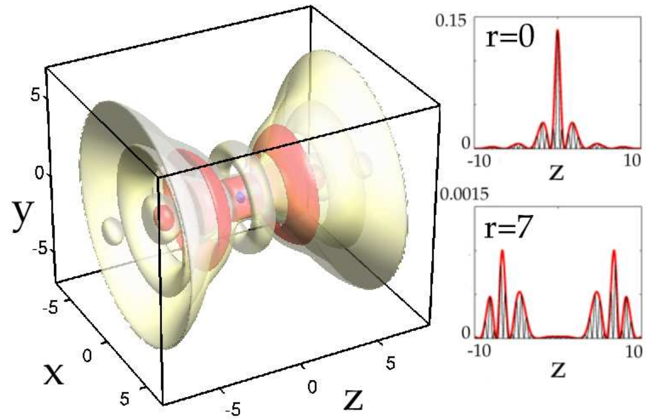

FIG. 1: (Color online) Surfaces of constant (high, intermediate, and low in blue, red, and bronze, respectively) envelope atom density $|\phi|^2$ of a matter envelope X wave solution ($\mu = 0$) of Eq. (6). The insets show on-axis ($r = 0$) and off-axis ($r = 7$) longitudinal profiles of the overall ($|\psi|^2$, black) and envelope ($|\phi|^2$, red) densities. We fix $z_0 = 10/k$ to have dimensionless units.

X-shaped BECs can be formed also in an *ideal noninteracting gas* ($a = 0$), which is realizable by exploiting Feshbach resonance [4]. This is in contrast with other known settings where BEC needs either a confining potential (as in pioneering experiments [1]), or the nonlinearities to balance kinetic spreading. Ultimately, this stems from the fact that X waves have a finite linear limit, unlike solitons of standard (elliptic) GP equation whose amplitude vanishes as $a \to 0$. To demonstrate, however, that *linear* matter X waves are observable, we need to address the fundamental issue of their norm finiteness. To this end let us solve Eq. (6) in dimensionless form with $a = 0$

$$\left(i\partial_t + \nabla_\perp^2 - \partial_z^2\right)\phi = 0, \quad (7)$$

by seeking for envelopes moving with velocity v

$$\phi(r, z, t) = \varphi(r, \zeta)\exp(-ivz/2 + itv^2/4), \quad (8)$$

where $\zeta \equiv z - vt$, and φ turns out to obey the equation

$$\partial_r^2\varphi + r^{-1}\partial_r\varphi - \partial_\zeta^2\varphi = 0. \quad (9)$$

The general solution of Eq. (9)

$$\varphi(r, \zeta) = \int_0^\infty f(\alpha)J_0(\alpha r)e^{i\alpha\zeta}d\alpha, \quad (10)$$

represents a class of envelope X waves specified by their spectrum $f(\alpha)$, which generalize to (Schrödinger-type) Eq. (7) the X wave solutions of Helmotz equation [21]. An exponentially decaying (with arbitrary inverse width Δ) spectrum $f_x(\alpha) = \exp(-\alpha\Delta)$, yields the simplest (or fundamental) X wave $\varphi_x = \left[r^2 + (\Delta - i\zeta)^2\right]^{-1/2}$, while $f_x^{(n)} = \alpha^n \exp(-\alpha\Delta)$ $(n = 1, 2, ...)$ defines the *derivative*

X waves $\varphi_x^{(n)} = d^n\varphi_x/d\Delta^n$. In general from Eq. (10), we obtain the full envelope X wave solution of Eq. (7) as

$$\phi(r,z,t) = \int_0^\infty f(\alpha)J_0(\alpha r)e^{i(\alpha-\frac{v}{2})z+i(\frac{v^2}{4}-\alpha v)t}d\alpha, \quad (11)$$

where $|\phi|^2$ clearly travels undistorted along z. However, we have to face the pitfall that these X waves do not represent physical objects since their norm diverges. This follows from the *transverse scalar product* $<\phi|\hat{\phi}>_\perp (z,t)$ [25] which yields for any pair of solutions $\phi, \hat{\phi}$ of Eq. (7) with different spectra $f, \hat{f}$ (but equal velocity v)

$$<\phi|\hat{\phi}>_\perp = 2\pi\int_0^\infty f(\alpha)\hat{f}^*(\alpha)\alpha^{-1}d\alpha. \quad (12)$$

Since $<\phi|\hat{\phi}>_\perp$ does not depend on z and t, the 3D norm $<\phi|\phi>$ of any X wave ϕ diverges, thus requiring an (unphysical) infinite number of atoms. Remarkably, however, finite norm beams can be generally constructed by introducing new orthogonal X-waves [24]. Inspired by Eq. (12), we can exploit the orthogonality of associated Laguerre polynomials $L_q^{(1)}(x)$ $(q = 0,1,2,...)$ with respect to the function $x\exp(-x)$, to introduce a numerable class of (transversally) orthogonal X waves $\phi_q^\perp(r,z,t|v)$ defined by the following spectra (and parametrically by their velocity v)

$$f_q^\perp(\alpha) = \frac{\Delta\alpha}{\pi\sqrt{2(q+1)}}L_q^{(1)}(2\Delta\alpha)e^{-\Delta\alpha}. \quad (13)$$

The X waves $\phi_q^\perp$ satisfy the orthogonality relation $<\phi_p^\perp|\phi_q^\perp>_\perp = \delta_{pq}/4\pi$, with δ_{pq} the Kronecker symbol. Importantly, the 3D scalar product shows that such waves are orthogonal also with respect to the velocity, i.e. any pair of waves with velocity u and v satisfies the relation

$$<\phi_p^\perp(r,z,t|v)|\phi_q^\perp(r,z,t|u)> = \delta_{pq}\delta(v-u). \quad (14)$$

From Eq. (14) it is natural to consider the solution $\phi = \phi_\Sigma$ of Eq. (7) given by the superposition of orthogonal X waves $\{\phi_q^\perp\}$ travelling with different velocities v as

$$\phi_\Sigma(r,z,t) = \sum_q\int_{-\infty}^\infty C_q(v)\phi_q^\perp(r,z,t|v)dv. \quad (15)$$

From the orthogonality relation (14), we find that the total number of atoms is

$$\mathcal{N}_\Sigma = <\phi_\Sigma|\phi_\Sigma> = \sum_q \mathcal{N}_q, \quad (16)$$

where $\mathcal{N}_q = \int_{-\infty}^\infty |C_q(v)|^2dv$ represents the atom number of the q-th X wave component $\phi_q^\perp(v)$ of the wave-packet. Therefore we obtain the remarkable result that, while the superposition ϕ_Σ generally describes atom wave-packets which evolve in time, such evolution preserves the distribution of atom number among the X wave components.

The importance of Eq. (15) stems from the fact that ϕ_Σ describes a wide class of physical atom beams. To show this, we start from the integral representation of $\phi_q^\perp$ in term of its spectrum (13), which yields

$$\phi_\Sigma = \int_{-\infty}^\infty\int_0^\infty F(\alpha,v)J_0(\alpha r)e^{i(\alpha-\frac{v}{2})\zeta-i\frac{v^2}{4}t}\alpha d\alpha dv, \quad (17)$$

where $F(\alpha,v) = \alpha^{-1}\sum_q C_q(v)f_q^\perp(\alpha)$. By introducing new variables k_t, k_z such that $\alpha = k_t$ and $v = 2(k_t - k_z)$, and setting $U(k_t,k_z) = 2F(k_t, 2k_t - 2k_z)$, Eq. (17) can be cast in the form

$$\phi_\Sigma = \int_{-\infty}^\infty\int_0^\infty U(k_t,k_z)J_0(k_t r)e^{ik_z z}e^{i(k_z^2-k_t^2)t}k_t dk_t dk_z,$$

which represents the generic axisymmetric solution of Eq. (7), expressed in 3D momentum space (k_t,k_z), k_t being the momentum transverse to the lattice direction z. Here $U(k_t,k_z)$ is the Fourier-Bessel (or plane-wave) spectrum of the initial atom distribution $\phi_0(r,z) \equiv \phi(r,z,t=0)$, and $U(k_t,k_z) = 2k_t^{-1}\sum_q C_q(2k_t - 2k_z)f_q^\perp(k_t)$ stands for its expansion in terms generalized Laguerre polynomials [any square-integrable $f(x)$ in $x \in [0,\infty)$ can be expanded in terms of $L_q^{(1)}(x)$]. This argument can be reversed by stating that, given the initial $(t = 0)$ distribution of atoms $U(k_t,k_z)$ in momentum space, if $F(\alpha,v) = U(\alpha,\alpha - v/2)/2$ is square integrable with respect to α (with $\alpha \in [0,\infty)$), then the atom wave-packet admits the representation (15). The expansion coefficient can be easily calculated as

$$C_q(v) = \frac{\sqrt{2}\Delta}{\pi\sqrt{q+1}}\int_0^\infty U\left(\alpha,\alpha-\frac{v}{2}\right)e^{-2\alpha\Delta}L_q^{(1)}(2\alpha\Delta)\alpha d\alpha,$$

Clearly, most of physically relevant wave-packets belong to the class ϕ_Σ [26]. For example a spectrally narrow gaussian beam can be described by few X waves (details will be given elsewhere).

Once the existence of finite norm linear X waves is established together with their potential to describe general BECs, the most intriguing question remains whether we can expect matter X-shaped atom distributions to be observable. Such waves correspond to a single fixed value $q = \bar{q}$ in Eq. (15) and an ideal velocity distribution $C_{\bar{q}}(v) = \delta(v - \bar{v})$. However, more generally, we can consider an atomic envelope beam ϕ_a constituted by several replicas of the single X wave $\phi_{\bar{q}}^\perp$ travelling with different velocities. Although, strictly speaking, such beam is not stationary, it can approximate such a state with an arbitrary degree of accuracy. In other words, it is possible to construct solutions that preserve their shape, for an arbitrary long time. Indeed if $C_{\bar{q}}(v)$ is a narrow function, e.g. peaked around $v = 0$, Eq. (15), using also Eq. (8), yields the following atom envelope ϕ_a

$$\begin{aligned}\phi_a(r,z,t) &= \int_{-\infty}^\infty \varphi_{\bar{q}}^\perp(r,z-vt)C_{\bar{q}}(v)e^{-i\frac{v}{2}z+i\frac{v^2}{4}t}dv \\ &\cong \varphi_{\bar{q}}^\perp(r,z)c(z,t) \qquad (18)\end{aligned}$$

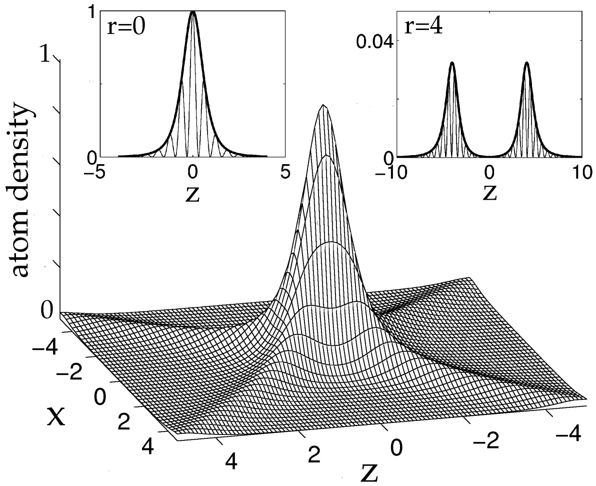

FIG. 2: Atom density of a pure X wave $|\phi_q^\perp(r,z)|^2$ with $q=0$ and $\Delta=1$, as seen on the (x,z) plane $(y=0)$. The insets show (same units as in Fig. 1) the overall (thin line) and envelope (bold line) density on-axis and off-axis, respectively.

where $c(z,t) = \int_{-\infty}^{\infty} C_{\bar{q}}(v)\exp(-i\frac{v}{2}z + i\frac{v^2}{4}t)dv$ is a solution of the dispersive wave equation $i\partial_t c - \partial_z^2 c = 0$. Eq. (18) represents an X wave modulated by a dispersing wave, which gives a sort of *adiabatic* dynamics of the finite norm X wave. Indeed the atom beam has an invariant spatial shape fixed by $\varphi_{\bar{q}}^\perp$, which we display in Fig. 2 as an example for $\bar{q}=0$. It spreads on a characteristic time which is longer, the narrower is the velocity distribution function $C_q(v)$. Finally, while in the linear regime we expect that such atom states should be somehow prepared, we envisage that atom collisions (nonlinear regime) can strongly favour the formation of X waves from more conventional ball-shaped atom clouds (e.g., obtained by a harmonic 3D trap which is then switched off) through instability mechanisms [17, 23], an issue which will be deepened elsewhere.

In conclusion we have shown that a periodic potential supports moving or still localized states of the GP model with envelope X-shape (strictly or nearly) preserved upon motion. A matter X waves entails localization both in momentum and configuration space and is a clear signature of a Bose condensed gas, so much as the anisotropy in the distribution function [2] detected in early experiments. However, unlike any other form of BEC including solitons, matter X waves can be observed in free-space and in the non-interacting regime, where they constitute a natural basis of expansion to describe the coherent properties of atom wave-packets. These results have strong implications also in optics, where the model (7) holds for normally dispersive bulk media [22], or in nonlinear optics of stratified media where Eqs. (3) describe propagation in a 1D bulk grating in the presence of diffraction, or propagation in a fully 3D photonic crystal (along proper directions).

We thank F. Cataliotti, M. Oberthaler, M. Salerno and P. Di Trapani for discussions. C.C. thanks the Fondazione Tronchetti Provera for the financial support.

* Electronic address: c.conti@ele.uniroma3.it; URL: `http://optow.ele.uniroma3.it`
† Also withDept. of Engineering, University of Ferrara, Via Saragat 1, 44100 Ferrara, Italy

[1] M. H. Anderson *et al.*, Science **269**, 198 (1995); K. B. Davis *et al.*, Phys. Rev. Lett. **75**, 3969 (1995).
[2] F. Dalfovo, S. Giorgini, L. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[3] A. Gorlitz *et al.*, Phys. Rev. Lett. **87**, 130402 (2001).
[4] L. Khaykovich *et al.*, Science **296**, 1290 (2002); K. Strecker, G. Partrige, A. Truscott, and R. Hulet, Nature (London) **417**, 150 (2002).
[5] L. Pitaevskii, Phys. Lett. A **221**, 14 (1996).
[6] L. Berge, T. J. Alexander, and Y. S. Kivshar, Phys. Rev. A **62**, 23607 (2000).
[7] H. Saito and M. Ueda, Phys. Rev. Lett. **90**, 40403 (2003).
[8] B. P.Anderson and M. Kasevich, Science **282**, 1686 (1998).
[9] E.A. Ostrovskaya and Y.S. Kivshar, Phys. Rev. Lett. **90**, 160407 (2003).
[10] S. Burger *et al.*, Phys. Rev. Lett. **86**, 4447 (2001); O. Morsch, J. H.Muller, M. Cristiani, D. Ciampini, and E. Arimondo, Phys. Rev. Lett. **87**, 140402 (2001); A. Trombettoni and A. Smerzi, Phys. Rev. Lett. **86**, 2353 (2001); F. S.Cataliotti *et al.*, Science **293**, 843 (2001); Greiner *et al.*, Nature (London) **419**, 51 (2002); B. Wu and Q. Niu, Phys. Rev. A **64**, R061603 (2001); H. P.Buchler, G. Blatter, and W. Zwerger, Phys. Rev. Lett. **90**, 130401 (2003).
[11] S. Pötting, P. Meystre, and E. M.Wright, in *Nonlinear photonic crystals* (Springer, 2003), vol. 10 of *Photonics*.
[12] C. Conti and S. Trillo, Phys. Rev. E **64**, 036617 (2001).
[13] A. Aceves, B. Costantini, and C. De Angelis, J. Opt. Soc. Am. B **12**, 1475 (1995).
[14] C. M. de Sterke and J. E. Sipe, Phys. Rev. A **42**, 550 (1990).
[15] H. Pu *et al.*, Phys. Rev. A **67**, 043605 (2003).
[16] I. Carusotto, D. Embriaco, and G. C. LaRocca, Phys. Rev. A **65**, R053611 (2002).
[17] C. Conti *et al.*, Phys. Rev. Lett. **90**, 170406 (2003); see also P. D.Trapani *et. al.*, physics/0303083 (2003).
[18] P. Saari and K. Reivelt, Phys. Rev. Lett. **79**, 4135 (1997).
[19] J. Lu and J. F.Greenleaf, IEEE Trans. Ultrason. Ferrelec. Freq. contr. **39**, 441 (1992).
[20] D. Mugnai, A. Ranfagni, and R. Ruggeri, Phys. Rev. Lett. **84**, 4830 (2000).
[21] J. Salo, J. Fagerholm, A. T.Friberg, and M. M. Salomaa, Phys. Rev. E **62**, 4261 (2000), and references therein.
[22] M. A.Porras, S. Trillo, C. Conti, and P. D.Trapani, Opt. Lett. **28**, 1090 (2003).
[23] C. Conti, physics/0302053; C. Conti, Phys. Rev. E **68**, 016606 (2003).
[24] J. Salo and M. M. Salomaa, J. Phys. A: Math. Gen., **34**, 9319 (2001).
[25] We define the 2D transverse scalar product between two

cylindrically symmetric functions ϕ, η as $<\phi|\eta>_\perp = 2\pi \int_0^\infty \phi(r,z,t)\eta^*(r,z,t) r dr$, as well as the 3D (in x,y,z) scalar product $<\phi|\eta> = \int_{-\infty}^{\infty} <\phi|\eta>_\perp dz$.

[26] Exceptions can be found, e.g. if $U(k_t, k_z) = U(k_z - k_t)$, then $F(\alpha, v)$ does not depend on α and cannot be expanded in terms of $L_q^{(1)}$.