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AN EPISTEMOLOGICAL USE OF NONSTANDARD ANALYSIS TO ANSWER ZENO'S OBJECTIONS AGAINST MOTION

ABSTRACT. Three of Zeno's objections to motion are answered by utilizing a version of nonstandard analysis, internal set theory, interpreted within an empirical context. Two of the objections are without force because they rely upon infinite sets, which always contain nonstandard real numbers. These numbers are devoid of 'numerical meaning', and thus one cannot render the judgment that an object is, in fact, located at a point in spacetime for which they would serve as coordinates. The third objection, 'an arrow never appears to be moving', is answered by showing that it only applies to a finite number of instants of time. A theory of motion is also advanced; it consists of a finite series of contiguous infinitesimal steps. The theory is immune to Zeno's first two objections because the number of steps is finite and each lies outside the domain of observation. 'Present motion' is hypothesized to be an unobservable process taking place within each step. The fact of motion is apparent through a summing (Riemann integration) of the steps.

1. ZENO'S OBJECTIONS AGAINST MOTION

Zeno of Elea (c. 490–30 B.C.) supported the monist philosophy of Parmenides (born c. 515 B.C.) and produced several arguments against the possibility of motion, arguments which have not survived in their original form but have come down to us through the works of Aristotle (384–22 B.C.). Although Zeno's objections are occasionally dismissed in textbooks of mathematics with a reference to his lack of knowledge of the methods of summation of infinite series, in fact, the Eleatic's influence on philosophical thought has been continuous and significant: "No one has ever touched Zeno without refuting him, and every century thinks it worthwhile to refute him" (Whitehead 1948, p. 87).

The arguments of interest for the present study are three in number. Their historical designations are enclosed within quotation marks.

(Z1) Consider a point moving in a straight line from A toward B. Before the point can reach B it must cover the first half of the distance from A to B. Then it must cover the first half of the remaining distance to B, etc., *ad infinitum*. Since

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it is not possible to complete an infinite number of acts, the moving point can never reach B. ("The Dichotomy")

(Z2) The same moving point, before it reaches the halfway point, must travel the first quarter of the distance from A to B, etc., *ad infinitum*. Thus, the point can never even start in motion from A. ("The Dichotomy", a second version)

(Z3) Consider an extended moving object, such as an arrow, at any instant. The arrow occupies a portion of space equal to its length and is not moving at that instant. Thus, the arrow moves at no time during its supposed flight. ("The Arrow")

The Zenonian vise progressively tightens from Z1 to Z3: motion cannot be completed, motion cannot be initiated, motion is a self-contradictory concept.

Zeno's other two arguments against motion, "Achilles and the Tortoise" and "The Stadium", have been omitted as being uninstructive for the present purposes. Zeno's arguments against plurality have also been neglected for this reason. (See Allen 1983; Cajori 1915; Salmon 1970; and Vlastos 1966a, 1966b, 1967, for textual and historical material.)

2. THE SIMPLE MATHEMATICAL THEORY OF MOTION

One set of responses to Zeno's objections appeals to the theory of infinite series to address Z1 and Z2 and to the concept of the derivative in calculus for Z3. For convenience this approach will be labeled 'the simple mathematical theory of motion' (SMT).

In terms of this theory, uniform motion along a straight line from A to B can be phrased in terms of elementary calculus.

In the remainder of the paper, the line segment from A to B will be represented by the closed unit interval, [0, 1], of the real line with 0 representing A and 1 representing B. The velocity V will be taken to be equal to 1, but the symbol "V" will be retained as a dimensional reminder. Then,

(1) $dx/dt = V$

with constant velocity V. The initial condition for this differential equation is

(2) $x(t = 0) = 0,$

and the solution is

$$(3) \qquad x(t) = \int_{s=0}^{t} (dx/ds)\, ds = Vt.$$

Then, SMT attempts to answer Z1 by an application of infinite series:

$$(4) \qquad x(t = 1) = \sum_{j=1}^{\infty} 2^{-j} = 1.$$

The answer of SMT to Z2 is given by summing through reverse steps from x = 1/2 to x = 0:

$$(5) \qquad x(t = 1/2) = \sum_{j=1}^{\infty} 2^{-j-1} = 1/2.$$

Similarly, Z3 seems to pose no problem for SMT since the idea of motion is carried effectively by the derivative expressed in (1): "effectively" because of the result given in (3).

The modern theory of infinite series permits a formal answer to Z1 and Z2: (4) and (5). However, the mathematical ability to manipulate infinite series does not relieve SMT of its burden of requiring an infinite number of objects to describe a process in the physical world. Also with regard to Z2, no first step from 0 is possible because the infinite series (5) has no last term. The answer of SMT to Z3 is unsatisfactory because the classical definition of a derivative appeals to behavior in a neighborhood of the point and, hence, is subject to the problems of SMT associated with Z1 and Z2.

Thus, SMT is rejected as an explanation, being nonresponsive to Zeno's objections.

3. INTERNAL SET THEORY

The most popular basis for the development of mathematical systems is the ten or so axioms of Zermelo–Fraenkel set theory. The primary idea behind Zermelo–Fraenkel set theory is the creation of new sets out of sets that are known already to exist – a technique designed to avoid the paradoxes that plagued the early days of the development of set theory. "Set" is not defined formally. It is a type of symbol whose meaning comes from the ways it can be used.

Thus, for example, one axiom of the system asserts the existence of the set whose members are all of the subsets of a given (known-to-exist) set. Another axiom guarantees the existence of a set whose members are exactly the members of two already existing sets. An axiom which has troubled many mathematicians is "the axiom of choice". Given a set of nonempty sets, the axiom says that there exists a set whose members are chosen one each from these nonempty sets. The troubling quality of the axiom is the lack of any specification as to the method of choice of an element from each breeding set.

Edward Nelson has created an elegant approach to nonstandard analysis, introduced by the logician Abraham Robinson (1918–74) (1966), by augmenting the axioms of Zermelo–Fraenkel set theory, and Nelson calls his system "internal set theory" (IST) (Davis 1983; Nelson 1977; and Robert 1988). The first step is to define a new unary relation symbol: a predicate (as one 'predicates' a quality for an object: 'the ball is red', 'the integer is even'). This new predicate is "standard"; that is, certain objects will be labeled as "standard", others are "nonstandard". A statement of IST is called "internal" if it does not contain the predicate "standard"; otherwise the statement is called "external". The simplest external statement is 'r is standard'.

The crucial point to keep in mind in IST is that it is forbidden to use external predicates to define subsets. Failure to comply with this restriction is the greatest source of error for the novitiate in IST. One of the most common acts in mathematics is to use a predicate to define a set: $\{x \in S | P(x)\}$, or, in words, "the set of all x in S such that P(x)", where the predicate P(x) might be "x is an integer". In IST one avoids obtaining a least upper bound on, say, the set of standard integers because this would represent a case of illegal set formation: applying the predicate "standard" to form a set.

The axioms of IST are those of Zermelo-Fraenkel set theory with three additional axioms: "transfer", "idealization", and "standardization" (whose first letters are an anagram for IST). Suppose that P(x) is an internal formula with free variable x; i.e., it is an 'announcement' in terms of conventional mathematics with 'unknown' x. For example, "P(x)" might stand for the formula "$x^2 + 22 > 0$". Then a simplified version of the transfer axiom is

$$(6) \qquad (\forall^s x)P(x) \rightarrow (\forall x)P(x),$$

where "$\forall^s$" means "for every standard object". In other words, an internal statement about standard objects can be generalized to a statement about unrestricted objects, which may be nonstandard.

The transfer principle has an important epistemological consequence. In the words of Nelson (1977, p. 1166): "Every specific object of conventional mathematics is a standard set". To see how this follows from the transfer principle, suppose that there is a unique ("specific") object x such that the internal statement $(\exists x)P(x)$ is true. Then by transfer, $(\exists^s_x)P(x)$ must be true (consider the contrapositive of the transfer axiom (6)). The x under discussion is, by assumption, unique, and so it must be standard. In particular, the natural numbers and the reals, as sets, are standard objects (but they may, and do, contain nonstandard members: just as a finite set can contain an infinite set as a member).

Thus far, no nonstandard objects have been exhibited. Suppose that S is a set that has two properties: (1) it is a standard set, e.g., it is the set of all positive integers less than 20 (an object defined in terms of conventional mathematics); and (2) S is a finite set. Further suppose that there exists an object x such that for every y belonging to S, $x > y$. In the example above, $x = 20$ would do, for then $20 > y$ for each of the nineteen elements y of S. The simplified version of the idealization axiom states that if, for every standard and finite set S there can be found an x_S (which may be chosen anew for each S) that is greater than all elements of S, then there exists an x (not necessarily standard) that is greater than *every* standard y.

This is a very strong axiom. After successfully finding customized upper bounds for finite sets, one is allowed to conclude there is an object that will do the job for *every* standard object. The existence of 'large' nonstandard numbers is immediately obvious from the example, but, formally, they are finite objects. Although this state of affairs may at first seem strange, it is similar to consequences of the physicist's "$\gg$" ("very much greater than") concept. The class of all x such that $x \gg 1$, for example, consists of finite reals, but these reals have a kind of epistemological disjointedness from 1 and its environs. As with the physicist's case it can be shown that there is no least of these large nonstandard numbers. Formally, a real number r is said to be 'limited' if $|r| < a$ for some standard real a, and it is 'unlimited', otherwise. All unlimited real numbers are nonstandard; the limited real numbers consist of standard and nonstandard real numbers.

To obtain an infinitesimal within IST, take the inverse of a positive unlimited real number. Straightforward arithmetic shows such an entity to be greater than 0 and less than every positive standard real: the definition of an infinitesimal (although "0" is also defined to be an infinitesimal).

These formal results of IST were prefigured by unease among certain philosophers and mathematicians that very large and very small numbers have no real meaning – they are incomprehensible. For example, van Dantzig (1955) claims that the difference between finite and infinite numbers is not an essential but a gradual one: a foretaste of Nelson's unlimited numbers.

Internal set theory axiomatically captures these feelings of unease and shows that the domain of reals is so large that it has room for infinitesimals and incomprehensibly large reals that are, still, truly reals. The epistemological consequences of this axiomatization are stunning and will be illustrated by their role in answering Zeno's arguments against motion after more than two millennia of effort.

The full-up version of the idealization axiom substitutes an arbitrary (but standard) binary relation R for the greater-than binary relation utilized above. In symbols,

$$(7) \qquad (\forall^{sf}A)(\exists x)(\forall y \in A)R(x, y) \leftrightarrow (\exists x)(\forall^{s}y)R(x, y),$$

where the notation "$\forall^{sf}$" means "for every standard and finite . . .".

The third axiom, standardization, provides a substitute for set-formation activities which were banned with respect to external predicates. Perhaps it is best grasped by looking at its formal specification. Let P(x) be a formula, internal or external, then:

$$(8) \qquad (\forall^{s}A)(\exists^{s}B)(\forall^{s}x)(x \in B \leftrightarrow x \in A \wedge P(x)).$$

The part "$x \in A \wedge P(x)$" is a subset-forming technique: "All elements of A that have property P as defined by the formula". The urge to form subsets by means of external predicates is, thus, satisfiable subject to the "standard" restrictions on the quantifiers in (8).

It is not difficult to prove the theorem that every element of a set is standard if and only if the set is standard and finite. Therefore, every infinite set has a nonstandard element, and, in particular, there exist nonstandard integers (this is a second way to obtain nonstandard objects). The IST integers are the usual well-ordered integers of classical mathematics with the difference that a new capability allows some to be called "standard" and some "nonstandard".

The most startling theorem of IST is: 'There is a finite set F that contains all standard elements' (see Nelson 1977, p. 1167) (this is readily deduced from the idealization axiom). The set F cannot be standard, for if it were, then by the transfer axiom F would contain all sets. In addition to containing all of the standard objects, there are nonstandard objects in F. It would be a case of illegal set formation to conclude that there are only finitely many standard real numbers, but there is certainly an epistemological flavor of finiteness of observables about the theorem, and this can only be good news for a Zenonian analysis. The set F will be employed in the development of a theory of motion and, in all subsequent usage, should be considered to be relativized to the closed unit interval [0, 1].

4. EPISTEMOLOGICAL PRINCIPLES

The purpose of this section is to state three epistemological principles which will be employed in answering Zeno's objections and in the development of a theory of motion.

The basic epistemological precept employed here is that a concrete event – observation or possible observation of the location of an object at a particular time – on the spacetime trajectory of the object should be labeled with a concrete real number. That is, the fact that an object is located at a point in spacetime should be commemorated only with a constructible real number: one whose numerical content is fully knowable.

While this is presented as an axiom, it seems to be a plausible one. It is difficult to see how one could meaningfully claim to have established a fact of kinematic history without being able, numerically, to characterize that fact; there is little else of empirical note in the simplified world of uniform motion in a straight line. Borel ([1950]/1963, pp. 91, 120–21) argues that certain very large and very small real numbers are not relevant to the description of matters of fact; they lie outside the parametric range employed in our scientific model of the world. The insistence here on the use of only constructible real numbers is a more relaxed constraint than Borel's circumscriptions and allows indefinite growth in the numerical scope of future scientific theories. Of course, to be responsive to the nature of the reigning scientific theories in any historical epoch and to be consistent with the expected value of errors in measurement, only a subset of the constructible numbers will, at any one point in history, be suitable for actual use. The concept of numerical meaning has been developed within the school of constructive mathematics, which has a long history going back through L. E. J. Brouwer (1881–1966), H. Poincaré (1854–1912), and others, and has been developed most recently by Errett Bishop (see Bishop 1967; and Mandelkern 1985). It is not asserted here that constructive mathematics must be employed for pure mathematical investigations. In fact, Nelson's developments do not restrict themselves by adherence to constructivist tenets, but it is asserted that, for describing matters of fact in the world, only real numbers with numerical meaning should be employed.

It is convenient to discuss numerical meaning in terms of decimal expansions. By "decimal expansion" it is meant that, in the ordinary sense, the appropriate integer symbols ('0' through '9') can be expressed to produce any desired degree of precision. For example, $\pi/4$, a member of the unit interval, is constructible in this sense. The symbol "$\pi/4$" serves as a guide in producing decimal expansions through calculations with power series or other appropriate methods. Of course "binary" or another base would serve as well as "decimal".

The concepts just discussed are distilled below into the epistemological principle E2. The principle E1 establishes the role of SMT within the analysis, and E3 provides a rule to be used in judging whether or not motion has taken place. "Spacetime", as used in the following, is represented by the Cartesian product of two unit intervals.

(E1) The set of all points in spacetime at which an object might be locatable is a subset of the points described by the simple mathematical theory of motion (SMT).

(E2) The fact that an object is located at a point in spacetime cannot be established if the coordinates describing the point are nonstandard real numbers.

(E3) The fact of motion of an object is established if the object has been located at two distinct points of space.

Principle E1 directs the present analysis of motion to build upon SMT in the sense that it may not propose new points to be added to the trajectory. For the present case of uniform motion with V = 1, the locus referenced in E1 is the diagonal of the unit square; other cases would be accommodated by other loci. The phrase "might be locatable" denotes that the locus referenced in E1 is a superset of the set of points where the object could be located by observation. That is, it is necessary but not sufficient for a point of spacetime to be on the locus in order to be eligible for verifiable occupancy by the object. (Zeno's arguments assume both necessity and sufficiency.)

The locus defined in E1 is epistemologically reduced by the stricture in E2 through the removal of points whose coordinates are nonstandard real numbers. The phrase "[t]he fact that" in E2 means that the object's location has been observationally verified or could have been observationally verified had one been sufficiently equipped and attentive to capture the requisite numerical description of the event. The principle E2 is based upon the theorem of IST that every specific object, and in particular a constructible real number, of classical mathematics is a standard object. Hence, a nonstandard real number cannot be a constructible real number, and therefore by the basic epistemological precept it cannot serve as a measure of location. Of course, not all standard real numbers are suitable as measures of location, e.g., those which have been specified only through existence arguments based upon the axiom of choice, and one could generate a stronger version of E2 by insisting upon the use of only constructible real numbers for coordinates. However, it suffices here to utilize the present form of E2, which eliminates the cogent offenders: nonstandard real numbers. Nonstandard real numbers have been employed in theories of measurement in non-Archimedean settings (Narens 1985), but the epistemological relevance of the theory has been questioned (Kyburg 1988; and Narens 1985, pp. 6–8). Perhaps the objections of Zeno can serve as a useful discriminator between classes of theories of measurement.

The kind of approach dictated by E2 is not the only one possible. Steps required to be greater than a fixed standard length could be postulated (reminiscent of the constraint imposed by the "Planck length" in physics), or a probabilistic measure could be furnished. But these approaches do not seem appropriate in a Zenonian context. The first ends further discussion of Z1 and Z2, and the second fails to respond to the precise numerical standards established by the specification of 'checkpoints' $(1 - 2^{-j})$ in Z1.

A corollary of E2 is the unsuitability of infinitesimal intervals of real numbers in the empirical description of motion. At most, an infinitesimal interval can contain one standard real number amongst its (uncountable) infinity of real numbers (two standard real numbers cannot be infinitesimally close). Thus, an infinitesimal interval would go unnoticed because its two boundary points could not both be measured.

The principle E3 provides a sufficient condition for the occurrence of motion to be announced as a fact. The criterion is retrospective and does not attempt to encompass present motion; this notion will be developed later.

5. ANSWERING ZENO'S OBJECTIONS

Let C be the set of spatial 'checkpoints' which are implied by the premise of Z1,

$$C = \{r \in [0, 1] \mid r = 1 - 2^{-j}, i \leqslant 1 < \infty\}. \tag{9}$$

Let L(t, r) be true if the object could, in fact, be located at the point ⟨t, r⟩ of spacetime, where t represents an instant of time, r represents a point of space, and ⟨t, r⟩ is their ordered pair. Otherwise, L(t, r) is false. The statement

$$(\forall t)(\forall r)((r = Vt) \wedge r \in C \rightarrow L(t, r)) \tag{10}$$

formalizes the 'checkpoint' implications from Z1. The universal quantifiers range over the unit interval (and, as usual, V = 1). The set C is (countably) infinite and therefore must contain at least one nonstandard representation of a point in spacetime, say $\langle t_n, r_n \rangle$. Then, by E2, $L(t_n, r_n)$ is false, and (10) must be false. This means that Z1 has no force as an objection; its premise has been shown to be false.

The argument of Z2 also depends upon an infinite set, and, in a manner very similar to the resolution of Z1, the argument can be dismissed as invalid on epistemological grounds. But, without appealing to the infinite set implied in Z2, one could still ask how a first step, from A, is possible without passing over 'unreported' points. The response is that a step of infinitesimal length initiates a departure from A and is not subject to this criticism since, as shown previously, it is not empirically accessible (and therefore its internal constitution is not subject to scrutiny). To demonstrate that such a departure results in motion, in the sense of E3, a theory of motion is developed and presented after the resolution of Z3.

The premise of Z3 is granted; the arrow is not seen to be moving at any instant t for which its 'tip' is in fact located at r. But, this set G of points in spacetime for which the premise of Z3 is granted is a subset of the finite set F × F, the Cartesian product of the set containing all standard real numbers (in the closed unit interval) with itself. Hence, the conclusion of Z3 is false because the argument of Zeno only demonstrates that the arrow does not appear to be moving for instants associated with points in spacetime represented by the finite set G (G is finite because it is a subset of the finite set $F \times F$). The objection Z3 can say nothing about an apparent lack of kinematics in the much larger set $[0, 1] \times [0, 1] - G$. Granting the premise of Z3, at points of observation, is compatible with the fact of motion, characterized in E3. Also, it does not seem to present an unnatural circumstance; consider a stroboscopic effect at isolated points along the path of a moving object. Note that no claim has been made that the "subset" of all standard pairs in $F \times F$ has been established. Any subset G of $F \times F$ corresponding to established facts would have been epistemologically specified through enumeration. It is not even assumed, *a priori*, that such an enumeration could not have been infinite. But *a posteriori* an infinite enumeration would not 'fit' in the finite superset $F \times F$. (Also, it would have to contain nonstandard real numbers.)

Although Zeno's objections against motion have been treated in the preceding analysis, it remains to demonstrate that the departure from A, by means of an infinitesimal step, does indeed lead to motion in the sense of E3. Since no adequate theory of motion remains, SMT having been rejected, one will be formulated. In the present context, the desiderata for a theory of motion are two: (1) that the trajectory calculated for the moving object include no points outside the trajectory defined by SMT; and (2) that the theory not be afflicted by any of Zeno's three objections against motion.

Let $P = \langle r_0 = 0, r_1, r_2, \ldots, r_k = 1\rangle$ be the ordered set of real numbers contained in the set F, considered as representatives of points in the unit (spatial) interval, and let $U = \langle u_1, u_2, u_3, \ldots, u_k\rangle$ represent the associated ordered set of open intervals of time, i.e., $r_j - r_{j-1} = V|u_j|$ for each j, $1 \leqslant j \leqslant k$, where $|u_j|$ denotes the length of the j^{th} interval of time. The instants t_j of time associated with U are:

$$(11) \qquad t_j = \sum_{i=1}^{j} |u_i|, \qquad \text{with } t_0 = 0 \quad \text{and} \quad 1 \leqslant j \leqslant k.$$

Then, for $1 \leqslant j \leqslant k$,

$$(12) \qquad x(t_0) = 0,$$

$$(13) \qquad x(t_j) = r_j,$$

and

$$(14) \qquad r_j = V \sum_{i=1}^{j} |u_i|.$$

The equations (12), (13), and (14) constitute the formal expression of the desired theory of motion. In words, the object is predicted to step through spacetime in a manner consistent with the predictions of SMT but restricted to points with coordinates in the finite set $F \times F$. A first step, of length r_1, takes place during the interval of time u_1. The theory satisfies the necessary condition E1 and is immune to the objections Z1, Z2, and Z3 because $F \times F$ is finite. The observer will be able to locate the object, in fact, at points with constructible coordinates that are within the scope of his or her powers of observation, and the remaining points are epistemologically superfluous for this observer. By E3, motion has indeed occurred (assuming that the observer's scope is sufficient to locate the object for at least two points).

This mode of explanation, mixing purely theoretical entities with observable entities, is exemplified within quantum theory where an unobserved wave function evolves in time and 'collapses' upon observation to yield numerical data.

The theory explains the fact of motion but does not describe the nature of 'present motion'. If there is a concept of 'present motion', it must refer to a process taking place during the infinitesimal open intervals u_i of time. It cannot be established, in fact, what process of 'present motion' is operative within the infinitesimal intervals u_i. The object could jump instantaneously from one end of an interval to the other, or it could move nonuniformly within an interval, or it could move uniformly within an interval. (In this last case, the process could be mathematically represented by the derivative of nonstandard calculus, this derivative being defined by the behavior of the distance-versus-time function in an infinitesimal interval about an instant of time.) More generally, the object might not be, during these time intervals, in any kind of spacetime.

The difference between this use of nonstandard objects in a theory of motion and Zeno's use of nonstandard objects, as it turned out, in Z1, Z2, and Z3, is that the former case introduces only the ambiguity of more than one possible theoretical explanation, while the latter case yields false statements concerning the treatment of matters of fact.

The epistemological roots of the mathematical theory can be detected by comparing (3) with (14). In IST, the (Riemann) integral in (3) would be obtained by partitioning the unit interval of time into arbitrary infinitesimal segments. A specific partitioning, for epistemological advantage, is given by U, and (14) is a Riemann sum. If the velocity of the object were, say, represented by any continuous function of time (rather than the very simple case where this function always has the constant value V), then the partition U would still serve, mathematically as well as epistemologically, as the basis for a Riemann sum in order to accomplish the integration to obtain position in the spatial interval as a function of time.

6. SUMMARY

The tactic of appealing to infinite sets to argue against the possibility of motion has been denied to Zeno by showing that such sets always contain references to nonverifiable occurrences; it had been believed that each member of the sets referred, at least potentially, to a matter of fact.

Also, a theory of motion has been advanced that agrees with predictions of the simple mathematical theory of motion and does not succumb to Zeno's objections. Basically, the theory represents motion as a finite series of infinitesimal steps. Mathematically this is equivalent to a Riemann sum phrased within the framework of internal set theory. If one wishes to define 'present motion', it is possible to do so in a manner consistent with this theory of motion. The fact that motion has occurred is verifiable without encountering Zeno's objections, but the fact of present motion does not appear to be verifiable, since it takes place inside unobservable infinitesimal intervals. The process of change is hidden but the effects of change are visible.

The linchpin of the arguments in this paper is the contention (E2), that nonstandard real numbers are not of value in characterizing matters of fact. Considering the mathematical structure of these arguments, it is not difficult to see that, with regard to responding to Zeno's objections, the ban on looking for infinitesimals within fields of experience could be partially lifted if, for example, a smallest epistemologically valid infinitesimal were postulated. However, there seems no reason to pursue this extension since its epistemological utility is doubtful.

Thus, if one accepts the epistemological principle that a description of motion is only responsible for accounting for numerically reportable events, then certain mathematical results from internal set theory disable Zeno's criticisms and, in addition, facilitate the construction of a theory of motion which circumvents his objections.

It should be noted that some of the sets utilized in the present work can be viewed "externally" (Nelson 1977) wherein they have infinite cardinality. However, this fact does not affect the validity of the treatment of Zeno's objections within an IST-modeled universe.

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