Nonstandard analysis and differentiable manifolds - Foundations

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Abstract. We approach the study of differentiable manifolds modeled on Banach spaces by means of Nonstandard Analysis. We stay inside the category of classical manifolds and using nonstandard analysis techniques, we present some new nonstandard characterizations for the tangent bundle, differentiable function, differential of a function, directional derivatives, etc. We establish some relations between our definitions and the classical ones.

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1 Introduction

In this paper we develop an analog of the classical theory of differentiable manifolds, formulated in terms of nonstandard analysis.

Some work has been done, connecting Nonstandard Analysis with Topology [2, 3, 4] and with Differentiable Manifolds [1, 5, 9, 11]. For more on Nonstandard Analysis, consult [6, 7, 8, 10, 12].

Many of the classical concepts that we deal with can be presented using a kind of internal functions, which we will call δ-infinitesimal transformations. The idea is that these functions move infinitely nearstandard points of the manifold, with some smoothness properties. We will stay inside the category of classical manifold theory, working with standard manifolds, and use nonstandard methods to present new definitions like Tangent Space, Derivative of a standard function, etc.

To start with we present a brief exposition on Nonstandard Analysis, and so we will omit proofs and technical details. In the references below the reader can find more on the subject.

Let E and F denote two Banach spaces and ${}^*E, {}^*F$ their nonstandard extensions.

Definition 1.1. Let x and y be two vectors of ${}^*E$.

1. x is **infinitesimal** if $|x|$ is infinitesimal, *ie*, $|x| < r$ for every $r \in \mathbb{R}^+$; the set of infinitesimal vectors of ${}^*E$ is denoted by $inf({}^*E)$ and for $x \in inf({}^*E)$ we write $x \approx 0$;

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2. x and y are **infinitely close** if $x - y \approx 0$ and we write $x \approx y$; if not we write $x \not\approx y$;
3. x is **finite** if $|x|$ is finite, *ie*, $|x| < r$ for some $r \in \mathbb{R}^+$; if not we say that x is **infinite**; the set of finite vectors of $^*E$ is denoted by $fin(^*E)$;
4. x is **nearstandard** if there exists some $a \in E$ such that $x \approx a$, we say that a is the **standard part** of x and we write $st(x) = a$; the set of nearstandard vectors of $^*E$ is denoted by $ns(^*E)$;

The **monad** of $x \in {}^*E$ is the set

$$\mu(x) := \{y \in {}^*E \,|\, x \approx y\}.$$

Theorem 1.1. *Let A be a subset of E. Then*

1. *A is open if and only if for all $a \in A$, $\mu(a) \subseteq {}^*A$*
2. *A is closed if and only if, whenever $a \in {}^*A$, if there exists $st(a)$ then $st(a) \in A$*
3. *A is compact if and only if for all $a \in {}^*A$, there exists $st(a)$ and $st(a) \in A$.*

Let U be a subset of E. In the following we denote

$$ns(^*U) := \{x \in {}^*U \,|\, x \in ns(^*E) \text{ and } st(x) \in U\}.$$

Given an internal linear operator $L \in {}^*L(E, F)$, we say that L is **finite** if $L(x) \in fin(^*F)$ whenever $x \in fin(^*E)$.

Definition 1.2. Let U be an open subset of E and $f : {}^*U \to {}^*F$ be an internal function. We say that f is

1. **S-continuous** if
$$\forall a \in U \, \forall x \in {}^*U \quad x \approx a \Rightarrow f(x) \approx f(a);$$
2. **SU-continuous** if
$$\forall x, y \in {}^*U \quad x \approx y \Rightarrow f(x) \approx f(y);$$
3. **S-differentiable** if
$$\forall a \in U \, \exists L_a \in {}^*L(E, F) \, \forall x \in {}^*U \, \exists \eta \in {}^*F$$
$$x \approx a \Rightarrow f(x) - f(a) = L_a(x - a) + |x - a|\eta$$
for some internal finite linear operator L_a and $\eta \approx 0$;
4. **SU-differentiable** if
$$\forall x \in ns(^*U) \, \exists L_x \in {}^*L(E, F) \, \forall y \in ns(^*U) \, \exists \eta \in {}^*F$$
$$x \approx y \Rightarrow f(x) - f(y) = L_x(x - y) + |x - y|\eta$$
for some internal finite linear operator L_a and $\eta \approx 0$.

For a standard function $f : U \to F$, we have

f is S-continuous	$\Leftrightarrow$	f is continuous
f is SU-continuous	$\Leftrightarrow$	f is uniformly continuous
f is S-differentiable	$\Leftrightarrow$	f is differentiable
f is SU-differentiable	$\Leftrightarrow$	f is of class C^1

2 Tangent Space to a Differentiable Manifold

For the sake of completeness, let us recall the definition of a differentiable manifold modeled on an arbitrary real Banach space E.

Definition 2.1. Let M be a nonempty Hausdorff topological space and $\{(U_i, x_i)\}$ $(i \in I)$ a family of pairs satisfying the following conditions:

1. Each U_i is an open subset of M and $x_i : U_i \to x_i(U_i) \subseteq E$ is a homeomorphism;
2. The U_i cover M: $\cup_{i\in I} U_i = M$;
3. When $U_i \cap U_j \neq \varnothing$, the function $x_i x_j^{-1} : x_j(U_i \cap U_j) \to x_i(U_i \cap U_j)$ is of class C^k;
4. The set $\{(U_i, x_i)_{i\in I}\}$ is maximal for the previous conditions, *i.e.*, the set contains all functions with these properties;

then we say that M is a **Differentiable Manifold** of class C^k. When $k = \infty$ the manifold is called **smooth**. If $dim(E) = n \in \mathbb{N}$ we say that M is a n-dimensional manifold.

The pair (U_i, x_i) is called a **chart** and $x_i x_j^{-1}$ the **transition** or **overlap function**; we say that the functions x_i and x_j are **smoothly compatible** . If a point p of M lies in U_i, then we say that (U_i, x_i) is a chart at p. The family of functions $\mathcal{A} := \{(U_i, x_i) \mid i \in I\}$ is an **atlas** on M. Observe that any chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$.

Given a manifold M we can describe the tangent space to M using a type of functions defined on M, the δ-infinitesimal transformations. The set obtained is a linear space isomorphic to the real Banach space. ¿From now on we will assume that M is a differentiable manifold of class C^k with $k \geq 2$.

Definition 2.2. Let δ be a fixed positive infinitesimal and $p \in M$. Let $A \subseteq {}^*M$ be an internal set, $X : A \to X(A) \subseteq {}^*M$ an internal bijection such that

- $\mu(p) \subseteq A \cap X(A)$;
- $X(q) \approx q$ for all $q \in ns(A)$,
- $X^{-1}(q) \approx q$ for all $q \in ns(X(A))$.

We say that X is a **δ-infinitesimal transformation** at p if there exists a chart (U, x) with $\mu(p) \subseteq {}^*U \subseteq A$ such that

$$\overline{X}(u) := \frac{xXx^{-1}(u) - u}{\delta} \quad \text{and} \quad \overline{X^{-1}}(u) := \frac{xX^{-1}x^{-1}(u) - u}{\delta}$$

are both SU-differentiable at

$$C := \{u \in {}^*x(U) \mid Xx^{-1}(u) \in {}^*U \wedge X^{-1}x^{-1}(u) \in {}^*U\}.$$

The set of all δ-infinitesimal transformations on M at p will be denoted by $\delta\Theta_p M$.

The set C contains all nearstandard points of ${}^*x(U)$. In fact, if $u \in ns({}^*x(U))$, that is, $x^{-1}(u) \in ns({}^*U)$, then

$$Xx^{-1}(u) \approx x^{-1}(u) \approx st(x^{-1}(u)) \in {}^\sigma U.$$

As U is open, it follows that $Xx^{-1}(u) \in {}^*U$. In a similar way for X^{-1}, one proves the desired.

As for notation simplification, in what follows, $\underline{X}(u) := xXx^{-1}(u)$. So

$$\overline{X}(u) = \frac{\underline{X}(u) - u}{\delta} \Leftrightarrow \underline{X}(u) = \delta\overline{X}(u) + u.$$

Consequently $\underline{X}$ is also SU-differentiable and $D\underline{X}_u = \delta D\overline{X}_u + I$. The same argument for $\underline{X}^{-1}$.

For example, let $\mathcal{S}^2 \subset \mathbb{R}^3$ be the 2-dimensional sphere and X be the function given by

$$X(\cos a \sin b, \sin a \sin b, \cos b) = (\cos(a+\epsilon)\sin(b+\eta), \sin(a+\epsilon)\sin(b+\eta), \cos(b+\eta)),$$

$$a \in {}^*\left]\frac{\pi}{2}, \frac{3\pi}{2}\right[, b \in {}^*\left]\frac{\pi}{4}, \frac{3\pi}{4}\right[, \epsilon, \eta \approx 0 \text{ with } \frac{\epsilon}{\delta}, \frac{\eta}{\delta} \in fin({}^*\mathbb{R})$$

where δ is a fixed positive infinitesimal.

Let $p = (\cos\pi\sin(\pi/2), \sin\pi\sin(\pi/2), \cos(\pi/2)) = (-1,0,0) \in \mathcal{S}^2$ and x be the chart on $U = \{(x_1,x_2,x_3) \in \mathcal{S}^2 \,|\, a \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[, b \in \left]\frac{\pi}{4}, \frac{3\pi}{4}\right[\}$ given by $x(x_1,x_2,x_3) := (x_2,x_3)$.

$$\begin{aligned}
\overline{X}(u) &= \frac{(\sin(a+\epsilon)\sin(b+\eta) - \sin(a)\sin(b), \cos(b+\eta) - \cos(b))}{\delta} \\
&= \left(\frac{\sin(a+\epsilon) - \sin(a)}{\delta}\sin(b+\eta) + \frac{\sin(b+\eta) - \sin(b)}{\delta}\sin(a),\right. \\
&\qquad \left.\frac{\cos(b+\eta) - \cos(b)}{\delta}\right) \\
&= \left(\frac{\epsilon}{\delta}\frac{\sin(a+\epsilon) - \sin(a)}{\epsilon}\sin(b+\eta) + \frac{\eta}{\delta}\frac{\sin(b+\eta) - \sin(b)}{\eta}\sin(a),\right. \\
&\qquad \left.\frac{\eta}{\delta}\frac{\cos(b+\eta) - \cos(b)}{\eta}\right) \\
&\approx \left(\frac{\epsilon}{\delta}\cos(a)\sin(b) + \frac{\eta}{\delta}\cos(b)\sin(a), -\frac{\eta}{\delta}\sin(b)\right).
\end{aligned}$$

For $a = \pi$ and $b = \pi/2$, we get

$$st(\overline{X}x(p)) = \left(st\left(-\frac{\epsilon}{\delta}\right), st\left(-\frac{\eta}{\delta}\right)\right).$$

For example, if $\epsilon = 0$ and $\eta = \pm\delta$ then $st(\overline{X}x(p)) = (0, \mp 1)$ and for $\epsilon = \pm\delta$ and $\eta = 0$, $st(\overline{X}x(p)) = (\mp 1, 0)$.

We note that the previous definition is independent of the chart: if $\overline{X}$ and $\overline{X^{-1}}$ are SU-differentiable at u for a certain chart (U,x), the same occurs for every other chart (V,y), because

$$\overline{X}_1(u) := \frac{yXy^{-1}(u) - u}{\delta} = \frac{(yx^{-1})\underline{X}(xy^{-1})(u) - (yx^{-1})(xy^{-1})(u)}{\delta}$$

where $\underline{X} = xXx^{-1}$, and so, for some infinitesimal function $\eta(\cdot)$ [1]:

$$D(\overline{X}_1)_u =$$
$$= \frac{D(yx^{-1})_{\underline{X}(xy^{-1})(u)} D\underline{X}_{(xy^{-1})(u)} D(xy^{-1})_u - D(yx^{-1})_{(xy^{-1})(u)} D(xy^{-1})_u}{\delta}$$
$$= \frac{D(yx^{-1})_{\underline{X}(xy^{-1})(u)} (\delta D\overline{X}_{(xy^{-1})(u)} + I) D(xy^{-1})_u - D(yx^{-1})_{(xy^{-1})(u)} D(xy^{-1})_u}{\delta}$$
$$= D(yx^{-1})_{\underline{X}(xy^{-1})(u)} D\overline{X}_{(xy^{-1})(u)} D(xy^{-1})_u + \frac{D(yx^{-1})_{\underline{X}(xy^{-1})(u)} D(xy^{-1})_u - D(yx^{-1})_{(xy^{-1})(u)} D(xy^{-1})_u}{\delta}$$
$$= D(yx^{-1})_{\underline{X}(xy^{-1})(u)} D\overline{X}_{(xy^{-1})(u)} D(xy^{-1})_u + \left[D^2(yx^{-1})_{(xy^{-1})(u)}(\overline{X}(xy^{-1})(u), \cdot) + |\overline{X}(xy^{-1})(u)|\eta(\cdot)\right] D(xy^{-1})_u$$

which is finite. It can be proven analogously that $\overline{X_1^{-1}}$ is also SU-differentiable.

Theorem 2.1. *Let $p \in M$ be a point and let ${}^*M \ni a \approx p$. Then there exist $\delta \approx 0$ and $X \in \delta\Theta_p M$ such that $X(p) = a$.*

Proof. Let (U, x) be a chart at p and define

$$\begin{aligned} \delta &:= |x(a) - x(p)| \approx 0; \\ X(q) &:= x^{-1}(x(q) + x(a) - x(p)). \end{aligned}$$

Then X is invertible with inverse $X^{-1}(r) := x^{-1}(x(r) - x(a) + x(p))$. Moreover,

$$\overline{X}(u) = \frac{xXx^{-1}(u) - u}{\delta} = \frac{x(a) - x(p)}{\delta}$$

and $\overline{X^{-1}}$ are SU-differentiable. Consequently, $X \in \delta\Theta_p M$ and $X(p) = a$. □

Also recall:

Definition 2.3. Let M and N be two differentiable manifolds. A function $f : M \to N$ is of class C^k if for each $p \in M$, and a chart (U, x) in M with $p \in U$ and a chart (V, y) with $f(U) \subseteq V$, the composite function $yfx^{-1} : x(U) \to y(V)$ is a C^k function.

Therefore a function $f : M \to \mathbb{R}$ is a C^k-function if and only if for every $p \in M$ there is some chart (U, x) at p so that $fx^{-1} : x(U) \to \mathbb{R}$ is a C^k-function.

Theorem 2.2. *Let $f : M \to \mathbb{R}$ be a function. Then f is of class C^1 if and only if for all $p \in ns({}^*M)$ there exists an internal finite linear operator $L_p \in {}^*L(E, \mathbb{R})$ such that*

$$\forall 0 \approx \delta \in {}^*\mathbb{R}^+ \, \forall X \in \delta\Theta_{st(p)}M \quad f(Xp) - f(p) = L_p(xX(p) - x(p)) + |xX(p) - x(p)|\eta$$

for some $\eta \approx 0$.

Proof. If $p \in ns({}^*M)$ then $x(p) \in ns({}^*E)$. By hypothesis, fx^{-1} is a C^1 function. Hence $D(fx^{-1})_{x(p)}$ exists in ${}^*L(E, \mathbb{R})$ and is a finite linear operator. Define $L_p := D(fx^{-1})_{x(p)}$. Fix now a positive $\delta \approx 0$ and $X \in \delta\Theta_{st(p)}M$. Then

$$\begin{aligned} f(X(p)) - f(p) &= (fx^{-1})(xX(p)) - (fx^{-1})(x(p)) \\ &= L_p(xX(p) - x(p)) + |xX(p) - x(p)|\eta \qquad (\eta \approx 0). \end{aligned}$$

[1] in the sense that $\eta(x) \approx 0$ whenever $x \in fin({}^*E)$

To prove the converse, let us see that fx^{-1} is differentiable at $x(p) \in ns(^*E)$, *i.e.*, there exists a finite linear operator $L' \in {}^*L(E, \mathbb{R})$ such that for all $0 \approx \epsilon \in {}^*E$,

$$(fx^{-1})(x(p) + \epsilon) - (fx^{-1})(x(p)) = L'(\epsilon) + |\epsilon|\eta,$$

for some infinitesimal η. To begin with, since $x(p) \in ns(^*E)$ then $p \in ns(^*M)$. Define $L' := L_p$ and fix any $\epsilon \approx 0$. Let $\delta := |\epsilon| \approx 0 \in {}^*\mathbb{R}^+$ (when $\epsilon = 0$ it is obvious). Define now

$$X(q) := x^{-1}(x(q) + \epsilon).$$

We will prove that $X \in \delta\Theta_{st(p)}M$: we have that X is an internal bijection with inverse

$$X^{-1}(r) = x^{-1}(x(r) - \epsilon).$$

Besides this, $\overline{X}(u) = \epsilon/\delta = \epsilon/|\epsilon|$ and $\overline{X^{-1}}(u) = -\epsilon/|\epsilon|$ are both SU-differentiable. As a result

$$\begin{aligned} fx^{-1}(x(p) + \epsilon) - fx^{-1}x(p) &= f(X(p)) - f(p) \\ &= L_p(xX(p) - x(p)) + |xX(p) - x(p)|\eta \\ &= L'(\epsilon) + |\epsilon|\eta. \end{aligned}$$

□

For $X, Y \in \delta\Theta_p M$, we say that they are δ-equivalent at $x(p)$ if $st(\overline{X}x(p)) = st(\overline{Y}x(p))$ and we write $X \equiv_{x(p)} Y$, or $X \equiv Y$ if there is no danger of confusion.

The set $\delta\Theta_p M$ forms a group under composition of functions. Although the operation is not commutative we have the following approximation: for $u \approx x(p)$,

$$\begin{aligned} \overline{XY}(u) &= \frac{\underline{X}\,\underline{Y}(u) - u}{\delta} \\ &= \frac{\underline{X}\,\underline{Y}(u) - \underline{Y}(u)}{\delta} + \frac{\underline{Y}(u) - u}{\delta} \\ &= \overline{X}(\underline{Y}(u)) + \overline{Y}(u) \\ &\approx \overline{X}(u) + \overline{Y}(u) \end{aligned}$$

because of the S-continuity of $\overline{X}$. This implies that

$$\overline{XY}(u) \approx \overline{YX}(u).$$

Moreover

$$\overline{X^{-1}}(u) \approx -\overline{X}(u)$$

since

$$0 = \overline{I}(u) = \overline{XX^{-1}}(u) \approx \overline{X}(u) + \overline{X^{-1}}(u).$$

Theorem 2.3. *$(\delta\Theta_p M, \circ)$ is a group.*

Proof. The proof of the theorem is identical to the one in [12] with the adequate adjustments.

To see that composition is well defined, take $X, Y \in \delta\Theta_p M$ with $X : A \to X(A)$ and $Y : B \to Y(B)$. Define

$$C := \{b \in B \,|\, Y(b) \in A\}.$$

The set C is internal and contains $\mu(p)$ (because $\mu(p) \subseteq B$ and, for $b \in \mu(p)$, $Y(b) \approx b \approx p$, it is also true that $Y(b) \in A$).

By the Cauchy's Principle (see [12], Theorem 8.1.4, pag. 196) there exists an open set W with $\mu(p) \subseteq {}^*W \subseteq C$. Define then $XY : {}^*W \to XY({}^*W)$.

It is also true that $\mu(p) \subseteq XY({}^*W)$ since, if we fix $a \in \mu(p)$, $Y^{-1}X^{-1}(a) \approx a \approx p$ will imply that $Y^{-1}X^{-1}(a) \in {}^*W$ and $XY(Y^{-1}X^{-1})(a) = a \in XY({}^*W)$.

Clearly XY is an internal bijection and

$$
\begin{aligned}
D\overline{XY}_u &= \frac{D\underline{X}_{\underline{Y}(u)} D\underline{Y}_u - I}{\delta} \\
&= \frac{(\delta D\overline{X}_{\underline{Y}(u)} + I)(\delta D\overline{Y}_u + I) - I}{\delta} \\
&\approx D\overline{X}_{\underline{Y}(u)} + D\overline{Y}_u
\end{aligned}
\tag{2.1}
$$

which is a finite operator. In conclusion $\overline{XY}$ is SU-differentiable. Similarly $\overline{(XY)^{-1}}$ is also SU-differentiable.

It is clear that the composition is associative, $I : {}^*M \to {}^*M$ is the identity element and $X^{-1} \in \delta\Theta_p M$. □

Remark: Since, by (2.1)

$$0 = D\overline{XX^{-1}}_u \approx D\overline{X}_{\underline{X}^{-1}(u)} + D\overline{X^{-1}}_u$$

it follows that

$$D\overline{X^{-1}}_u \approx -D\overline{X}_{\underline{X}^{-1}(u)}.$$

We can define **sum** and **scalar multiplication** on $\delta\Theta_p M$ in the following way: For $X, Y \in \delta\Theta_p M$ and $a \in \mathbb{R}$:

$$(X+Y)(q) := XY(q) \text{ and } aX(q) := x^{-1}(x(q) + a\delta\overline{X}x(p)).$$

Note that it is still true that

$$(X+Y)(q) = x^{-1}(x(q) + \delta\overline{XY}x(q)).$$

By Theorem 2.3, the sum is an internal operation. About the scalar multiplication let $Y(q) := x^{-1}(x(q) + a\delta\overline{X}x(p))$. Then Y is injective with inverse $Y^{-1}(r) = x^{-1}(x(r) - a\delta\overline{X}x(p))$. Besides this, $\overline{Y}(u) = a\overline{X}x(p)$ and $\overline{Y^{-1}}(u) = -a\overline{X}x(p)$ are SU-differentiable at $u \approx x(q)$. To sum up, $aX \in \delta\Theta_p M$.

Now we may define tangent vectors on a manifold.

Definition 2.4. For $p \in M$ and (U, x) a chart at p, we define the δ-tangent space of M at p as

$$\delta T_p M := \{(p, st(\overline{X}x(p))) \mid X \in \delta\Theta_p M\}$$

and $(p, st(\overline{X}x(p)))$ is called a **tangent vector** on M at p.

We say that $(p, st(\overline{X}x(p))) \equiv (p, st(\overline{Y}x(p)))$ if $X \equiv_{x(p)} Y$. The **tangent space** to the manifold at p is

$$T_p M := \delta T_p M / \equiv$$

and the **tangent bundle** of M is given by the disjoint union

$$TM := \bigcup_{p \in M} T_p M.$$

This definition of tangent vectors has a number of advantages: it makes the local nature of the tangent space clearer, without requiring the use of bump functions, and it is very intuitive. But it also has an inconvenient: it depends on the choice of charts; nevertheless:

Theorem 2.4. *The set T_pM does not depend on the choice of the infinitesimal δ.*

Proof. Let δ and β be two positive infinitesimal numbers and fix $X \in \delta\Theta_p M$.

Define Y as being

$$\begin{aligned} Y(q) &:= x^{-1}\left(x(q)+\beta\overline{X}x(p)\right) \\ &= x^{-1}\left(x(q)+\beta\frac{xX(p)-x(p)}{\delta}\right). \end{aligned}$$

It is clear that Y is injective with inverse

$$Y^{-1}(r) = x^{-1}\left(x(r)-\beta\overline{X}x(p)\right).$$

Besides this,

$$\overline{Y}(u) := \frac{xYx^{-1}(u)-u}{\beta} = \overline{X}x(p),$$

which is SU-differentiable. Similarly we can prove that $\overline{Y^{-1}}$ is also SU-differentiable. So, in conclusion, $Y \in \beta\Theta_p M$.

Since $\overline{X}x(p) = \overline{Y}x(p)$ then

$$\delta T_pM/\equiv \;=\; \beta T_pM/\equiv$$

□

If we define sum and scalar multiplication on T_pM by

$$(p, st(\overline{X}x(p))) + (p, st(\overline{Y}x(p))) := (p, st(\overline{XY}x(p)))$$

and

$$a(p, st(\overline{X}x(p))) := (p, st(\overline{aX}x(p))),$$

it follows that the set T_pM is a linear space, where $(p,0) = (p, st(\overline{I}x(p)))$ is the identity element. Observe that we also have

$$(p, st(\overline{X}x(p))) + (p, st(\overline{Y}x(p))) = (p, st(\overline{X}x(p)) + st(\overline{Y}x(p)))$$

and

$$a(p, st(\overline{X}x(p))) = (p, a\cdot st(\overline{X}x(p))).$$

Theorem 2.5. *There exists an isomorphism between T_pM and E.*

Proof. Consider the function Φ_p defined by

$$\begin{array}{llll} \Phi_p : & T_pM & \to & E \\ & (p, st(\overline{X}x(p))) & \mapsto & st(\overline{X}x(p)) \end{array}$$

Clearly Φ_p is injective. Fix now $u \in E$ and let $X(q) := x^{-1}(x(q)+\delta u)$. The function X is invertible with inverse $X^{-1}(r) = x^{-1}(x(r)-\delta u)$. Once $\overline{X}(v) = u$ and $\overline{X^{-1}}(v) = -u$ it follows that $X \in \delta\Theta_p M$.

Furthermore $\Phi_p(p, st(\overline{X}x(p))) = u$ and so Φ_p is also onto.

Finally the operator is linear as can easily be seen. □

In the classical literature we may find several definitions of tangent space. We are going to present a brief presentation of one of those.

A tangent vector at p is an equivalence class of C^k paths $\alpha :]-\epsilon,\epsilon[\to M$, with $\alpha(0) = p$ where two paths $\alpha_1 :]-\epsilon,\epsilon[\to M$ and $\alpha_2 :]-\epsilon,\epsilon[\to M$ are called equivalent, $\alpha_1 \equiv_1 \alpha_2$, if $(x\alpha_1)'(0) = (x\alpha_2)'(0)$ for some (and hence for any) chart (U,x) on M with $p \in U$. The tangent space of M at p is the set of all tangent vectors at p, $\Gamma/\equiv_1$, where Γ denotes the set of paths with $\alpha(0) = p$. If we define sum and scalar multiplication by

$$\begin{aligned}(\alpha+\beta)(t) &:= x^{-1}(x(p) + t((x\alpha)'(0) + (x\beta)'(0)))\\ (a\alpha)(t) &:= \alpha(at)\end{aligned}$$

it follows that the tangent space has a linear structure.

Theorem 2.6. *The sets T_pM and $\Gamma/\equiv_1$ are isomorphic.*

Proof. Let

$$\begin{aligned}\Phi : \quad \Gamma/\equiv_1 &\to T_pM\\ \alpha &\mapsto (p,(x\alpha)'(0))\end{aligned}$$

The δ-infinitesimal transformation associated to α in T_pM is $X(q) := x^{-1}(x(q) + \delta(x\alpha)'(0)) \in \delta\Theta_pM$. The operator Φ is well defined since for $\alpha \equiv_1 \beta$, $\Phi(\alpha) = \Phi(\beta)$.

Φ is a linear operator because for $\alpha, \beta \in \Gamma(x)/\equiv_1$ and $a \in \mathbb{R}$,

$$\begin{aligned}\Phi(\alpha+\beta) &= \Phi(x^{-1}(x(p) + t((x\alpha)'(0) + (x\beta)'(0))))\\ &= (p,(x\alpha)'(0) + (x\beta)'(0))\\ &= \Phi(\alpha) + \Phi(\beta)\end{aligned}$$

and

$$\begin{aligned}\Phi(a\alpha) &= \Phi(\alpha(at))\\ &= (p, a(x\alpha)'(0))\\ &= a\Phi(\alpha)\end{aligned}$$

Clearly Φ is injective. To prove that is also onto let

$$\Phi^{-1}(p, st(\overline{X}(p))) := x^{-1}(x(p) + t\, st(\overline{X}x(p))), \text{ for } (p, st(\overline{X}x(p))) \in T_pM.$$

The curve $t \mapsto x^{-1}(x(p) + t\, st(\overline{X}x(p)))$ is well defined in a neighbourhood of zero since $x(U)$ is an open set.

In addition,

$$\Phi\Phi^{-1}(p, st(\overline{X}x(p))) = (p, st(\overline{X}x(p)))$$

and

$$\Phi^{-1}\Phi(\alpha) = x^{-1}(x(p) + t(x\alpha)'(0)) \equiv_1 \alpha(t),$$

as desired. □

Apart from this, if $\alpha(t) := x^{-1}(x(p) + t\, st(\overline{X}x(p)))$ then

$$(x\alpha)'(0) = st\frac{x\alpha(\delta) - x\alpha(0)}{\delta} = st(\overline{X}x(p)).$$

Moreover, if

$$(x\beta)'(0) = st(\overline{Y}x(p))$$

then $\alpha \equiv_1 \beta$ if and only if $X \equiv Y$.

This tangent bundle is a smooth manifold in its own right. Let (U, x) and (V, y) be two charts at $p \in U \cap V$ and $X \in \delta\Theta_p M$.

Once the overlap function yx^{-1} is of class C^1, there exists $\eta \approx 0$ such that

$$\frac{yX(p) - y(p)}{\delta} = \frac{(yx^{-1})xX(p) - (yx^{-1})x(p)}{\delta}$$
$$= D(yx^{-1})_{x(p)} \frac{xX(p) - x(p)}{\delta} + \left| \frac{xX(p) - x(p)}{\delta} \right| \eta.$$

If we take the standard part of both members of the last equation one gets

$$st(\overline{X}y(p)) = D(yx^{-1})_{x(p)} st(\overline{X}x(p)).$$

Definition 2.5. For $p \in M$ and (U, x) a chart at p, let

$$\tilde{U} := \{(p, st(\overline{X}x(p))) \mid p \in U \wedge X \in \delta\Theta_p M\}$$

and

$$\begin{array}{rccl} \tilde{x}: & \tilde{U} & \to & E^2 \\ & (p, st(\overline{X}x(p))) & \mapsto & (x(p), st(\overline{X}x(p))) \end{array}$$

The function $\tilde{x}$ is injective because of the *1 to 1* condition of x and also by the δ-equivalent definition on $\delta\Theta_p M$. Moreover, $\tilde{x}(\tilde{U}) = x(U) \times E$.

Theorem 2.7. *Let M be a differentiable manifold and $\{(U_i, x_i)\}$ $(i \in I)$ an atlas on M. Then $\{(\tilde{U}_i, \tilde{x}_i)\}$ $(i \in I)$ is an atlas on TM. Furthermore, if M is a n-dimensional manifold then TM is a $2n$-dimensional manifold.*

Proof. Simply note that

$$\begin{array}{rccl} \tilde{y}\tilde{x}^{-1}: & \tilde{x}(\tilde{U} \cap \tilde{V}) & \to & \tilde{y}(\tilde{U} \cap \tilde{V}) \\ & (v, st(\overline{X}x(p))) & \mapsto & (yx^{-1}(v), st(\overline{X}y(p))) \end{array}$$

is differentiable. □

3 Stationary Transformations

Definition 3.1. Let $X \in \delta\Theta_p M$. We say that X is a **stationary transformation** at p if $\overline{X}x(p) \approx 0$ for some chart (U, x) at $p \in U$. The set of all stationary transformations at p will be denoted by $\delta I_p M$.

When x and y are charts at p,

$$\begin{aligned} \overline{X}y(p) &= \frac{yX(p) - y(p)}{\delta} \\ &= \frac{(yx^{-1})xX(p) - (yx^{-1})x(p)}{\delta} \\ &= D(yx^{-1})_{x(p)} \overline{X}x(p) + |\overline{X}x(p)|\eta \end{aligned}$$

for some infinitesimal η, it follows that

$$\overline{X}x(p) \approx 0 \Leftrightarrow \overline{X}y(p) \approx 0,$$

i.e., the definition does not depend on the choice of charts.

Theorem 3.1. *The set $\delta I_p M$ is a subgroup of $\delta\Theta_p M$.*

Proof. Since

$$\overline{XY}x(p) \approx \overline{X}x(p) + \overline{Y}x(p) \approx 0 \text{ if } \overline{X}x(p) \approx \overline{Y}x(p) \approx 0$$

we proved that $XY \in \delta I_p M$ if $X, Y \in \delta I_p M$. It is clear that $I \in \delta I_p M$ and, given $X \in \delta I_p M$, $X^{-1} \in \delta I_p M$ because

$$\overline{X^{-1}}x(p) \approx -\overline{X}x(p) \approx 0.$$

□

However, $\delta I_p M$ is not an ideal of $\delta\Theta_p M$. As a matter of fact, $I \in \delta I_p M$ and if we define $X(q) := x^{-1}(x(q) + \delta u)$, with $u \in E \setminus \{0\}$, it follows that $X \in \delta\Theta_p M$ but $IX \notin \delta I_p M$.

We define a relation $\sim$ on $\delta\Theta_p M$ in the following way: given $X, Y \in \delta\Theta_p M$, we say that $X \sim Y$ if there exists $Z \in \delta I_p M$ with $X(p) = YZ(p)$.

Theorem 3.2. *$\sim$ is an equivalence relation.*

Proof. Clearly $X \sim X$ because $I \in \delta I_p M$.

Assume now that $X \sim Y$ and let $Z \in \delta I_p M$ be such that $X(p) = YZ(p)$. Therefore $p = Z^{-1}Y^{-1}X(p)$ and so it is also true that

$$Y(p) = X(X^{-1}YZ^{-1}Y^{-1}X)(p).$$

If we define $Z_1 := X^{-1}YZ^{-1}Y^{-1}X$, since

$$\overline{Z_1}x(p) \approx -\overline{X}x(p) + \overline{Y}x(p) - \overline{Z}x(p) - \overline{Y}x(p) + \overline{X}x(p) \approx 0,$$

it follows that $Z_1 \in \delta I_p M$ and since $Y(p) = XZ_1(p)$, $Y \sim X$.

Lastly, suppose that $X(p) = YZ_1(p)$ and $Y(p) = WZ_2(p)$, with $X, Y, W \in \delta\Theta_p M$ and $Z_1, Z_2 \in \delta I_p M$. Then $p = Y^{-1}WZ_2(p)$ and so

$$X(p) = YZ_1(p) = W(W^{-1}YZ_1Y^{-1}WZ_2)(p).$$

Define now $Z := W^{-1}YZ_1Y^{-1}WZ_2$. With similar calculations as done before, we conclude that $Z \in \delta I_p M$, which ends the proof. □

Theorem 3.3. *There exists an isomorphism between $\delta\Theta_p M/\sim$ and E.*

Proof. Let

$$\begin{array}{rccl} \Phi: & \delta\Theta_p M/\sim & \rightarrow & E \\ & X & \mapsto & st(\overline{X}x(p)) \end{array}$$

The operator Φ is well defined because if $X \sim Y$ then

$$\Phi(X) = st(\overline{X}x(p)) = st(\overline{YZ}x(p))$$

$$= st(\overline{Y}x(p)) + st(\overline{Z}x(p)) = \Phi(Y),$$

for some $Z \in \delta I_p M$. Let us see that Φ is injective. Suppose that $\Phi(X) = \Phi(Y)$, for some $X, Y \in \delta\Theta_p M$. Then there exists an infinitesimal $\epsilon \in {}^*E$ with $xX(p) = xY(p) + \delta\epsilon$, which is equivalent to say that

$$X(p) = Y(Y^{-1}x^{-1}(xY(p) + \delta\epsilon)).$$

Let $Z(q) := Y^{-1}x^{-1}(xY(q)+\delta\epsilon)$. Then $Z \in \delta\Theta_p M$ with inverse $Z^{-1}(r) = Y^{-1}x^{-1}(xY(r) - \delta\epsilon)$ because

$$\begin{aligned} \overline{Z}(u) &= \frac{xY^{-1}x^{-1}(xYx^{-1}(u) + \delta\epsilon) - u}{\delta} \\ &= \frac{\underline{Y}^{-1}(\underline{Y}(u) + \delta\epsilon) - u}{\delta} \end{aligned}$$

and

$$\begin{aligned} D\overline{Z}_u &= \frac{D\underline{Y}^{-1}_{\underline{Y}(u)+\delta\epsilon} D\underline{Y}_u - I}{\delta} \\ &= \frac{(\delta D\overline{Y^{-1}}_{\underline{Y}(u)+\delta\epsilon} + I)(\delta D\overline{Y}_u + I) - I}{\delta} \\ &\approx D\overline{Y^{-1}}_{\underline{Y}(u)+\delta\epsilon} + D\overline{Y}_u \end{aligned}$$

which is a finite operator. Similarly, replacing $-\delta\epsilon$ for $\delta\epsilon$, we can prove that $\overline{Z^{-1}}$ is also SU-differentiable. Let us prove now that $Z \in \delta I_p M$.

$$\begin{aligned} \overline{Z}x(p) &= \frac{xY^{-1}x^{-1}(xY(p) + \delta\epsilon) - x(p)}{\delta} \\ &= \frac{xY^{-1}x^{-1}(xX(p)) - x(p)}{\delta} \\ &= \overline{Y^{-1}X}x(p) \approx -\overline{Y}x(p) + \overline{X}x(p) \approx 0 \end{aligned}$$

In conclusion, $X(p) = YZ(p)$ with $Z \in \delta I_p M$ and so $X \sim Y$. As done in Theorem 2.5, we can prove analogously that Φ is onto and linear. □

Theorem 3.4. *If $X, Y \in \delta\Theta_p M$ then $X \sim Y$ if and only if $X \equiv Y$.*

Proof. Suppose that $X \sim Y$ and let $Z \in \delta I_p M$ with $X(p) = YZ(p)$. Then

$$\begin{aligned}
\overline{X}x(p) &= \frac{xX(p) - x(p)}{\delta} \\
&= \frac{xYZ(p) - x(p)}{\delta} \\
&= \overline{YZ}x(p) \approx \overline{Y}x(p) + \overline{Z}x(p) \\
&\approx \overline{Y}x(p).
\end{aligned}$$

With similar calculations as done in the proof of the previous theorem we can prove the converse. □

4 Conjugation between δ-infinitesimal Transformations

Let M and N be two differentiable manifolds and $f : M \to N$ a standard diffeomorphism.

Given a δ-infinitesimal transformation on M, we can define a new one on N in the following way: for $X \in \delta\Theta_p M$ let $Y := fXf^{-1}$. Then $Y \in \delta\Theta_{f(p)}N$. In fact, Y is clearly an internal bijection with inverse $Y^{-1} = fX^{-1}f^{-1}$.

If $q \in ns(^*N)$ then $Y(q) \approx q$ since

$$Y(q) \approx q \Leftrightarrow fXf^{-1}(q) \approx q \Leftrightarrow Xf^{-1}(q) \approx f^{-1}(q)$$

and $f^{-1}(q) \in ns(^*M)$.

Finally let us prove that $\overline{Y}$ and $\overline{Y^{-1}}$ are both SU-differentiable. Let (U, x) be a chart at p and define $y := xf^{-1}|_V$, where V is an open set in N with $V \subseteq f(U)$ and $f(p) \in V$. Then (V, y) is a chart on N at $f(p)$ (simply note that y is compatible with the other charts on N). Moreover, we have seen that the SU-differentiability of $\overline{Y}$ does not depend of the choice of charts. So, if we fix this chart on N, we obtain

$$\overline{Y}(u) = \frac{yYy^{-1}(u) - u}{\delta} = \frac{xf^{-1}fXf^{-1}fx^{-1}(u) - u}{\delta} = \overline{X}(u)$$

and

$$\overline{Y^{-1}}(u) = \overline{X^{-1}}(u),$$

which are SU-differentiable.

Define then

$$\begin{array}{rccc}
\mathfrak{F}_p f : & T_pM & \to & T_{f(p)}N \\
& (p, st(\overline{X}x(p))) & \mapsto & (f(p), st(\overline{fXf^{-1}}yf(p)))
\end{array}$$

Let us begin by proving that this function is well defined. For $X, Y \in \delta\Theta_p M$ with $\overline{X}x(p) \approx \overline{Y}x(p)$ we have $\overline{fXf^{-1}}yf(p) \approx \overline{fYf^{-1}}yf(p)$. In fact

$$\overline{X}x(p) \approx \overline{Y}x(p) \Leftrightarrow \frac{xX(p) - xY(p)}{\delta} \approx 0.$$

On the other hand

$$\overline{fXf^{-1}}yf(p) - \overline{fYf^{-1}}yf(p) = \frac{yfX(p) - yfY(p)}{\delta} = \frac{xX(p) - xY(p)}{\delta} \approx 0.$$

If we choose $y := xf^{-1}|_V$ then

$$\mathfrak{F}_p f(p, st(\overline{X}x(p))) = (f(p), st(\overline{X}x(p))).$$

With simple calculations we can prove the following theorems:

Theorem 4.1. *Let $f : M \to N$ and $g : N \to R$ be two diffeomorphisms. Then is well defined $\mathfrak{F}_p gf : T_pM \to T_{gf(p)}R$ and $\mathfrak{F}_p gf = \mathfrak{F}_{f(p)} g \mathfrak{F}_p f$.*

Theorem 4.2. *The function $\mathfrak{F}_p f$ is linear.*

Theorem 4.3. *The function $\mathfrak{F}_p f$ is invertible with inverse $(\mathfrak{F}_p f)^{-1} = \mathfrak{F}_{f(p)} f^{-1}$.*

We can generalize the previous definition to the tangent bundle of a manifold. Let $f : M \to N$ be a diffeomorphism and define

$$\begin{array}{rccl} \mathfrak{F} f : & TM & \to & TN \\ & (p, st(\overline{X}x(p))) & \mapsto & (f(p), st(\overline{fXf^{-1}}yf(p))) \end{array}$$

Similarly we have

Theorem 4.4. *The following is verified*

1. *The function $\mathfrak{F} f$ is invertible and $(\mathfrak{F} f)^{-1} = \mathfrak{F} f^{-1}$;*
2. *If $f = I$ then $\mathfrak{F} f = I$;*
3. *$\mathfrak{F} gf = \mathfrak{F} g \mathfrak{F} f$;*
4. *The following diagram is commutative*

$$\begin{array}{ccccc} & & \mathfrak{F} f & & \\ & TM & \longrightarrow & TN & \\ \pi_M & \downarrow & & \downarrow & \pi_N \\ & M & \longrightarrow & N & \\ & & f & & \end{array}$$

i.e., *$f\pi_M = \pi_N \mathfrak{F} f$, where π_M and π_N are the canonical projections.*

5 The Differential of a Function

Let M and N be two differentiable manifolds. With a function $f : M \to N$ of class C^k and for a fixed $p \in M$ we can associate a linear operator $T_pf : T_pM \to T_{f(p)}N$ that maps tangent vectors into tangent vectors. Indeed, define

Definition 5.1. The **differential** of f at p is the function

$$\begin{array}{rccl} T_pf : & T_pM & \to & T_{f(p)}N \\ & (p, st(\overline{X}x(p))) & \mapsto & (f(p), D(yfx^{-1})_{x(p)} st(\overline{X}x(p))) \end{array}$$

where (U, x) is a chart on M at p and (V, y) a chart on N at $f(p)$, with $f(U) \subseteq V$.

The δ-infinitesimal transformation associated on $T_{f(p)}N$ is

$$Y(q) := y^{-1}(y(q) + \delta D(yfx^{-1})_{x(p)} st(\overline{X}x(p))).$$

Since f is a function of class C^k, T_pf is a function of class C^{k-1}. If f is the identity function then T_pf is also the identity function.

Theorem 5.1. *Let $f : M \to N$ and $g : N \to P$ be two functions of class C^k. Then $T_p(gf) = T_{f(p)}gT_pf$.*

Proof. Let (U, x) be a chart at p, (W, z) a chart at $f(p)$ and (V, y) another chart at $gf(p)$, with $f(U) \subseteq W$ and $g(W) \subseteq V$. Then

$$\begin{aligned} T_pgf(p, st(\overline{X}x(p))) &= (gf(p), D(ygfx^{-1})_{x(p)} st(\overline{X}x(p))) \\ &= (gf(p), D(ygz^{-1})_{zf(p)} D(zfx^{-1})_{x(p)} st(\overline{X}x(p))) \\ &= T_{f(p)}g(f(p), D(zfx^{-1})_{x(p)} st(\overline{X}x(p))) \\ &= T_{f(p)}gT_pf(p, st(\overline{X}x(p))) \end{aligned}$$

□

The following properties hold:

Theorem 5.2. *The operador T_pf is linear.*

Theorem 5.3. *If f is a diffeomorphism then T_pf is an isomorphism and $(T_pf)^{-1} = T_{f(p)}f^{-1}$.*

Proof. The inverse of T_pf is

$$(T_pf)^{-1}(f(p), st(\overline{Y}yf(p))) = (p, D(xf^{-1}y^{-1})_{yf(p)} st(\overline{Y}yf(p))).$$

□

Theorem 5.4. *The following diagram is commutative.*

$$\begin{array}{ccccc} & & T_pf & & \\ & T_pM & \longrightarrow & T_{f(p)}N & \\ \pi_M & \downarrow & & \downarrow & \pi_N \\ & M & \longrightarrow & N & \\ & & f & & \end{array}$$

where π_M and π_N are the canonical projections.

6 Directional Derivative of a Function

Let M be a differentiable manifold, F a normed space and $f : M \to F$ a function of class C^1.

Definition 6.1. For $p \in M$, we define the **directional derivative** of f at p as being

$$\begin{array}{rccl} Df_p : & T_pM & \to & F \\ & (p, st(\overline{X}x(p))) & \mapsto & st\left(\dfrac{fX(p) - f(p)}{\delta}\right) \end{array}$$

Observe that, for some $\eta \approx 0$,

$$\begin{aligned} Df_p(p, st(\overline{X}x(p))) &= st\left(\frac{(fx^{-1})xX(p) - (fx^{-1})x(p)}{\delta}\right) \\ &= st[D(fx^{-1})_{x(p)}\overline{X}x(p) + |\overline{X}x(p)|\eta] \\ &= D(fx^{-1})_{x(p)} st(\overline{X}x(p)) \end{aligned}$$

Consequently, Df_p is well defined, *i.e.*, if

$$(p, st(\overline{X}x(p))) \equiv (p, st(\overline{Y}x(p)))$$

then

$$Df_p(p, st(\overline{X}x(p)))) = Df_p(p, st(\overline{Y}x(p))).$$

As one might expect,

Theorem 6.1. *The operator Df_p is linear.*

7 Functionals defined on a Manifold

In this section we will study some properties of functionals of class C^∞ on M.

Definition 7.1. Let $p \in M$, (U, x) a chart for M whose domain contains p and $X \in \delta\Theta_p M$. We define

$$\begin{aligned} X' : \quad C^\infty(M) &\to \mathbb{R} \\ f &\mapsto st\left(\frac{fX(p) - f(p)}{\delta}\right) \end{aligned}$$

The function X' will be called the **derivative** of X at p.

The function X' is well defined since

$$\frac{fX(p) - f(p)}{\delta} \approx D(fx^{-1})_{x(p)}\overline{X}x(p) \in fin(^*\mathbb{R})$$

and so

$$X'(f) = D(fx^{-1})_{x(p)} st(\overline{X}x(p)).$$

The following properties hold (see [12]):
For all $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$,

1. $X'(f + g) = X'(f) + X'(g)$;
2. $X'(af) = aX'(f)$;
3. $X'(fg) = f(p)X'(g) + g(p)X'(f)$.

To these properties we add a fourth:

4. $X'(f/g) = \dfrac{g(p)X'(f) - f(p)X'(g)}{g^2(p)}$ if $g(p) \neq 0$. In fact,

$$X'\left(\frac{1}{g}\right) = st\left(\frac{\frac{1}{gX(p)} - \frac{1}{g(p)}}{\delta}\right) = -st\left(\frac{gX(p) - g(p)}{\delta g(p) gX(p)}\right).$$

Since g is a continuous function and $X(p) \approx p$, it follows that $st(gX(p)) = g(p) \neq 0$. Hence

$$X'\left(\frac{1}{g}\right) = -X'(g)\frac{1}{g^2(p)}$$

and

$$X'\left(\frac{f}{g}\right) = X'\left(f \cdot \frac{1}{g}\right) = \frac{g(p)X'(f) - f(p)X'(g)}{g^2(p)}.$$

The first two conditions prove that X' is a linear operator of $C^\infty(M)$ to $\mathbb{R}$. The third condition justifies the term *derivative* (the "Leibniz rule").

The set of derivatives at $p \in M$ will be denoted by D_pM:

$$D_pM := \{X' : C^\infty(M) \to \mathbb{R} \,|\, X'(f) = st\left(\frac{fX(p) - f(p)}{\delta}\right) \wedge X \in \delta\Theta_p M\}$$

If we define for $X', Y' \in D_pM$, $f \in C^\infty(M)$ and $a \in \mathbb{R}$,

$$(X' + Y')(f) = X'(f) + Y'(f)$$

$$(aX')(f) = aX'(f)$$

the set D_pM becomes a real linear space. If $I : M \to M$ denotes the identity function then $I'(f) = 0$, for every $f \in C^\infty(M)$. Moreover, $-(X') = (-X)'$ (recall the scalar multiplication on $\delta\Theta_p M$). In fact,

$$X'(f) + (-X)'(f) = D(fx^{-1})_{x(p)} st(\overline{X}x(p)) + D(fx^{-1})_{x(p)} st(\overline{-X}x(p)) = 0.$$

Observe that we can also write

$$(X' + Y')(f) = D(fx^{-1})_{x(p)} st\, \overline{XY}x(p)$$

and

$$(aX')(f) = D(fx^{-1})_{x(p)} st\, \overline{aX}x(p).$$

¿From the previous observations it follows that

Theorem 7.1. *[12] It is true that for $X, Y \in \delta\Theta_p M$ and $f \in C^\infty(M)$,*

$$(XY)'(f) = X'(f) + Y'(f).$$

Theorem 7.2. *For $X \in \delta\Theta_p M, f \in C^\infty(M)$ and $a \in \mathbb{R}$,*

$$(aX)'(f) = aX'(f).$$

Theorem 7.3. *The following properties hold:*

1. *If f is constant then $X'(f) = 0$;*
2. *If $f(p) = g(p) = 0$ then $X'(fg) = 0$;*
3. *If $f = g$ in a neighbourhood of p then $X'(f) = X'(g)$.*

Proof. The second condition follows from the Leibniz rule. The other two are clear. □

Theorem 7.4. *There exists an isomorphism between D_pM and E.*

Proof. Let

$$\begin{array}{rccc} \Omega: & D_pM & \rightarrow & E \\ & X' & \mapsto & st(\overline{X}x(p)) \end{array}$$

It is clear that the operator Ω is linear. Let us see that it is bijective.

1. it is injective: Let $X', Y' \in D_pM$ with $\Omega(X') = \Omega(Y')$, *i.e.*,

   $$st(\overline{X}x(p)) = st(\overline{Y}x(p)).$$

   Now let $f \in C^\infty(M)$. Then

   $$\begin{array}{rl} X'(f) & = D(fx^{-1})_{x(p)} st(\overline{X}x(p)) \\ & = D(fx^{-1})_{x(p)} st(\overline{Y}x(p)) \\ & = Y'(f). \end{array}$$

   Thus $X' = Y'$.

2. it is onto: Fix $u \in E$ and define $X(q) := x^{-1}(x(q) + \delta u)$. Then $X \in \delta\Theta_pM$ and $\Omega(X') = u$.

□

Theorem 7.5. *The sets T_pM and D_pM are isomorphic.*

Proof. Follows from the previous theorem and from Theorem 2.5. □

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