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Mathematics

Douglas' solution of the Plateau problem

[minimal surface/representation of SL(2, R)]

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ABSTRACT **Using ideas suggested by some recent developments in string theory, we give here an elementary demonstration of one of the key steps in Douglas' celebrated proof of the existence of solutions of the Plateau problem in *n* dimensions.**

It was recently suggested to us by T. Regge that one should reexamine the celebrated solution by Douglas (1) of Plateau's problem in light of recent developments in string theory. We have followed his suggestion and wish to show in this note how some simple facts in the representation theory of Sl(2, **R**) give an elementary proof of a key step in Douglas' ingenious argument.

We begin with an outline of the Douglas proof. It is a theorem due to Weierstrass that a surface in $\mathbf{R}^n$ is minimal if and only if it can be represented as the real part of a complex n-dimensional holomorphic null curve. That is, a surface is minimal if and only if one can introduce (isothermal) coordinates u_1, u_2 such that each of the coordinate functions $x_k(u_1, u_2)$ is given as

$$x_k(u_1, u_2) = \mathrm{Re}F_k(z), \qquad z = u_1 + iu_2,$$

where the Fs are holomorphic and

$$\sum_k F_{k'}(z)^2 \equiv 0. \qquad \textbf{[W]}$$

An extremely clear proof of this result starting from first principles can be found in the book by Osserman (ref. 2, cf. the lemma on page 30). Let us take this as our definition of a minimal surface.

Plateau's problem (in its simplest form) is this: given a simple closed curve in $\mathbf{R}^n$, find a minimal surface spanning it. Now if we had a solution of the problem, we would have a parametrized map, h, of the unit circle, S^1, into $\mathbf{R}^n$, a parametrization of our curve, and the coordinates of h would be the real parts of the boundary values of the F_k. On the other hand, if we knew these boundary values, we could reconstruct the F_ks by use of the Poisson integral formula giving a harmonic function in terms of its boundary values. So, as Douglas points out, in order to solve Plateau's problem we have to find the "correct" parametrization of the curve. Put another way, suppose we are given a map, g, of S^1 into $\mathbf{R}^n$. Find, among all reparametrizations of g, an h whose harmonic extension into the interior is the real part of a holomorphic F satisfying **[W]**.

Douglas solves this problem as follows: For each map g and each pair of distinct angles θ_1 and θ_2, let $D(\theta_1, \theta_2)$ denote the distance, in $\mathbf{R}^n$ from $g(\theta_1)$ to $g(\theta_2)$ so

$$D(\theta_1, \theta_2)^2 = \sum_k [g_k(\theta_1) - g_k(\theta_2)]^2.$$

Let $C(\theta_1, \theta_2)$ denote the distance in the $z = u_1 + iu_2$ plane between the corresponding points on the unit circle, so

$$C(\theta_1, \theta_2)^2 = |e^{i\theta_1} - e^{i\theta_2}|^2.$$

Now define

$$A(g) = \iint \frac{D^2}{C^2} d\theta_1 d\theta_2.$$

Here the integration is over all θ_1, θ_2 with $\theta_1 \neq \theta_2$. Let $\mathbb{H}$ denote the set of all such θ_1, θ_2 so that $\mathbb{H}$ is the torus with the diagonal removed. The functional $A(g)$ is invariant under reparametrization by Sl(2, **R**). Indeed Sl(2, **R**) acts transitively on $\mathbb{H}$ and it is easy to check (or see ref. 3) that

$$C^{-2} d\theta_1 d\theta_2$$

is the Sl(2, **R**) invariant measure on $\mathbb{H}$ (determined up to multiplicative constant).

The key observation of Douglas is that the parametrization, g, which minimizes $A(g)$ is the "correct" parametrization giving the minimal surface. It is easy to show that $A(g)$ achieves a minimum. Using the Sl(2, **R**) invariance it is easy to prove that the minimum is achieved at a nondegenerate reparametrization.

The proof of this key observation is an immediate consequence of the following formula reparaphrasing a portion of Douglas' argument: For each w in the interior of the unit disk let ξ_w denote the complex vector field on the unit circle given by

$$\xi_w = \frac{-2i}{1 - e^{-i\theta}w} \frac{d}{d\theta}.$$

Then

$$\xi_w A(g) = w^2 \sum_k F_{k'}(w)^2. \qquad \textbf{[D}_\mathbf{n}\textbf{]}$$

This formula is to be understood as follows: We can think of $A(g)$ as a function on Diff S^1, the group of all reparametrizations of the circle. The vector field ξ_w can be thought of as lying in a completion of the complexification of the Lie algebra of Diff S^1, and this is the meaning of the left-hand side of **[D$_\mathbf{n}$]**, depending on the complex parameter w, lying in the unit disk. Each function F_k on the right is the holomorphic function on the disk whose real part is the harmonic extension into the disk of the kth coordinate, g_k, of g given as a function on the unit circle. If g is a minimum of $A(g)$, then any vector field on the circle must yield zero when applied to $A(g)$, since all variations must vanish. So the left-hand side of **[D$_\mathbf{n}$]** vanishes at a minimum. The right-hand side must then vanish, so Weierstrass' condition, **[W]**, is satisfied.

Thus the heart of the proof is to establish **[D$_\mathbf{n}$]**. The subscript **n** refers to n dimensions. But the formula **[D$_\mathbf{n}$]** is a

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consequence of the corresponding formula (D_1) in one dimension, simply by summing over the coordinates. Let V denote the space of real (smooth) functions on the circle modulo the constants. This space is real irreducible under Sl(2, **R**) and hence has at most a one-dimensional space of invariant quadratic forms under Sl(2, **R**), which must then coincide with scalar multiples of the quadratic function $A(.)$ defined above (with now $n = 1$). On the other hand V carries a symplectic form (,) invariant under all of Diff S^1, namely,

$$(f, g) = \int f dg.$$

Let W denote the complexification of V. We will continue to denote the complex extension of the symplectic form by (,). Any quadratic form on W will be given by

$$Q(f) = (f, Kf),$$

where K is an element of the symplectic algebra of W. If Q is to be Sl(2, **R**) invariant, then K must lie in the centralizer of Sl(2, **R**) in the symplectic algebra of W. It is easy to see that this centralizer is one dimensional, consisting of all multiples of the operator whose block decomposition is given by

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

relative to the decomposition

$$W = W_{\text{hol}} + W_{\text{antihol}}$$

of W into holomorphic and antiholomorphic parts. Thus, up to scalar factors, we must have $A(f) = (f, Kf)$, where

$$K = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Now $d/d\theta$ commutes with K since $d/d\theta$ belongs to Sl(2, **R**). Hence, up to scalar factors, we have

$$\xi_w A(f) = 2(\xi_w f, Kf) = \oint \frac{1}{1 - e^{-i\theta}w} f' K f' d\theta.$$

The integration is over the unit circle and f is real. So we may write

$$f = F + \overline{F},$$

where F is the holomorphic part of f (the sum of the positive Fourier components). Now

$$\frac{d}{d\theta} = iz\frac{d}{dz} \text{ and } d\theta = \frac{dz}{z}$$

and so

$$\frac{df}{d\theta} = i\left(z\frac{dF}{dz} - z\frac{\overline{dF}}{dz}\right)$$

for $z = e^{i\theta}$. Hence, up to overall scalar factors we have

$$\oint \frac{1}{1 - e^{-i\theta}w} f' K f' d\theta = \oint \frac{1}{z - w}$$
$$(z^2 F'(z)^2 - z^2 F'(z)^2) dz = 2\pi i w^2 F'(w)^2,$$

by the Cauchy integral formula. This completes the proof of $[\mathrm{D_n}]$.

1. Douglas, J. (1931) *Trans. Am. Math. Soc.*, Vol. 37.
2. Osserman, R. (1969) *A Survey of Minimal Surfaces* (van Nostrand Reinhold, New York).
3. Kostant, B. & Sternberg, S. (1988) *Lett. Math. Phys.*, in press.