Introduction to Geometric Measure Theory
Fall 04-Spring 05

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Contents

1 Preamble and books

This is a set notes for an introductory seminar course in geometric measure theory. The books referred to are

Morgan: Frank Morgan; *Geometric measure theory: a beginner's guide.3rd Ed.*

LY: Lin and Yang: *Geometric measure theory: an introduction. 1st Ed. International Press. Boston. (series in advnaced Matheamtics Volume 1)*

Other references include *Geometric Measure Theory*: Federer. (reprinted by Springer verlag

Lectures on Geometric Measure Theory by Leon Simon. available from the Centre for Mathematical Analysis, Australian National University, Volume 3, 1983

2 Introductory survey

2.1 Differential Geometry vs Geometric measure theory: Oct 5th

Differential geometry deals with differentiable maps into an ambient space in extrinsic geometry or intrinsically in Riemannian geometry with atlases of maps from $\mathbb{R}^n$ with differentiable transition functions.

There is a branch of Geometric Measure Theory (GMT) which operates in this intrinsic context, Harvey and Lawson use currents and forms for example with calibrations or to represent connections and hence curvature. This has applications in complex geometry.

We will be studying GMT in an extrinsic context with $\mathbb{R}^n$ as the ambient space. The term geometric measure theory derives from the fact that we only require our objects to be integrable with respect to specific measures. There is no need for differentiable structure, although many solutions such as minimal surfaces do have differentiable structure.

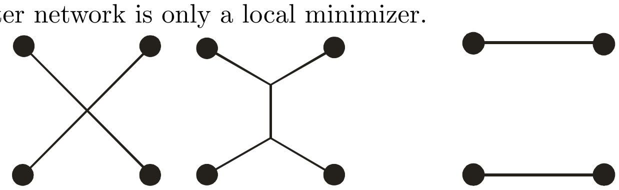
2.1.1 Local vs global minimizers.

Local minimization means that in some topology on the space of candidates the candidate is the minimum in a neighborhood of the candidate in the space of candidates. Global minimizers have the minimum mass over all candidates in the space. Note it is not trivial that a global minimizer exists.

2.1.2 Network example

All these networks are local minimizers. They are left to right order of decreasing mass.

The topological class of candidates determines whether or not they are global minimizers as well as local minimizers. If the network must be connected, the center network is a global minimizer. If disconnected candidates are allowed the center network is only a local minimizer.



Stationary Lower length Least length, not connected

We will see later how currents and varifolds can be used to define different topological classes of solution.

Notice that the more symmetric solution is not the one of least length. This also occurs when considering the minimal surfaces bounded by a base ball seam shaped boundary. The most symmetric one goes through the where the center of the baseball would be. A least area surface approximates half the surface of the baseball made up of one piece of leather.

2.2 First variation

The action of vector field $\boldsymbol{\phi}$ on an n-dimensional set M after time t is given by

$\phi_t\#M = \{y : y = x + t\boldsymbol{\phi}(x), x \in M\}$

The first derivative of mass under the action of a vector field

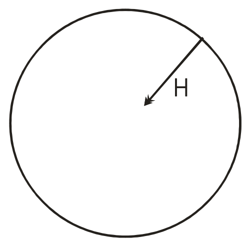
$= \frac{d}{dt} Mass(\phi_t\#M)_{t=0}$

$$=-\int_M \mathbf{H}.\boldsymbol{\phi} dH^n - \int_{\partial M} \boldsymbol{\nu}.\boldsymbol{\phi} dH^{n-1}$$

where M is a union of C^2 n-dimensional manifolds in $\mathbb{R}^{n+k}$. $\mathbf{H}$ is the mean curvature vector, $\boldsymbol{\nu}$ is the inward pointing unit normal to the boundary and $\boldsymbol{\phi}$ is a smooth compactly supported vector field in $\mathbb{R}^{n+k}$.

M is defined as stationary if this quantity is zero for all $\boldsymbol{\phi}$.

The first term can be seen in an n-dimensional sphere of radius r collapsing to the center at the origin.



Let $\boldsymbol{\phi}(\mathbf{x}) = \frac{-\mathbf{x}}{|\mathbf{x}|}$ for $|\mathbf{x}| > \varepsilon > 0$, an inward pointing radial vector field away from the origin.

$|\mathbf{H}| = \frac{n}{r}$ as there are n principle curvatures all equal to $1/r$. $\mathbf{H}$ points toward the origin.

$$\frac{d}{dt} Mass(\phi_t \# M)_{t=0} = -\int_M \mathbf{H}.\boldsymbol{\phi} dH^n - \int_{\partial M} \boldsymbol{\nu}.\boldsymbol{\phi} dH^{n-1}$$

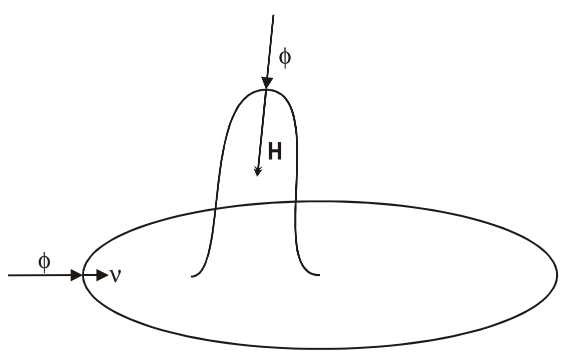
$=-\varpi_n r^n \frac{n}{r} - 0$

$=-\varpi_n r^{n-1}$

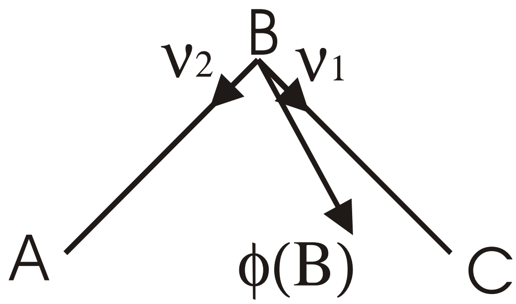
ϖ_n is the n-volume of S^n

Note for n=1 this quantity is constant. That is the first variation of a C^1 curve is equal to the integral of its curvature.

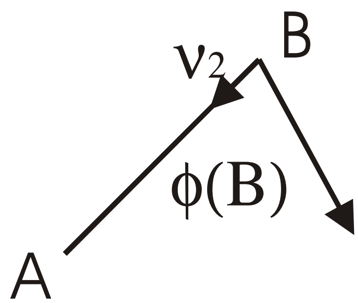
Now consider the shape below to see both the effects of mean curvature contribution and boundary contribution to first variation.

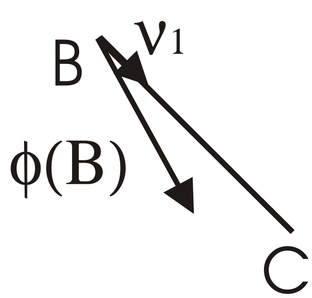


2.2.1 Piecewise C^1 curves

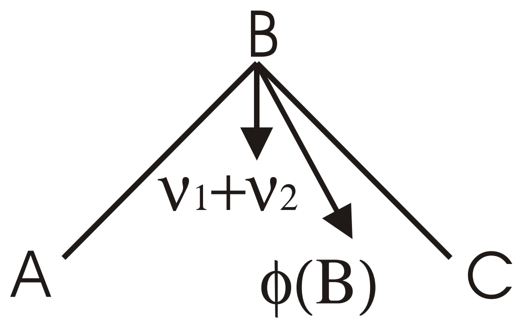


Consider the first variation at vertex B of the piecewise linear curve ABC. The mass is the sum of the length of segment AB and segment BC. So we can look at their first variations separately.





First variation with respect to $\boldsymbol{\phi}$ at B is given by $\boldsymbol{\nu}_1.\boldsymbol{\phi}(\mathbf{B}) + \boldsymbol{\nu}_2.\boldsymbol{\phi}(\mathbf{B}) = (\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2).\boldsymbol{\phi}(\mathbf{B})$■ which can be shown as

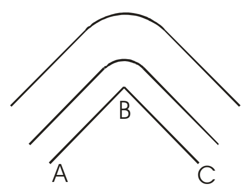


2.2.2 First variation is lower semi continuous.

We can define a function

$$\|M\| = \sup_{\phi} \left[-\int_{M} \mathbf{H}.\boldsymbol{\phi} dH^n - \int_{\partial M} \boldsymbol{\nu}.\boldsymbol{\phi} dH^{n-1} \right], |\phi| \leq 1$$

If we place a topology on the space of objects whose first variation is defined, and treat first variation as a function on these spaces we can examine the continuity of the function. Consider the 1-parameter family of smooth curves with the piecewise straight limit.



The first variation of the smooth curves at the curve which tends to B is given by the angle, approx $\pi/2 = 1.6$. The first variation of the piecewise straight line curve at B is $\sqrt{2} \approx 1.4$. Another more trivial example is two segments of the same line expanding in length until they meet. Then first variation suddenly drops by 2.

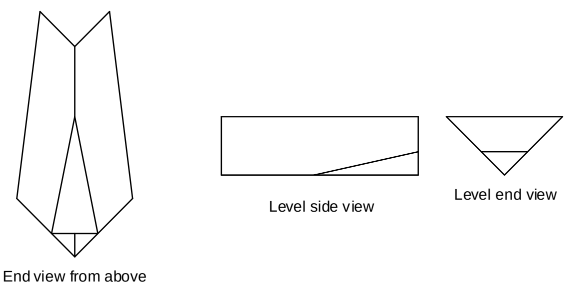
Therefore first variation is not continuous. We can speculate that it is lower semi continuous in any reasonable topology. However in the pathological example in section 6.2, we see that for general varifolds first variation is not lower semicontinuous. The topology on the space of curves used here is the Hausdorff set topology. Later we will use another topology for this purpose based on the flat norm.

2.2.3 Stationary does not imply local minimization

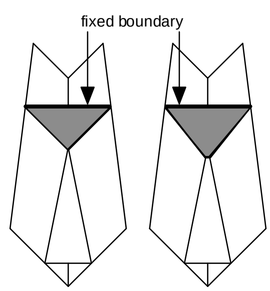
The ramp in the valley example.

We might assume that first variation of area of a soap film in a polyhedral boundary problem would be a situation where stationary, that is the first derivative of area with respect to a smooth vector field deformation would be sufficient to indicate local minimization. It is not.

A free boundary problem is one where a surface may have any boundary in a given set, such as the sides of a region. Consider the region in $\mathbb{R}^3$ above the planes $z = x$ and $z = -x$. An area stationary surface will meet these planes orthogonally, and will therefore have the formula $y = k$, for some k. In fact this is a one parameter family of surfaces parameterized by k.



Now modify the region to be the region above the planes $z = x, z = -x$ and $z = tan(a)y$. The plane $y = 0.$ is area stationary. It is shown in the figure with a fixed boundary making the surface compact.



Under the local deformation shown $\phi(z) = \mathbf{j}(h - z)$. For some h, area loss $=(at)^2$ .So second variation is non zero in this example.

2.3 Global Minimization

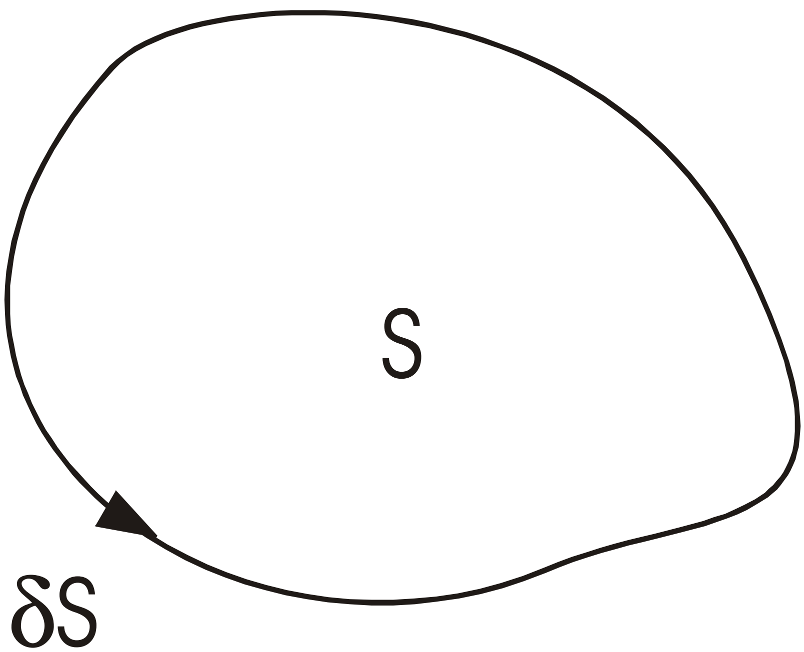
There are two techniques from geometric measure for indicating that a candidate is a global minimizer. The first is a direct geometric comparison using a calibration form for a specific candidate and Stokes' theorem. Variants of this method use vector fields and flow arguments across surfaces, usually divergence free vector fields.

The second method, is that of proving that the space of candidates is compact thus proving a minimizer must exist as a limit of a mass decreasing sequence of candidates. This is non constructive and gives us little information about the minimizer, nor does it verify a specific candidate is or is not a minimizer.

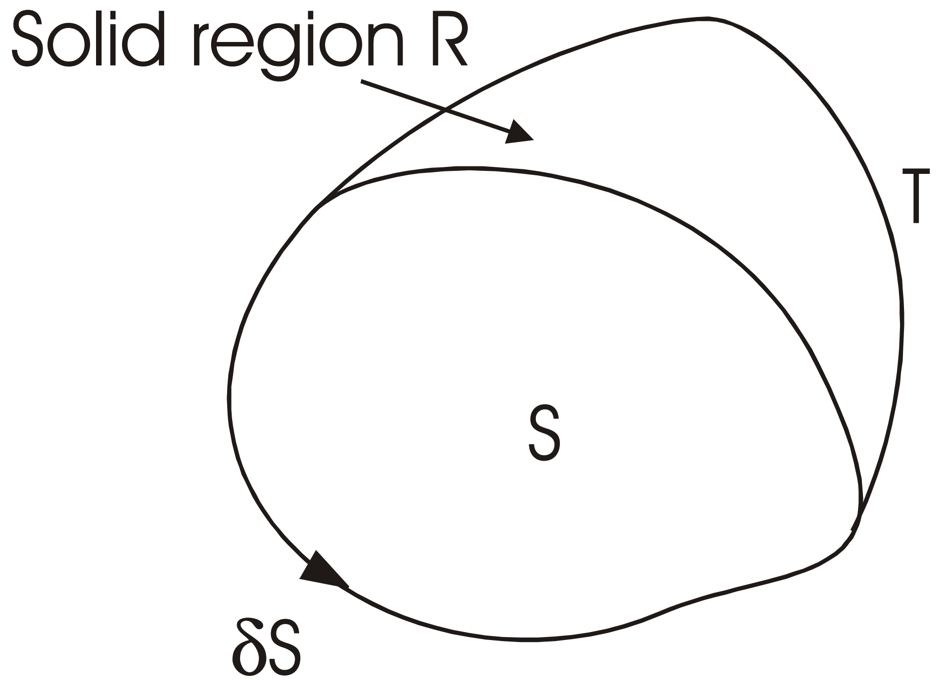
2.3.1 Calibration.(geometric proof of global minimization) Oct 12th

Given a candidate oriented surface S with boundary in $\mathbb{R}^3$, and a closed 2 form on $\mathbb{R}^3$ ϕ, where $|\phi| \leq 1$.and $\int_S \phi = area(S)$, we can conclude that S is a global minimizer with respect to its oriented boundary, in the class of orientable surfaces (A Möbius band is an example of a non-orientable surface which has a boundary which can also bound orientable discs. See below). This method generalizes to any volume form as it is an application of Stokes' theorem.

Proof



The oriented surface S has oriented boundary ∂S.Say another surface T has the same boundary $\partial S = \partial T$, then $S - T$ will bound a solid region R.



Using the fact that ϕ is closed and Stokes' theorem we obtain

$$0 = \int\limits_R d\phi = \int\limits_{\partial R = S - T} \phi$$

$$\Rightarrow$$

$$\int\limits_S \phi = \int\limits_T \phi$$

Now using the fact that $|\phi| \leq 1$ and the hypothesis we obtain

$$area(S) = \int\limits_S \phi = \int\limits_T \phi \leq area(T).$$

Note $\phi = adx \wedge dy + bdy \wedge dz + cdz \wedge dx$, where a,b and c are real valued functions on $\mathbb{R}^3$. $|\phi| = \sqrt{a^2 + b^2 + c^2}$. So if $\mathbf{n}$ is the positively oriented normal to S then $< a, b, c >= \mathbf{n}$, so that $area(S) = \int\limits_S \phi$.

Other forms of calibration type argument exist with flows of divergence free vector fields across surfaces.

2.3.2 Graphs in $\mathbb{R}^3$ that are minimal surfaces are global minimizers.

The unit normal to a graph $f(x, y)$ is

$$\frac{-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$$

giving rise to the 2 form in $\mathbb{R}^3$

$$\phi = \frac{-f_x dy \wedge dz - f_y dz \wedge dx + dx \wedge dy}{\sqrt{1 + f_x^2 + f_y^2}}$$

.

Calculation will verify that ϕ is closed when the minimal surface equation $(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0$ is satisfied:

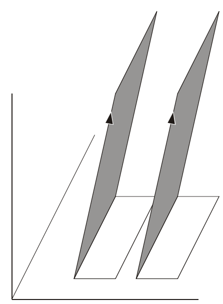
$d\phi = \left(-\frac{\partial}{\partial x} f_x (f_x^2 + f_y^2 + 1)^{-1/2} - \frac{\partial}{\partial y} f_y (f_x^2 + f_y^2 + 1)^{-1/2} \right) dxdydz$

$= (f_x^2 + f_y^2 + 1)^{-3/2} ((1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy}) dxdydz$

$= 0$

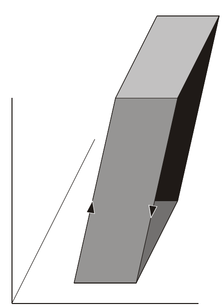
2.3.3 Example of calibrations with pathological orientations .

Consider two subsets of parallel planes. As a surface their union is a graph

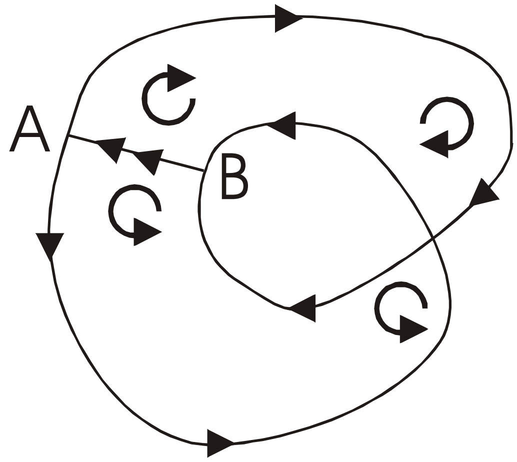


where each component is the graph of a minimal surface. There exists a calibration form for the union. It is constant, based on the unit normal to the surface.

For the orientations induced on the boundary the union is the minimizer, but there is a lower area surface with the same boundary but with a different boundary orientation pattern.



2.3.4 Möbius band example of homological boundary



If we place an orientation locally on a part of a Möbius band and propagate it around the band (shown by the clockwise and counter clockwise arrows) there will be a line where the orientations do not match up shown above as AB. The orientation induced on the boundary is shown by the arrows on the boundary and see that there is a double weighting of the induced boundary on AB. Stokes theorem could now be applied to the möbius band with a weighting of two on the line from B to A. Thus the homological boundary induced by Stokes' thm is the normal point set topology boundary union the extra curve BA with a multiplicity of 2.

2.4 Current and Varifold compactness(non-constructive proof of existence of global minimizers in classes)

2.4.1 Currents:

The technical definition of an n-dimensional current is that it is an element of the dual space D_n to smooth compactly supported n-forms D^n. The class of currents we are concerned with here are called n-rectifiable currents. These can be represented as the integral of a form over and integrable n dimensional subset M of $\mathbb{R}^{n+k}$.These subsets are called n-rectifiable sets.

If M is rectifiable then

$M = M_0 \cup \bigcup_{i=1}^{\infty} M_i$ where each $M_i \subset C_i$ and each C_i is a C^1submanifold, and M_0 is a set of zero n-dimensional I

If we have such an M endowed with an orientation ϕ and a density function ρ.

Then we define the current T as a linear functional. Let $\phi \subset D^n$

$T(\phi) = \int_M \rho \langle \xi, \phi \rangle \, dH^n$

The pairing $\langle \xi, \phi \rangle$ is explained by seeing that the orientation ξ represents not just the sign of the orientation on M, but also the tangent space. ξ is therefore described as a vector which is then paired with the form ϕ. So for example consider a surface in $\mathbb{R}^3$.

$\phi = adydz + bdzdx + cdxdy$

Let the positively oriented unit normal be (e, f, g) represented as

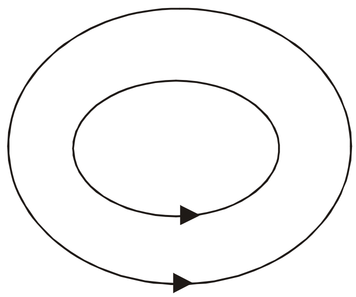
$\xi = e\frac{\partial}{dy}\frac{\partial}{dz} + f\frac{\partial}{dz}\frac{\partial}{dx} + g\frac{\partial}{dx}\frac{\partial}{dz}$.

$\langle \xi, \phi \rangle = ae + bf + cg$

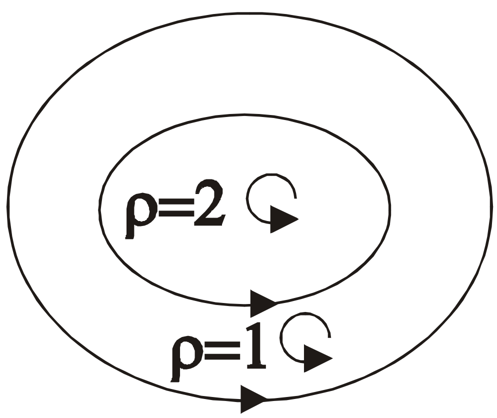
If ρ is restricted to integer values then we can see that it operates like a multiplicity function on the manifold.

2.4.2 Example of boundary and multiplicity. 19th Oct

Consider the two curves in $\mathbb{R}^2$ below with the assigned orientations.



We can ask what oriented integer multiplicity surface in $\mathbb{R}^2$ could have these oriented curves as its boundary? Here is one solution. The inner ellipse has multiplicity 2 and the annulus has multiplicity 1. Both have counter-clockwise orientations.



We could find this by considering the outside curve first and obtaining the desired orientation by assigning the density and orientation to the region of the annulus. This induces a clockwise orientation on the inner circle, which is in the wrong direction. To compensate for this we assign a multiplicity 2 density which induces a multiplicity 2 inner curve with the correct orientation. The net orientation on the inner curve is therefore as desired with multiplicity 1.

We can also see this as a large ellipse combined with a small ellipse where both have the same orientations. The boundary of the sum is then the sum of the boundaries.

2.4.3 Boundary current.

We can now define the boundary current in accordance with Stokes' thm.

$$\partial T(\phi) = T(d\phi)$$

The geometric interpretation is given by the above examples for the integer multiplicity case. It is the homological boundary induced by the set and the orientation weighted by the density.

2.4.4 Current Mass

The mass of a current $M(T) = sup(T(\phi))\,|\phi| \leq 1, \phi \subset D^n$

This can be used to set up a mass norm giving $d(T_1, T_2) = M(T_1 - T_2)$. This however does not indicate the closeness of tow parallel surfaces coming together.

2.4.5 Weak* topology

The natural topology on currents is the weak* topology. That is $T_i \to T \Leftrightarrow T_i(\phi) \to T(\phi)$ for all $\phi \in D^n$. It is called weak* instead of weak because currents are duals to forms, rather than D^n being dual to D_n.There are linear functionals on n-currents which are not smooth compactly supported forms.

2.4.6 The current and varifold compactness theorems

Currents without signed orientation can still be integrated with unsigned volume integration. These are called varifolds.

Subject to uniform bounds, the two classes of object are compact, integer multiplicity n-rectifiable currents and integer multiplicity n-rectifiable varifolds

Table of hypotheses

n-rectifiable currents T_i	n-rectifiable varifolds V_i
Underlying set is n-rectifiable	Underlying set is n-rectifiable
Integer multiplicity ($\rho \in \mathbb{Z}^+$)	Integer multiplicity ($\rho \in \mathbb{Z}^+$)
Uniformly bounded Mass (T_i)	Uniformly bounded Mass (V_i)
Uniformly bounded boundary mass:	Uniformly bounded first variation:
-point set boundary mass	-point set boundary mass
-extra induced homological boundary,	-integral of mean curvature

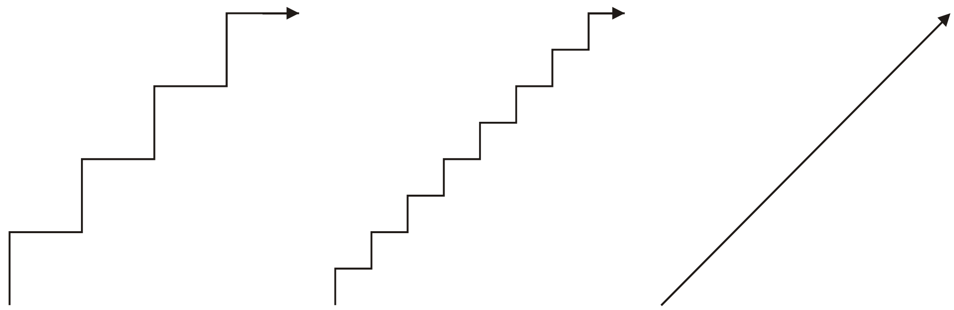
Notice that the difference between the hypothesis is homological boundary for the current vs first variation for the varifold.

First variation is point set boundary mass plus integral of mean curvature.

Homological boundary is point set boundary mass plus induced homological boundary from singularities such as Ys and non-orientability.

2.4.7 Examples of current and varifold compactness

Current compactness can deal with infinitessimal undulations.



Consider a sequence of staircases from (0,0) to (1,1), with progressively smaller step sizes as shown. The limit is the diagonal line. Note that the mass of the sequence drops in the limit from 2 to $\sqrt{2}$.So current mass is lower semi continuous. Varifold compactness cannot deal with this example. If each staircase was represented as a rectifiable varifold, then the first variation would be unbounded as each step adds a constant to the first variation.

See also general varifolds which can represent the sequence as a measure in the Grassman bundle of $\mathbb{R}^2$. However we do not have an easy compactness theorem for these objects.

Example of varifold only convergence

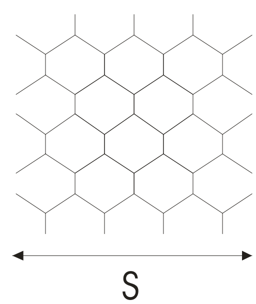


Figure 10

Consider a honeycomb, figure 10, in a cube of edge length s in $\mathbb{R}^3$. It projects down to a tessellation of hexagons in the $x - y$ plane and has height h in the z direction. Along each side of the honeycomb there are n vertical edges. This will correspond to the order of n^2 vertical Y-singularities on the interior of the honeycomb, where three faces come together.

The total mass of the honeycomb is of the order $2nsh$. As s does not affect any other quantity, we can now ignore mass.

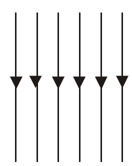
The homological boundary mass, from the Y's is of the order of hn^2. The first variation, from the outer ends of the honeycomb is of order $4nh$. We can set up a sequence where $h = 1/n$ of honeycombs. The first variation is uniformly bounded and the homological boundary is not.

Notice that if $h = 1/n^3$ we can take the unions of all the honeycombs and obtain a rectifiable set which can be represented as an integer multiplicity rectifiable varifold, but not as an integer multiplicity rectifiable current.

This particular example can be represented as a current mod 3, but one can easily add extra faces to the honeycomb to create an infinite measure of singularities with 3 and infinite measure with 5 faces coming together. Thus in general currents mod n do not eliminate the need for varifold compactness.

2.4.8 Current compactness non-examples

If density is not integer multiplicity. We get a series of n vertical lines in the unit square with density 1/n. This converges to dy with Lebesgue measure on the unit square. e.g.:



If mass is not uniformly bounded we can get n lines on the unit square, each with density 1 and this tends to infinite density on the unit square integrated with respect to dy.

If boundary mass is not uniformly bounded. We can get a series segments on the real line $[1/n, 2/n]$,$[3/n, 4/n]$...$[(n-1)/n, 1]$, n even. This will approach the unit interval with density 1/2.

2.4.9 Boundary of the continuously variable density currents (Oct 26th) and distributional density gradients $< \nabla\rho \times \xi, \phi >$.

We will now derive from first principles a geometric interpretation for boundaries of 2-currents with positive real valued densities. As a linear model we will take the current:

$T(\varphi)=\int_0^1 \int_0^1 pxf\,dxdy$ where $\varphi = f dxdy, \rho = px, , p$ is constant and $\xi = \frac{\partial}{dx}\frac{\partial}{\partial y}$.

We will use $\partial T(\phi) = T(d\phi)$. Now ∂T will be a 1-current so ϕ must be a 1-form, and $d\phi$ a 2 form.

$\phi = ady + bdx$

$d\phi = (a_x - b_y)\, dxdy$

$$T(d\phi) = \int_0^1 \int_0^1 px\left(a_x - b_y\right) dx dy$$

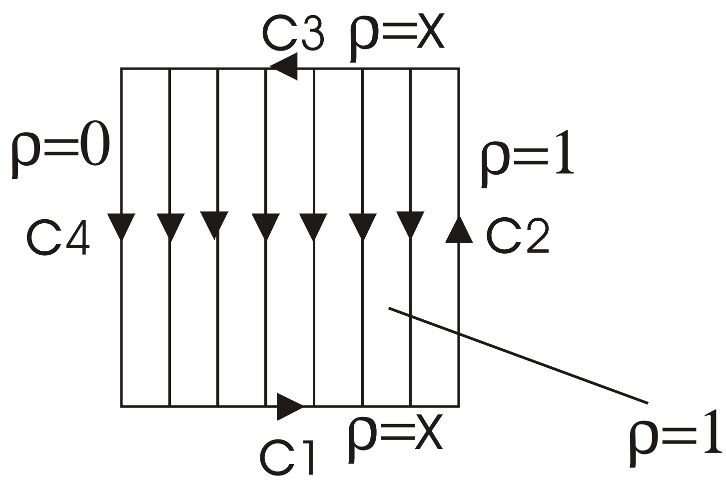
$$= -\int_0^1 \int_0^1 pxb_y dy dx + \int_0^1 \int_0^1 pxa_x dx dy$$

$$= -p \int_0^1 [xb]_{y=0}^{y=1} dx + p \int_0^1 \left(\int_0^1 -a dx + [xa]_{x=0}^{x=1} \right) dy$$

$$= -p \int_0^1 [xb(x,1) - xb(x,0)]\, dx + p \int_0^1 \left(\int_0^1 -a dx + a(1,y) \right) dy$$

$$= p \left\{ -\int_0^1 xb(x,1) dx + \int_0^1 xb(x,0) dx + \int_0^1 a(1,y) dy - \int_0^1 \int_0^1 a(x,y) dx dy \right\}$$

This now has a geometric interpretation as three line integrals and a kind of smeared out line integral over the unit square.



$c1 + c2 + c3 + c4$ with $\rho = 1$ is the oriented boundary of the oriented unit square with $\rho = 1$.

∂T in the case of $\rho = px$ on the unit square is $c1 + c2 + c3 + c4$ with $\rho = x$, combined with the smeared out 1 form related to $\nabla \rho = (p, 0)$.

This suggests that when

$$T(\phi) = \int_M \rho < \xi, \phi > dH^2 = T(\phi) = \int_M < \rho\xi, \phi > dH^2$$

We can interpret

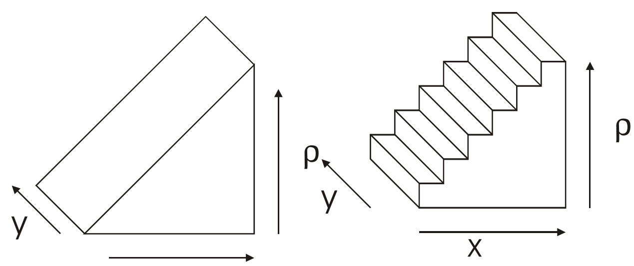
$$T(d\phi) = \int_{\partial M} < \rho\partial\xi, \phi > dH^1 + \int_M < \nabla\rho \times \xi, \phi > dH^2$$

where $\partial\xi$ is the usual homological boundary. In fact we can even interpret this as a distributional $\nabla\rho\ \xi$, where $\nabla\rho$ is distribution valued and hence has support on, and is integrated on, a set of H^{n-1} measure. We can now write this as.

$T\,(d\phi) = \int\limits_M < \nabla\rho \times \xi, \phi >_D \, dH_D^2$, where the pairing $< \nabla\rho \times \xi, \phi >_D$and the integral are defined to include distributional values of $\nabla\rho$. For the case $n = 2$ in $\mathbb{R}^3$ the wedge product is the $\times$ product. Where $\mathbf{i} = \frac{\partial}{dy}\frac{\partial}{dz}, \mathbf{j} = \frac{\partial}{dz}\frac{\partial}{dx}, \mathbf{k} = \frac{\partial}{dx}\frac{\partial}{dy}$

$\mathbf{k} \times \mathbf{i} = -\mathbf{j}$

For other dimensions the $\times$ product will have to be replaced by another kind of product that can be derived from Stokes' thm just as we did in dimension 2.



T

Approximation to T

This example can be modelled and illustrated directly by taking a sequence of currents which give step-wise approximations to the current. So in the figure T can be approximated by the sum of seven currents, each with $\rho = 1/7$. Their underlying sets will be $[0,1] \times [0,1]$, $[1/7,1] \times [0,1]$, $[2/7,1] \times [0,1]$, $[3/7,1] \times [0,1]$, $[4/7,1] \times [0,1]$, $[5/7,1] \times [0,1]$, $[6/7,1] \times [0,1]$. Their boundaries will be approximately as shown with each downward arrow having a density of 1/7.

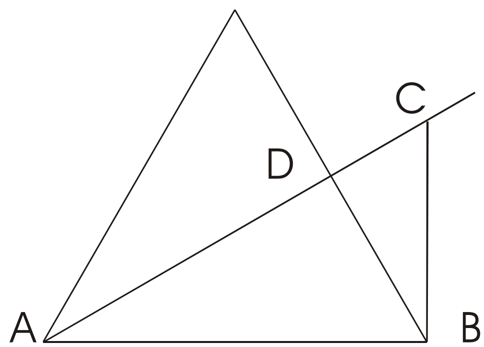
Exercise. Determine the boundary current of the 1-current on the real line given by

$T\,(\phi) = \int\limits_0^1 \rho < \xi, \phi > dH^1$, where $\phi = f dx$, and $< \xi, \phi >= +f dx$.

Note for $\rho = 1$. A zero form $\psi = f(x)$ is a function and $d\psi = f' dx$ is a 1-form.

$$\partial T(\psi) = T(d\psi) = \int\limits_0^1 f' dx = f(1) - f(0)$$

See also BV functions.



3 Hausdorff Measure and Fractals

3.1 Some preliminaries about measures(LY 1.1)

Definition of topological space and Σ algebra

Definition of Borel set.

Definition of a measure over a space X

Definition of Measurable subset via Caratheodory approach

Definition of Borel measure

Definition of Borel regular measure

Definition of Radon measure

Weak convergence of Radon measures

$\mu_k \rightharpoonup \mu$ in the space M(U) of Radon measures on U if

$\int_U f d\mu_k \to \int_U f d\mu, \forall f \in C_0(U)$, continuous functions with compact support.

Theorem

$\mu_k \rightharpoonup \mu$ in the space $M(U)$ Then

$\limsup_{k\to\infty} \mu_k(C) \leq \mu(C)$

for all compact $C \subset U$

and

$\liminf_{k\to\infty} \mu_k(O) \leq \mu(O)$

for all open $O \subset C$

Note the process of approximating compact sets from within and open sets from outside commutes with weak convergence of measures.

3.2 Hausdorff measure (M ch.2 & LY p.6)

Definition of diameter of set

Definition of Hausdorff measure for integer dimensions

Comparison of set diameter with diameter of smallest ball containing set. e.g. triangle. Besicovitch example of a set whose spherical based measure is greater than its Hausdorff measure. (LY p 6) see section below

Diameter of triangle = 2, $DC = \frac{1}{\sqrt{3}}$, $AC = \frac{4}{\sqrt{3}} > 2$.

Non integer values of Hausdorff measure. Function to give volume coefficient of non-integer dimension unit ball. (Morgan ch 2)

3.3 Fractal dimension

In general when we scale all linear distances by an integer such as 2, we get 2^d copies of the original set, where d is the set dimension.

$copies = (linear\ scale)^d$

$d = \frac{\log(copies)}{\log(linear\ scale)}$

Example Sierpinski sponge Morgan ch 2,

scale factor =3, number of copies = 27 -6-1=20

dimension = $\frac{\log(20)}{\log(3)}$

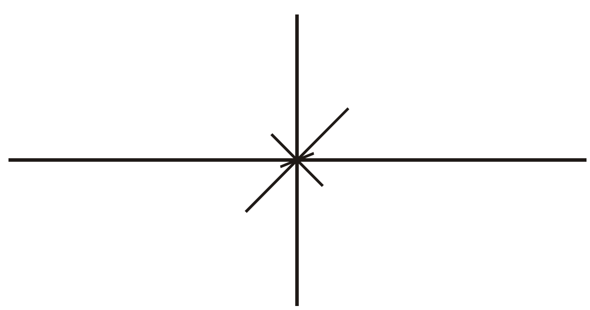
3.4 Hausdorff Measure related theorems (Morgan ch 2, LY p17)

Density definition

$$\Theta^m(E,a) = \lim_{\rho\to 0} \frac{\mu(E \cap B_{(a,\rho)})}{\alpha_m \rho^m}$$

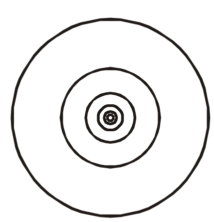
does this density exist everywhere? well limit can go to infinity on sequence of crossing line segments (finite mass), $sin(1/n)$(infinite mass) more pathological

examples below



$\Theta^1 = \infty. \qquad H^1(E) = \Sigma \frac{1}{2^n}$

A finite mass rectifiable set with no density at one point.



$$E = \bigcup_{i=1}^{\infty} \partial(B(0, \tfrac{1}{2^i}))$$
$$\Theta^1 \text{does not exist } H^1(E) = \Sigma \tfrac{2\pi}{2^n}$$
$$H^1(E \cap B(0, \varepsilon)) = \tfrac{\varepsilon}{2^n} \Sigma \tfrac{2\pi}{2^n}, \qquad \text{for} \qquad \tfrac{1}{2^n} \leq \varepsilon \leq \tfrac{1}{2^{n-1}}, \qquad n \in \mathbb{N}$$

For certain purposes it might worth defining density as a liminf or a limsup check Simon for upper and lower density.

$$\Theta^{*m}(E, a) = \limsup_{\rho \to 0} \frac{\mu(E \cap B_{(a,\rho)})}{\alpha_m \rho^m}$$

$$\Theta_*^m(E, a) = \liminf_{\rho \to 0} \frac{\mu(E \cap B_{(a,\rho)})}{\alpha_m \rho^m}$$

If there is an approximate tangent cone then density is defined because it is homothety invariant. see below

relationship of density of a cone to mass of link

Approximate limits of functions.

$E = (Ball \cup Rectangle)$

$f = \chi(E)$ Approximate limit of f at intersection, a, is 1, as $\Theta^2(E^c, a) = 0$

characteristic function of $\{1/n\}$ on the real line

4 Rectifiable sets (Nov 2nd)

We must first show that measurability of sets is not a trivial thing.

4.1 Existence of non-measurable sets.

The quotient of $\mathbb{R}/\mathbb{Q}$ is E. Let $\mathbb{Q} = \cup_{i=1}^{\infty} q_i$.

Now $\mathbb{R} = \cup_{i=1}^{\infty}(q_i + E)$. Where E is a representative set of numbers and $q_i + E$ is E translated by q_i.

Assume E is measurable.

The measure of E must be greater than zero, otherwise the measure of $\mathbb{R}$ would be zero as a countable union of sets of zero measure.

Consider $\mathbb{Q} \cap [0, 1] = \cup_{i=1}^{\infty} p_i$ therefore $I = \cup_{i=1}^{\infty}(p_i + E)$ has bounded measure, which is the measure of the unit interval. But if E has measure greater than zero, then the measure of I will be infinite, as a countably infinite sum of sets of fixed positive measure. Therefore E cannot be measurable.

This means any set contains a non-measurable subset, and we can construct non measurable sets for sets of any integer dimension (by taking products of E with intervals).

4.2 Definitions of rectifiable sets (Morgan ch 3, LY ch 3)

4.3 Structure theorem (Morgan ch 3, LY ch 3)

Integral geometric measure.
compute H_1 of set constructed
unrectifiable sets can hide mass without shadows.

The self similar set given by replacing a square by four squares within it of 1/4 linear size touching the vertices of the original square, then iterating,.gives a purely unrectifiable set with zero integral-geometric measure and Hausdorff dimension 1.

If we do the same construction with a triangle, replacing a triangle by three 1/3 edge length triangles, then we get 1 dim Hausdorff measure 1 with spherical one dimensional measure $2/\sqrt{3}$ (see diagrm above both set diameter of triangle vs. diameter of smallest ball containing the triangle.

Density and tangent cones (Morgan ch 3, LY ch 3)

zig zag staircase /concentric rings. at $1/2^n$ see that mass with intersection of unit ball is saw tooth function of r. These were rectifiable sets.

5 November 9th through FebruarySections from books, Current theory

Frank Morgan chapters 4, 5 and 9
Lin Chapters 4 and 7

6 March: Varifold Theory

Motivation and introduction Frank Morgan ch 11
Lin ch 6

6.1 Additional notes on varifolds

6.1.1 Why is the measure μ on the varifold based upon Hausdorff measure in $\mathbb{R}^n$instead of in $\mathbb{R}^n \times G(k,n)$?

There are things that might go wrong if we base the measure μ on $\mathbb{R}^n \times G(k,n)$ on Hausdorff measure on $\mathbb{R}^n \times G(k,n)$.

1) the lift of a rectifiable set need not be rectifiable, and so the measurability of sets in $\mathbb{R}^n \times G(k,n)$ cannot be based on them.

2) rectifiable sets in $\mathbb{R}^n$ are equivalence classes modulo sets of zero measure in $\mathbb{R}^n$. If positive measure or mass is associated with sets of zero measure in $\mathbb{R}^n$, then we can no longer work with equivalence classes. To make the lift rectifiable, the set in $\mathbb{R}^n$ needs to be C^2 rectifiable, that is a set of measure zero union a countable number of sets contained in C^2 embedded submanifolds.

6.1.2 How do varifolds deal with cone points?

Cone points or sets on k-varifolds have dimension k-1 or less. As sets of measure zero they have no significance in $\mathbb{R}^n$. However if we wish to consider measures of first, second or higher order variations lower dimensional cone points contribute to that variation.

First variation involves codimension 1 sets which can be interpreted as boundary.

Second variation can involve codimension 2 sets.....and so on.

Polyhedral examples make this clear.

6.1.3 What is $Div_s X$? (Lin p 163)

S is a smooth submanifold of $\mathbb{R}^n$, and X is a vector field on $\mathbb{R}^n$.

This is the contribution to divergence given by the components of the vector field X in the directions of the tangent space TM_S.

Why is this the correct way to approach first variation of S under the action of S?

Let's consider two components of X, parallel to S and perpendicular to S. Now if S curves, the perpendicular component does contribute to $Div_S X$.

Example, a radial vector field on a sphere

$X(x) = x, S = \{x : |x| = R\}$. The vector field is perpendicular to S

Let's calculate $div_S(X)$ for $n = 2$

$X(x, y) = x\overrightarrow{i} + y\overrightarrow{j}$

$TM(0, R) = t\overrightarrow{i}$

$Div_S(X(0, R)) = \frac{\partial}{\partial x} Xi = 1$

Warning! Equation 6.2.1 on Lin p 163. See next example where we have problems interpreting this equation in non-standard situations where the measure of the varifold is not just the lift to $\mathbb{R}^n \times Gl(k, n)$ of the tangent space of the manifold in $\mathbb{R}^n$.

6.2 A pathological sequence of general varifolds and first variation.

Version 1 of this example is commonplace in the current literature (e.g. Normal currents; [Morgan] p 40, and example S_2 figure 4.5.1 p 48).

We can take a series of stationary 1 varifolds $V_n = (\{(x, y) : -1 \leq x \leq 1, y = \frac{a}{n}, -1 \leq y \leq 1, a \in \mathbb{Z}\}$ endowed with density $= \frac{1}{n})$. The limit varifold will have a first variation term on the set $\{(x, y); -1 \leq x \leq 1, y = \pm 1\}$. This will only appear in the limit and will cause the first variation to jump up in the limit.

Geometrically this corresponds to the line segments in the sequence which are stationary on their interiors becoming the boundary of a set with 2 dimensional Hausdorff measure. This causes a new boundary term of first variation to appear in the limit on these line segments. As 2 dimensional Hausdorff measure is a Radon measure on $\mathbb{R}^2$, the limit object is still a general 1-varifold. So the problem here for the Equation 6.2.1 on Lin p 163 is that the underlying manifold in $\mathbb{R}^n$ is 2-dimensional, but the fiber bundle is that of a 1-dimensional general varifold.

Version 2 of this example was developed with Joao Boavida. In polar coordinates take the sets $\{(r, \theta) : 0 \leq r \leq 1, \theta = \frac{2\pi a}{n}, 0 \leq \theta \leq 2\pi, a \in \mathbb{Z}\}$ endowed with density $= \frac{1}{n}$. In the limit we have the unit disc with density $= \frac{1}{2\pi r}$ with the density concentrated on a section of the bundle, at the $\pm\theta$ position in each fiber. Note that each 1-varifold is stationary on the interior of the disc, but in the limit the interior of the disc is not stationary. Under a radial vector field away from the origin, the mass of the varifold will increase linearly. This is because the density decreases as radius increases.

6.3 Geometric implications of the monotonicity formula

Monotonicity concerns the intersection of a varifold with a Ball radius r in the ambient space. If we rescale every such intersection to make the ball radius 1, we get a 1-parameter family of varifolds in the unit ball. As r increases the mass of this rescaled varifold cannot decrease. In the general case this means that as r increases the varifold cannot look more and more like an affine subspace. In specific cases it may not look more and more like a cone of lower mass as r increases.

Example 1 varifolds. You cannot have a stationary integer multiplicity varifold with three lines coming together outside a region in a ball, and have any point with more than three rays coming into it inside the ball.

Note this relates to the isoperimetric inequality.