The Haar measure on $SU(2)$

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Abstract

We give a way to find the Haar measure on $SU(2)$

1 Introduction

From some abstract mathematics we know that one a compact Lie group G there exists an up to scaling unique left-invariant integration measure $d\mu$:

$$\int_G f(gh)d\mu(h) = \int_G f(h)d\mu(h)\,. \tag{1}$$

The measure $d\mu$ is called the Haar measure.

Knowing existence is of course nice, but in practice (as in physics and chemistry) one needs a way to construct the Haar measure. For $SU(2)$ there are enough nice properties that make it tractable to find the Haar measure without too much effort. We will use a number of facts:

- We assume the reader knows what the Lie group $SU(2)$ is; for example see the text "Notes on SU(2) and SO(3)" on my website `http://www.mat.univie.at.at/ westra/physmath.html`.
- Any element g of $SU(2)$ can be written as

$$g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}\,, \quad |a|^2 + |b|^2 = 1. \tag{2}$$

  In this way the underlying manifold of $SU(2)$ is the three-sphere S^3. See the same text of my website for a derivation.
- $SU(2)$ has a natural representation on $\mathbb{C}^2$ where it leaves the inner product $\langle z, w\rangle = \bar{z}_1 w_1 + \bar{z}_2 w_2$ invariant. Note that we write $z, w, \ldots$ for elements of $\mathbb{C}^2$

2 Representations on homogeneous polynomials

Consider the linear complex vector space V_j of dimension $j+1$ of polynomials of homogeneous degree j in the coordinates z_1 and z_2. We get a representation of $SU(2)$ on V_j by defining the action

$$(U_g p)(z) = p(g^{-1}z)\,, \quad p \in V_j\,, g \in SU(2)\,. \tag{3}$$

We thus have a map $SU(2) \to GL(j, \mathbb{C})$ given by $g \mapsto U_g$. It is easy to see that $U_{gh} = U_g U_h$ and $U_e = \mathrm{id}_{V_j}$.

We equip V_j with the following inner product: Let D be the solid three-sphere embedded in $\mathbb{C}^2$

$$D = \left\{ z \in \mathbb{C}^2 \mid \langle z, z\rangle \leq 1 \right\}\,. \tag{4}$$

Then for $p, q \in V_j$ we define

$$(p, q) = \int_D \mathcal{D}z\, \overline{p(z)}\, q(z)\,, \tag{5}$$

where $\mathcal{D}z = dz_1 d\bar{z}_2 dz_2 d\bar{z}_2$. The integration measure $\mathcal{D}z$ satisfies

$$\int_D \mathcal{D}z = 4\pi^2 \int_{x^2+y^2\geq 1} xy\ dxdy = \frac{\pi^2}{2}\,. \tag{6}$$

For further reference it is useful to note the following

$$\begin{aligned}\int_{x^2+y^2\geq 1} x^{2n+1}y^{2m+1}\ dxdy &= \tfrac{1}{4}\int_{0\geq s+t\geq 1} s^n t^m\ dsdt \\ &= \tfrac{1}{4}\int_0^1 s^n(1-s)^m\ ds = \tfrac{1}{4}\frac{n!m!}{(n+m+1)!}\,.\end{aligned} \tag{7}$$

We then find

$$\int_D \mathcal{D}z \bar{z}_1^a \bar{z}_2^b z_1^c z_2^d = \pi^2 \frac{a!b!}{(a+b+1)!}\delta_{ac}\delta_{db}\,, \tag{8}$$

so that the monomials $z_1^a z_2^{j-a}$ form a basis for V_j where $0\leq a\leq j$. We can thus define $p_a^j = z_1^a z_2^{a-j}$ and find

$$(p_a^j, p_b^j) = \frac{\pi^2}{j+1}\begin{pmatrix} j \\ a\end{pmatrix}^{-1}\delta_{ab}\,. \tag{9}$$

Although this representation is nice already in itself, it is not the object of study. The reason we expanded was to give the reader some preparation of how to think of what follows.

The measure $\mathcal{D}z$ is invariant under $SU(2)$-transformations. Indeed, under $z\mapsto g\cdot z$ we have $dz_1\ dz_2 \mapsto \det g dz_1\ dz_2 = dz_1\ dz_2$.

The group $SU(2)$ is inside $\mathbb{C}^2$ as the subset $S^3 = \{z\in\mathbb{C}^2 \mid \langle z,z\rangle = 1\}$, which is just the boundary of D. In mathematical words: $SU(2) = \partial D$. In $\mathbb{C}^2$ we can use spherical coordinates r,θ,φ,χ and then $SU(2)$ corresponds to the subset wherre $r=1$. Given a smooth function, we can restrict it to a smooth function on $SU(2)$. In fact, we may parameterize $z_1 = r\cos\theta e^{i\varphi}$, $z_2 = r\sin\theta e^{i\chi}$ with $0\leq\theta\leq\pi$ and $0\leq\varphi,\chi\leq 2\pi$. Then the polynomials p_a^j restrict to the functions P_a^j on $SU(2)$ defined by

$$P_a^j(\theta,\varphi,\chi) = (\cos\theta)^a(\sin\theta)^{j-a}e^{ia\varphi}e^{ia\chi}\,. \tag{10}$$

Hence for each j we get a class of functions on $SU(2)$; the restriction of V_j to $SU(2)$. In fact, it is a theorem of Hermann Weyl and his student Peter that says that every integrable function on can be approximated by the functions P_a^j, where integrable is with respect to the Haar measure. Now, ... what can this Haar measure be???

3 The Haar measure

The functions on $SU(2)$ can be extended to functions on D by decomposing a function into its V_j components and then multiplying with the radial coordinate r. For functions on D we have an $SU(2)$-invariant measure. Hence we obtain an invariant measure on $SU(2)$ by restricting the one on D to its boundary. Note that an $SU(2)$ transformation only changes the angles in a function and not its r-dependence. Hence when we split off the radial part of the measure

$$\mathcal{D}z = rdr\mu(\theta,\varphi,\chi)d\theta d\varphi d\chi\,, \tag{11}$$

we have that

$$\int_0^1 r^3 dr \int_{S^3}\mu(\theta,\varphi,\chi)d\theta d\varphi d\chi\ f(r,\theta,\varphi,\chi)\,, \tag{12}$$

is invariant under rotating the angles by means of an $SU(2)$-transformation. The $SU(2)$-invariance means

$$\int_D \mathcal{D}z(U_g f)(z_1,z_2) = \int_D \mathcal{D}z f(z_1,z_2)\,. \tag{13}$$

Since $SU(2)$ only acts on the radial part of the functions, we must have that

$$\int_{S^3}\mu(\theta,\varphi,\chi)d\theta d\varphi d\chi\ f(r,\theta,\varphi,\chi) \tag{14}$$

is invariant under $SU(2)$-transformations for each r. Hence also for $r = 1$, which gives back the original function f on $SU(2)$.

We find $\mu(\theta, \varphi, \chi)$ by writing out

$$dz_1 d\bar{z}_1 dz_2 d\bar{z}_2 = r^3 |\det J| \; d\theta d\varphi d\chi \,, \tag{15}$$

where

$$r^3 J = \begin{pmatrix} \frac{\partial z_1}{\partial r} & \frac{\partial z_1}{\partial \theta} & \frac{\partial z_1}{\partial \varphi} & \frac{\partial z_1}{\partial \chi} \\ \frac{\partial \bar{z}_1}{\partial r} & \frac{\partial \bar{z}_1}{\partial \theta} & \frac{\partial \bar{z}_1}{\partial \varphi} & \frac{\partial \bar{z}_1}{\partial \chi} \\ \frac{\partial z_2}{\partial r} & \frac{\partial z_2}{\partial \theta} & \frac{\partial z_2}{\partial \varphi} & \frac{\partial z_2}{\partial \chi} \\ \frac{\partial \bar{z}_2}{\partial r} & \frac{\partial \bar{z}_2}{\partial \theta} & \frac{\partial \bar{z}_2}{\partial \varphi} & \frac{\partial \bar{z}_2}{\partial \chi} \end{pmatrix} \tag{16}$$

A small calculation reveals

$$\mu(\theta, \varphi, \chi) d\theta d\varphi d\chi = \sin 2\theta \; d\theta d\varphi d\chi \,. \tag{17}$$