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Minimal Surfaces of the 3-Dimensional Lorentz-Heisenberg Space

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Abstract

In this paper we study some geometric properties of the three-dimensional Heisenberg space H_3 endowed with a left-invariant Lorentzian metric. We write the equation of mininimal surfaces in H_3 and we show that the plane, helicoid and hyperbolic paraboloid and other surfaces are defined by elliptic integrals verifying the equation of minimal surfaces in H_3.

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1. Introduction

It is show in [11] and [12] that modulo an automorphism of the Lie algebra of the Heisenberg group there exist three classes of invariant lorentzian metrics on the Heisenberg group one of which is flat.

The Lorentzian Heisenberg space H_3 can be seen as the space $\mathbb{R}^3$ endowed with a left-invariant Lotrentzian metric g_ξ given by

$$g_\xi = dx^2 + dy^2 - (dz + \xi(ydx - xdy))^2, \quad \xi \in \mathbb{R}$$

This metric is invariant under the transformation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ A & B & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Where θ, a, b and c are real numbers and

$$\begin{aligned} A &= \xi(a\sin\theta - b\cos\theta), \\ B &= \xi(a\cos\theta + b\sin\theta). \end{aligned}$$

In the [1], M. Bekkar studied the minimal surfaces in the Riemannian-Heisenberg space $\mathbb{H}_3$. In particular, in [2], M. Bekkar and T. Sari give a classification of all these minimal surfaces ruled by lines in the space $\mathbb{H}_3$.

On the other hand, in [13], I. Van De Woestijne gave the equation of minimal surfaces in the three-dimensional Minkowski space $\mathbb{R}^3_1$ and he showed that the plane, the catenoid, the helicoid and the surface of Enneper are minimal in $\mathbb{R}^3_1$.

It should be remarked that L. McNertney completely classified timelike minimal sufaces in Minkowski space $\mathbb{R}^3_1$ in her Ph.D. thesis [8] and R. Lopez studied timelike minimal surfaces and timelike sufaces with constant mean curvature that are foliated by circles in [7]. In particular, he showed that if a timelike surface with non-zero constant mean curvature is foliated by circles in parallel planes, it must be rotational.

In order to know better the Lorentz-Heisenberg space H_3, we give some geometric properties and we write the equation of minimal sufaces in this space for a graph surface $z = f(x, y)$

$$\left(1 - (f_y - \xi x)^2\right) f_{xx} + (1 - (f_x + \xi y)^2) f_{yy} + 2(f_x + \xi y)(f_y - \xi x) f_{xy} = 0.$$

and we give somme paticular solutions.

2. Preliminaries

Let H_3 be the Lorentz-Heisenberg group endowed with a left-invariant lorentzian metric g_ξ given by

$$g_\xi = dx^2 + dy^2 - (dz + \xi(ydx - xdy))^2, \quad \xi \in \mathbb{R}$$

We recall that the product of H_3 is given by

$$(\overline{x}, \overline{y}, \overline{z})(x, y, z) = (\overline{x} + x, \overline{y} + y, \overline{z} + z - \overline{x}y + x\overline{y}).$$

H_3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lie algebra of H_3 has an orthonormal basis $\varepsilon = (e_1, e_2, e_3)$ defined by

$$e_1 = \frac{\partial}{\partial x} - \xi y \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + \xi x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}$$

One can easily check that ε satisfies $g_\xi(e_i, e_j) = \delta_{ij}$. Here δ_{ij} denotes the kronecker's symbol

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

We have the Lie products

$$[e_1, e_2] = 2\xi e_3, \quad [e_1, e_3] = [e_2, e_3] = 0$$

with

$$g_\xi(e_1, e_1) = g_\xi(e_2, e_2) = 1, \quad g_\xi(e_3, e_3) = -1$$

The Levi-Civita connection ∇ of g_ξ is explicitly given as follows

$$\begin{cases} \nabla_{e_1} e_1 = 0, \ \ \nabla_{e_2} e_1 = -\xi e_3, \ \nabla_{e_3} e_1 = \xi e_2 \\ \nabla_{e_1} e_2 = \xi e_3, \ \nabla_{e_2} e_2 = 0, \ \nabla_{e_3} e_2 = -\xi e_1 \\ \nabla_{e_1} e_3 = \xi e_2, \ \nabla_{e_2} e_3 = -\xi e_1, \ \nabla_{e_3} e_3 = 0 \end{cases}$$

The dual coframe field $\theta = (\theta^1, \theta^2, \theta^3)$ associated to $\varepsilon = (e_1, e_2, e_3)$ is a triplet of 1-forms which satisfies the condition $\theta^i(e_j) = \delta_{ij}$. This coframe field is given by

$$\theta^1 = dx, \quad \theta^2 = dy, \quad \theta^3 = dz + \xi(ydx - xdy)$$

Note that the 1-form θ^3 is a contact form on H_3; $d\theta^3 \wedge \theta^3 \neq 0$ if and only if $\xi \neq 0$.

Further the connection forms are determined by

$$g_\xi(\nabla_X e_i, e_j) = \theta^{ij}(X),$$

we obtain then

$$\theta^{13} = -\xi\theta^2, \quad \theta^{12} = -\xi\theta^3, \quad \theta^{23} = \xi\theta^1.$$

In the other hand the Ricci tensor is defined by

$$Ricc(X,Y) = \sum_{i=1}^{3} \epsilon_i g_\xi(R(X,e_i)Y,e_i),$$

where X, Y are vectors fields on H_3 and $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = -1$.

The Ricci tensor components are

$$R_{11} = R_{22} = -2\xi^2, \quad R_{33} = 2\xi^2, \text{ and } R_{ij} = 0 \text{ for } i \neq j$$

The scalar curvature $k = \sum_{i=1}^{3} R_{ii}$ is $k = -2\xi^2$.

3. Minimal surface equation

Let S be an immersed surface in H_3 which is given as the graph of the function $z = f(x,y)$. The position vector $\mathbf{X}(x,y)$ of S is expressed as a vector valued function $\mathbf{X}(x,y) = (x,y,f(x,y))$.

The tangent vector $\mathbf{X}_x = \frac{\partial X}{\partial x}$ and $\mathbf{X}_y = \frac{\partial X}{\partial y}$ are described by

$$\begin{cases} \mathbf{X}_x(x,y) = \frac{\partial}{\partial x} + f_x\frac{\partial}{\partial z} = e_1 + Pe_3 \\ \mathbf{X}_x(x,y) = \frac{\partial}{\partial y} + f_y\frac{\partial}{\partial z} = e_2 + Qe_3 \end{cases}$$

in terms of the orthonormal frame ε. Here the functions P and Q are defined by

$$P = f_x + \xi y, \quad Q = f_y - \xi x.$$

The first fundamental form I of S is defined by

$$I = Edx^2 + 2Fdxdy + Gdy^2,$$

where

$$E = g_\xi(\mathbf{X}_x, \mathbf{X}_x), \quad F = g_\xi(\mathbf{X}_x, \mathbf{X}_y) \text{ and } G = g_\xi(\mathbf{X}_y, \mathbf{X}_y).$$

The coefficient functions E, F and G aregiven by

$$E = 1 - P^2, \quad F = -PQ \text{ and } G = 1 - Q^2$$

Take a unit vector field $\mathbf{N}$ normal to S. Namely $\mathbf{N}$ is vector field along S which satisfies

$$g_\xi(\mathbf{X}_x, \mathbf{N}) = g_\xi(\mathbf{X}_y, \mathbf{N}) = 0 \text{ and } g_\xi(\mathbf{N}, \mathbf{N}) = 1.$$

The *second fundamental form* II derived from $\mathbf{N}$ is defined by

$$II = Ldx^2 + 2Mdxdy + Ndy^2,$$

where

$$L = g_\xi(\nabla_{\mathbf{X}_x}\mathbf{X}_x, \mathbf{N}), \quad M = g_\xi(\nabla_{\mathbf{X}_y}\mathbf{X}_x, \mathbf{N}) \text{ and } L = g_\xi(\nabla_{\mathbf{X}_y}\mathbf{X}_y, \mathbf{N}).$$

Since S is a graph of function f, we can choose a unit normal vector field $\mathbf{N}$ as

$$\mathbf{N} = \frac{Pe_1 + Qe_2 + e_3}{W}, \quad W = \sqrt{P^2 + Q^2 - 1}.$$

The second fundamental form derived from this unit normal vector field is given by

$$L = \frac{1}{W}(f_{xx} + 2\xi PQ), \quad M = \frac{1}{W}(f_{xy} - \xi P^2 + \xi Q^2) \text{ and } N = \frac{1}{W}(f_{yy} - 2\xi PQ).$$

Let us denote the following matrix-valued functions associated to I and II by the same letters I and II, respectively;

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad II = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

The solutions λ_1 and λ_2 to the characteristic equation $\det(II - \lambda I) = 0$ are called *principal curvatures* of S. Recall that the average $\mathbf{H} = (k_1 + k_2)/2$ of k_1 and k_2 is called the *mean curvature* of S. The mean curvature $\mathbf{H}$ is computed by the formula

$$\mathbf{H} = \frac{EN + GL - EFM}{2\,|EG - F^2|}.$$

A surface S: $z = f(x, y)$ is said to be *minimal* if $\mathbf{H} = 0$.

On the other hand, the Gaussian curvature $\mathbf{K}_G$ of S is given by the formula

$$\mathbf{K}_G = g_\xi(\mathbf{N}, \mathbf{N})\frac{\det\ II}{\det\ I}$$

A surface S: $z = f(x, y)$ is said to be *flat* if $\mathbf{K}_G = 0$.

The differential equation $\mathbf{H} = 0$ for a surface S defined as a graph $(x, y; f(x, y))$ is called the *minimal surface equation* in H_3. The minimal surface equation is given explicitly by

$$(E_\xi) \qquad \left(1-Q^2\right) f_{xx} + (1-P^2) f_{yy} + 2PQ f_{xy} = 0.$$

Clearly, if $\xi = 0$, this equation reduces to the minimal surface equation of the Minkowski space $\mathbb{R}^3_1$, [13].

The minimal surface equation (E_ξ) can be rewritten as the following *divergence form:*

$$\frac{\partial}{\partial x}(\frac{P}{W}) + \frac{\partial}{\partial y}(\frac{Q}{W}) = 0.$$

4. Minimal surfaces in Lorentz-Heisenberg space H_3

Minimal surfaces theory in Euclidian 3-space E^3 started with constructing and classifying funamental examples of minimal surfaces: minimal surfaces of revolution, ruled minimal surfaces, or translation minimal surfaces ect. (For more informatins about the minimal surface theory in E^3, we refer to Nitsche's book [6] and Osserman's book [10]). On the other hand in [13] I. Van De Woestijne studied and classified the minimal surfaces of the 3-dimensional Lorentz-Minkowski space $\mathbb{R}^3_1$.

In this section we study elementary and fundamental examples of minimal surfaces of the 3-dimensional Lorentz-Heisenberg space H_3.

It is easy to observe that in H_3, the linear function $f(x,y) = ax + by + c$ is solution to the minimal surface equation (E_ξ).

Proposition 1 *Let H_3 be a 3-dimensional Lorentz-Heisenberg space. Then the spacelike surface $z = f(x,y) = ax + by + c$ is minimal for arbitrary a, b and c.*

By analogy as in Heisenberg space $\mathbb{H}_3$, where the hyperbolic paraboloid is an minimal surface [1], we have in H_3 also the particular minimal surface $z = f(x,y) = \pm\xi xy$, $\xi \in \mathbb{R}$.

Theorem 2 *In H_3, the surface $z = f(x,y) = \pm\xi xy$, $\xi \in \mathbb{R}$, are minimal in H_3*

4.1. Helicoids in H_3

Euclidean helicoids can be characterised as minimal surface in E^3 which is a graph of a function of the form $f(x,y) = g(\frac{x}{y})$. On the other hand, the minimal spacelike helicoids of $\mathbb{R}^3_1$ were examined in [5] and the minimal Lorentzian helicoids of $\mathbb{R}^3_1$ were examined in [woestijne]. In this subsection we look for minimal surfaces determined by solution $f(x,y) = g(\frac{x}{y})$ to the minimal surface equation in H_3.

Let $\mathbb{S}$ be a surface which is graph of a function in the form $f(x,y) = g(\dfrac{y}{x})$. Put $u = \dfrac{y}{x}$ for $x \neq 0$. Then we have

$$\begin{aligned} f_x &= -\frac{y}{x^2}g', \quad f_y = \frac{1}{x}g', \\ f_{xx} &= \frac{2y}{x^3}g' + \frac{y^2}{x^4}g'', \; f_{xy} = -\frac{1}{x^2}g' - \frac{y}{x^3}g'', \quad f_{yy} = \frac{1}{x^2}g'' \end{aligned}$$

Here g' et g'' are the derivatives with respect to u.

Now we insert these data to the minimal surface equation (E_ξ), then we have the differential equation

$$(1+u^2)g'' + 2ug' = 0.$$

One can check easily that the general solution to this O. D. E. is given explicitly by $f(x,y) = g(\dfrac{y}{x}) = \alpha \tan^{-1}(\dfrac{y}{x}) + \beta$; where $\alpha, \beta \in \mathbb{R}$.

Theorem 3 *The only minimal surface in H_3 which has the form $z = f(x,y) = g(\dfrac{y}{x})$ are the surfaces $f(x,y) = \alpha \tan^{-1}(\dfrac{y}{x}) + \beta$, $\alpha, \beta \in \mathbb{R}$.*

4.2. Axially symmetric minimal surfaces

It is easy to see the metrics g_ξ are invariant under rotations about the z-axis and translation along the same axis. Based on this fundamental property, in this subsection, we will be studying axially symmetric minimal graphs in H_3.

A surface $S : z = f(x,y)$ is said to be *axially symmetric* if f depends only on $r = \sqrt{x^2+y^2}$.

Now let S be an axially symmetric graph of function $f(x,y) = T(r)$. Then we have

$$\begin{aligned} f_x &= \frac{x}{r}T', \; f_y = \frac{y}{r}T', \; f_{xx} = \frac{y^2}{r^3}T' + \frac{x^2}{r^2}T'', \\ f_{yy} &= \frac{x^2}{r^3}T' + \frac{y^2}{r^2}T'', \; f_{xy} = \frac{-xy}{r^3}T' + \frac{xy}{r^2}T'' \end{aligned}$$

Here T', T'' are the derivatives with respect to r. From these, we get the following minimal surface equation:

$$r(1-\xi^2 r^2)T'' + T'(1-T'^2) = 0.$$

To solve this equation we put $T' = u$. Since $r(1-\xi^2 r^2) \neq 0$, then we have

$$u' + \frac{u}{r(1-\xi^2 r^2)} = \frac{u^3}{r(1-\xi^2 r^2)}.$$

This is a Bernoulli's equation. Now we put $v = \frac{1}{u^2}$ then the preceding equation is rewritten as:

$$v' - \frac{2u}{r(1-\xi^2 r^2)} = \frac{-2}{r(1-\xi^2 r^2)}.$$

General solutions to this equation are given by

$$v = \frac{1+cr^2}{1-\xi^2 r^2}, \qquad c > 0.$$

Hence

$$(T')^2 = \frac{1-\xi^2 r^2}{1+cr^2} = \frac{\xi^2}{c} \cdot \frac{1/\xi^2 - r^2}{1/c + r^2}, \quad c > 0.$$

To solve this equation, we need separate discussions according to the values of ξ. Our general reference on the elliptic integrals is [4].

(1) $\xi = 0$: In this case we have $(T')^2 = \frac{1}{1+cr^2}$, $c > 0$. The solution is the axially symmetric surface

$$T(r) = \frac{1}{\sqrt{c}} \cosh^{-1} \sqrt{c} r + c_1.$$

Hence the surface is a catenoid in Minkowski space.

(2) $\xi \neq 0$: The solution is the axially symmetric surface

$$T(r) = \frac{|\xi|}{\sqrt{c}} \int_r^{\frac{1}{|\xi|}} \sqrt{\frac{1/\xi^2 - t^2}{1/c + t^2}} dt + c_2$$

This elliptic integral can be expressed by the elliptic integrals in Legendre form of first and second kind. In fact, the integral

$$I(t) = \int_b^t \sqrt{\frac{b^2 - x^2}{a^2 + x^2}} dx$$

is represented as

$$I(u) = \sqrt{a^2+b^2}(F(\gamma, s) - E(\gamma, s)) + t\sqrt{\frac{b^2 - t^2}{a^2 + t^2}}$$

with $b \geq t > 0$. Here $F(\gamma, s)$ and $E(\gamma, s)$ are the elliptic integrals in Legendre form of first and second kind, respectively;

$$F(\gamma, s) = \int_0^{\gamma} \left(1 - s^2 \sin^2 \alpha\right)^{-\frac{1}{2}} d\alpha, \quad E(\gamma, s) = \int_0^{\gamma} \left(1 - s^2 \sin^2 \alpha\right)^{\frac{1}{2}} d\alpha.$$

The modulus γ and the variable s are given by $\gamma = \sin^{-1}(\frac{t}{b})\sqrt{\frac{b^2+a^2}{a^2+t^2}}$, $s = \frac{b}{\sqrt{a^2+b^2}}$.

Theorem 4 *The only axially minimal surfaces in H_3 are graphs of functions $f(x, y) = T(r)$ with $r^2 = x^2 + y^2$, where*

(1) $T(r) = \frac{1}{\sqrt{c}} \cosh^{-1} \sqrt{c}r + c_1$, $c > 0$ *for* $\xi = 0$.

(2) $T(r) = \frac{|\xi|}{\sqrt{c}} \int_r^{\frac{1}{|\xi|}} \sqrt{\frac{1/\xi^2 - t^2}{1/c + t^2}} dt + c_2$, $c > 0$ *for* $\xi \neq 0$.

4.3. Minimal translation surfaces

A similar approach to that of finding Sherch's sufaces. We set $f(x, y) = u(x) + v(y) - \xi xy$ and we substitute in equation (E_ξ). We obtain the following equation

$$\left(1 - (v' - 2\xi x)^2\right) u'' + (1 - u'^2)v'' - 2\xi u'(v' - 2\xi x) = 0.$$

This equation becomes when taking $v(y) = cte$

$$\left(1 - 4\xi^2 x^2\right) u'' + 4\xi^2 x u' = 0$$

Integrating gives us

$$u(x) = \frac{c_1}{4\xi}(\arcsin 2\xi x + 2\xi x\sqrt{1 - 4\xi^2 x^2}) + c_2.$$

The surfaces of equations

$$f(x, y) = \frac{c_1}{4\xi}(\arcsin 2\xi x + 2\xi x\sqrt{1 - 4\xi^2 x^2}) - \xi xy + c_2.$$

are minimal in H_3.

Theorem 5 *In H_3, the translation surfaces $f(x, y) = \frac{c_1}{4\xi}(\arcsin 2\xi x + 2\xi x\sqrt{1 - 4\xi^2 x^2}) - \xi xy + c_2$, c_1, c_2 and $\xi \in \mathbb{R}$, are minimal .*

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