

Cosmological Spacetimes with $\Lambda > 0$

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1 Introduction

This paper is based on a talk given by the author at the BEEMFEST, held at the University of Missouri, Columbia, in May, 2003. It was indeed a great pleasure and privilege to participate in this meeting honoring John Beem, whose many outstanding contributions to mathematics, and leadership in the field of Lorentzian geometry has enriched so many of us.

In this paper we present some results, based on joint work with Lars Andersson [2], concerning certain global properties of asymptotically de Sitter spacetimes satisfying the Einstein equations with positive cosmological constant. There has been increased interest in such spacetimes in recent years due, firstly, to observations concerning the rate of expansion of the universe, suggesting the presence of a positive cosmological constant in our universe, and, secondly, due to recent efforts to understand quantum gravity on de Sitter space via, for example, some de Sitter space version of the AdS/CFT correspondence; cf., [4] and references cited therein. In fact, the results discussed here were originally motivated by results of Witten and Yau [21, 5], and related spacetime results on topological censorship [11], pertaining to the AdS/CFT correspondence. These papers give illustrations of how the geometry and/or topology of the conformal infinity of an asymptotically hyperbolic Riemannian manifold, or an asymptotically anti-de Sitter spacetime, can influence the global structure of the manifold. In this vein, the results described here establish connections between the geometry/topology of conformal infinity and the occurrence of singularities in asymptotically de Sitter spacetimes.

2 Asymptotically de Sitter spacetimes

We use Penrose's notion of conformal infinity [19] to make precise the notion of "asymptotically de Sitter". Recall, this notion is based on the way in which the

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standard Lorentzian space forms, Minkowski space, de Sitter space and anti-de Sitter space, conformally embed into the Einstein static universe $(\mathbb{R} \times S^n, -du^2 + d\omega^2)$. For concreteness, throughout the paper we restrict attention to spacetimes (M^{n+1}, g) (of signature $(- + \cdots +)$), which are globally hyperbolic, with *compact* Cauchy surfaces. We shall refer to such spacetimes as **cosmological spacetimes**. We refer the reader to the core references [3, 15, 18] for basic results, terminology and notation in Lorentzian geometry and causal theory.

Recall, de Sitter space is the unique geodesically complete, simply connected spacetime of constant *positive* curvature ($= 1$ in equation (2.1) below). As such, it is a vacuum solution to the Einstein equations with positive cosmological constant (see the next section for further discussion of the Einstein equations). It can be represented as a hyperboloid of one-sheet in Minkowski space, and may be expressed in global coordinates as the warped product,

$$M = \mathbb{R} \times S^n, \quad ds^2 = -dt^2 + \cosh^2 t d\omega^2. \quad (2.1)$$

Under the transformation $u = \tan^{-1}(e^t) - \pi/4$, the metric (2.1) becomes

$$ds^2 = \frac{1}{\cos^2(2u)}(-du^2 + d\omega^2). \quad (2.2)$$

Thus, de Sitter space conformally embeds onto the region $-\pi/4 < u < \pi/4$ in the Einstein static universe; see Figure 1. Future conformal infinity \mathcal{J}^+ (resp., past conformal infinity \mathcal{J}^-) is represented by the *spacelike* slice $u = \pi/4$ (resp., $u = -\pi/4$). This situation serves to motivate the following definitions.

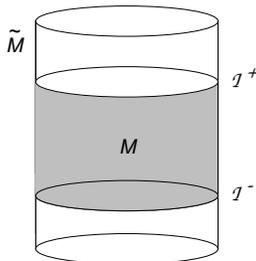


Figure 1. Conformal embedding of de Sitter space.

Definition 2.1 *A cosmological spacetime (M, g) is said to be **future asymptotically de Sitter** provided there exists a spacetime-with-boundary (\tilde{M}, \tilde{g}) and a smooth function Ω on \tilde{M} such that (a) M is the manifold interior of \tilde{M} , (b) the manifold boundary $\mathcal{J}^+ = \partial\tilde{M}$ ($\neq \emptyset$) of \tilde{M} is spacelike and lies to the future of M , $\mathcal{J}^+ \subset I^+(M, \tilde{M})$, and (c) \tilde{g} is conformal to g , i.e., $\tilde{g} = \Omega^2 g$, where $\Omega = 0$ and $d\Omega \neq 0$ along \mathcal{J}^+ .*

We refer to \mathcal{J}^+ as future (conformal) infinity. In general, no assumption is made about the topology of \mathcal{J}^+ ; in particular it need not be compact. The conformal factor Ω is also referred to as the *defining function* for \mathcal{J}^+ .

Definition 2.2 A future asymptotically de Sitter spacetime M is said to be **future asymptotically simple** provided every future inextendible null geodesic in M reaches future infinity, i.e. has an end point on \mathcal{J}^+ .

It is a basic fact [2] that a cosmological spacetime M is future asymptotically simple if and only if \mathcal{J}^+ is compact; in this case \mathcal{J}^+ is diffeomorphic to the Cauchy surfaces of M . **Past asymptotically de Sitter** and **past asymptotically simple** are defined in a time-dual manner. M is said to be asymptotically de Sitter provided it is both past and future asymptotically de Sitter.

Schwarzschild-de Sitter space. Schwarzschild-de Sitter (SdS) space is a good example of a spacetime that is asymptotically de Sitter but not asymptotically simple. Roughly speaking it represents a Schwarzschild black hole sitting in a de Sitter background. Its metric in static coordinates (and in four dimensions) is given by,

$$ds^2 = ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{1}{\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)}dr^2 + r^2d\omega^2 \quad (2.3)$$

where $\Lambda > 0$, and $d\omega^2$ is the standard metric on the unit 2-sphere. SdS space has Cauchy surface topology $S^1 \times S^2$, while \mathcal{J}^+ for this spacetime has topology $\mathbb{R} \times S^2$. This spacetime fails to be asymptotically simple (either to the past or future), since there are null geodesics which enter the black hole region to the future, and the white hole region to the past. The Penrose diagram for this spacetime is given in Figure 2.

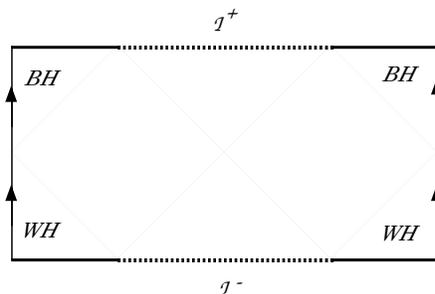


Figure 2. Penrose diagram for Schwarzschild-de Sitter space.

The de Sitter cusp. By the de Sitter cusp, we mean the spacetime, given by,

$$M = \mathbb{R} \times T^n, \quad ds^2 = -dt^2 + e^{2t}d\sigma_0^2 \quad (2.4)$$

where $d\sigma_0^2$ is a flat metric on the n -torus T^n . This spacetime is obtained as a quotient of a region in de Sitter space. In fact, the universal cover of this spacetime is isometric to the “half” of de Sitter space shown in Figure 3. With the exception of the t -lines, all timelike geodesics in the de Sitter cusp are past incomplete. This spacetime is future asymptotically de Sitter (and future asymptotically simple, as well), but is not past asymptotically de Sitter.

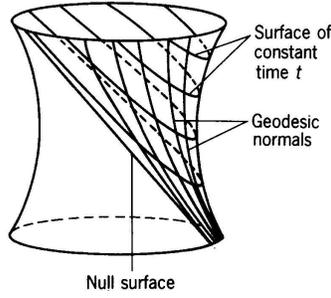


Figure 3. The universal cover of the de Sitter cusp.

The examples considered above are vacuum solutions to the Einstein equations with positive cosmological constant. In the next section we consider some matter filled models.

3 Occurrence of singularities and the geometry of conformal infinity

In this section we consider spacetimes M^{n+1} which obey the Einstein equations,

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = 8\pi T_{ij}, \quad (3.5)$$

with *positive* cosmological constant Λ , where the energy-momentum tensor T_{ij} will be assumed to satisfy certain energy inequalities.

Following (more or less) conventions in [15], M is said to obey

- (a) the *strong energy condition* provided,

$$(T_{ij} - \frac{1}{n-1}Tg_{ij})X^iX^j \geq 0 \quad (3.6)$$

for all timelike vectors X , where $T = T_i^i$,

- (b) the *weak energy condition* provided,

$$T_{ij}X^iX^j \geq 0 \quad (3.7)$$

for all timelike vectors X ,

- (c) the *dominant energy condition* provided, $T_{ij}X^iY^j \geq 0$ for all causal vectors X and Y that are either both future pointing or both past pointing, and

- (d) the *null energy condition* provided (3.7) holds for all *null* vectors X .

Setting $\Lambda = n(n-1)/2\ell^2$, the strong energy condition (3.6) is equivalent to,

$$\text{Ric}(X, X) = R_{ij}X^iX^j \geq -\frac{n}{\ell^2} \quad (3.8)$$

for all unit timelike vectors X . Finally, note, the null energy condition is equivalent to,

$$\text{Ric}(X, X) = R_{ij}X^iX^j \geq 0 \quad (3.9)$$

for all null vectors X .

We now consider some classical dust filled FRW models which are solutions to the Einstein equations (3.5); see, e.g., [6, chapter 23]. Thus, let M be a warped product spacetime of the form,

$$M = \mathbb{R} \times \Sigma^3, \quad ds^2 = -dt^2 + R^2(t)d\sigma_k^2 \quad (3.10)$$

where $(\Sigma^3, d\sigma_k^2)$ is a compact Riemannian manifold of constant curvature $k = -1, 0, +1$. Let the energy-momentum tensor of M be that corresponding to a dust, i.e., pressureless perfect fluid,

$$T_{ij} = \rho u_i u_j, \quad (3.11)$$

where $\rho = \rho(t)$ and $u = \partial/\partial t$. The Einstein equations imply that the scale factor $R = R(t)$ obeys the Friedmann equation,

$$(R')^2 = \frac{C}{R} + \frac{1}{3}\Lambda R^2 - k, \quad (3.12)$$

where,

$$\frac{8}{3}\pi R^3 \rho = C = \text{constant} . \quad (3.13)$$

The table in Figure 4 summarizes the qualitative behavior of the scale factor $R(t)$ for all choices of the sign of the cosmological constant Λ and of the sign of k . Focussing attention on the column $\Lambda > 0$, and those solutions for which the scale factor is unbounded to the future, we observe that the positively curved case $k = +1$ is distinguished from the nonpositively curved cases $k = -1, 0$. In the case $k = +1$, if the ‘‘mass parameter’’ C (which we assume to be positive) is sufficiently small, relative to the cosmological constant Λ , then M is timelike geodesically complete to the past, as well as the future. (In Figure 4, $\Lambda_c = 4/9C^2$.) In this regard, M behaves like de Sitter space, which in fact corresponds to the limiting case $C = 0$. On the other hand, in the cases $k = -1, 0$, no matter how small the mass parameter C , M begins with a big bang singularity, and hence all timelike geodesics in M are past incomplete.

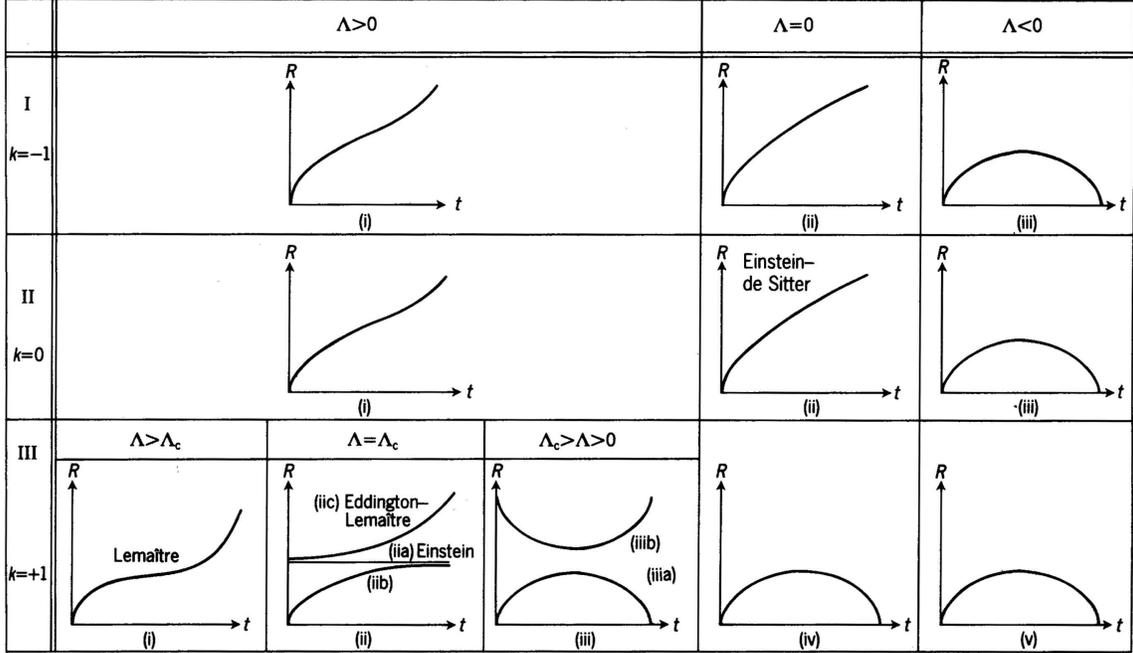


Figure 4. Behavior of the scale factor $R(t)$.¹

The aim of our first results is to show that the past incompleteness in the cases $k = -1, 0$ holds in a much broader context, one that does not require exact symmetries, assumptions of constant curvature, etc. In all (unbounded) cases the scale factor behaves like, $R(t) \sim e^{\sqrt{\frac{\Lambda}{3}}t}$ for large t , just as in de Sitter space. In fact, and this is the key to the generalization, all of these models are future asymptotically simple and de Sitter, in the sense of section 2.

Let (M^{n+1}, g) be a cosmological spacetime which is future asymptotically simple and de Sitter, in the sense of section 2. Then there exists a spacetime with boundary (\tilde{M}, \tilde{g}) such that $\tilde{M} = M \cup \mathcal{J}^+$, where \mathcal{J}^+ is compact (and connected), and $\tilde{g} = \Omega^2 g$, as in Definition 2.1. The Riemannian metric \tilde{h} induced by \tilde{g} on \mathcal{J}^+ changes by a conformal factor with a change in the defining function Ω , and thus \mathcal{J}^+ is endowed with a natural conformal structure $[\tilde{h}]$. By the positive resolution of the Yamabe conjecture [20], the conformal class $[\tilde{h}]$ contains a metric of constant scalar curvature $-1, 0$, or $+1$, exclusively, in which case we will say that \mathcal{J}^+ is of negative, zero, or positive Yamabe class, respectively. In this way, every future asymptotically simple and de Sitter spacetime falls into precisely one of three classes. We note that future conformal infinity \mathcal{J}^+ for the dust-filled FRW model (3.10) - (3.13) is of negative, zero, or positive Yamabe class precisely when the metric $d\sigma_k^2$ has curvature $k = -1, 0$ or $+1$, respectively.

¹We thank Ray D’Inverno [6] and Oxford University Press for permission to reproduce this chart.

We now present a singularity theorem for future asymptotically simple and de Sitter spacetimes of negative Yamabe class (which is a slight variation of Theorem 3.2 in [2]).

Theorem 3.1 *Let (M^{n+1}, g) be a future asymptotically simple and de Sitter spacetime satisfying the Einstein equations with $\Lambda > 0$, such that*

- (i) *the strong and weak energy conditions hold, and*
- (ii) *the fall-off condition, $T_i^i \rightarrow 0$ on approach to \mathcal{J}^+ , holds.*

Then, if \mathcal{J}^+ is of negative Yamabe class, every timelike geodesic in (M, g) is past incomplete.

Proof: Theorem 3.1 may be viewed as a Lorentzian analogue of the main result of Witten-Yau [21]; its proof is essentially a ‘‘Lorentzification’’ of the proof of the Witten-Yau result given in [5].

The Einstein equations and the assumptions on the energy-momentum tensor imply that (3.8) holds and that $R \rightarrow n(n+1)/\ell^2$ on approach to \mathcal{J} , where $R = R_i^i$ is the spacetime scalar curvature. Thus, by a constant rescaling of the metric, we may assume without loss of generality that (M, g) satisfies,

$$\text{Ric}(X, X) \geq -n, \quad (3.14)$$

for all unit timelike vectors X , and

$$R \rightarrow n(n-1) \quad \text{on approach to } \mathcal{J}^+. \quad (3.15)$$

The first step of the proof involves some gauge fixing. Fix a metric γ_0 in the conformal class $[\tilde{h}]$ of \mathcal{J}^+ of constant scalar curvature -1 . By the formula relating the scalar curvature functions of conformally related metrics, (3.15) implies that $\tilde{g}(\tilde{\nabla}\Omega, \tilde{\nabla}\Omega)|_{\mathcal{J}^+} = -1$. Then, as is shown in [2] (see also [13] for the similar Riemannian case), the conformal factor Ω can be chosen so that this equality holds in a neighborhood of \mathcal{J}^+ . To put it in a slightly different way, $\Omega = t$ can be chosen so that in a \tilde{g} -normal neighborhood $[0, \epsilon) \times \mathcal{J}^+$ of \mathcal{J}^+ , the physical metric g takes the form

$$g = \frac{1}{t^2} \left(-dt^2 + \tilde{h}_t \right), \quad (3.16)$$

where \tilde{h}_t , $t \in [0, \epsilon)$, is the metric on the slice $\Sigma_t = \{t\} \times \mathcal{J}^+$ induced by \tilde{g} , such that, in addition, $\tilde{h}_0 = \gamma_0$. (If Ω is the given defining function, one seeks a new defining function of the form $e^u\Omega$ with the requisite properties; this leads to a first order non-characteristic PDE for u , with appropriate boundary condition, which can always be solved; see [2, 13] for details.)

Let $H = H_t$, $0 < t < \epsilon$, denote the mean curvature of Σ_t in (M, g) . By our conventions, $H = \operatorname{div} u$, where $u = -t \frac{\partial}{\partial t}$ is the future pointing unit normal to the Σ_t 's. $H = H_t$ and $\tilde{H} = \tilde{H}_t$ (the mean curvature of Σ_t in $(M, \tilde{g}|_M)$) are related by,

$$H = t\tilde{H} + n. \quad (3.17)$$

In particular, $H|_{\Sigma_t} > 0$ for t sufficiently small.

The Gauss equation (in the physical metric g) applied to each Σ_t , together with the Einstein equations, yields the constraint,

$$\begin{aligned} H^2 &= 2\mathcal{T}(u, u) + 2\Lambda + |K|^2 - r \\ &= 2t^2\mathcal{T}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + n(n-1) + |K|^2 - t^2\tilde{r}, \end{aligned} \quad (3.18)$$

where $r = r_t$ (resp., $\tilde{r} = \tilde{r}_t$) is the scalar curvature of Σ_t in the metric induced from g (resp., \tilde{g}), $K = K_t$ is the second fundamental form of Σ_t in (M, g) , and $\mathcal{T} = T_{ij}$ is the energy-momentum tensor. Since, by the Schwarz inequality, $|K|^2 \geq (\operatorname{tr} K)^2/n = H^2/n$, equation (3.18) implies,

$$\begin{aligned} H^2 &\geq n^2 + \frac{n}{n-1}t^2 \left(\mathcal{T}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) - \tilde{r} \right) \\ &\geq n^2 - \frac{n}{n-1}t^2 \tilde{r}_t, \end{aligned} \quad (3.19)$$

where in the second inequality we have used the weak energy condition (3.7). Thus, since $\tilde{r}_0 =$ the scalar curvature of $(\mathcal{J}^+, \tilde{h}_0 = \gamma_0) = -1$, (3.19) implies that $H|_{\Sigma_t} > n$ for all $t > 0$ sufficiently small. This is the essential consequence of the assumption that \mathcal{J}^+ is of negative Yamabe class. Theorem 3.1 is now an immediate consequence of the following result. \square

Proposition 3.2 *Let M^{n+1} be a spacetime satisfying the energy condition,*

$$\operatorname{Ric}(X, X) \geq -n$$

for all unit timelike vectors X . Suppose that M has a smooth compact Cauchy surface Σ with mean curvature H satisfying $H > n$. Then every timelike geodesic in M is past incomplete.

Proof: This result is a straightforward extension of an old singularity theorem of Hawking; see e.g., [18, Theorem 55A, p. 431]. Its proof makes use of basic comparison theory.

Fix $\delta > 0$ so that the mean curvature of Σ satisfies $H \geq n(1+\delta)$. Let $\rho : I^-(\Sigma) \rightarrow \mathbb{R}$ be the Lorentzian distance function to Σ ,

$$\rho(x) = d(x, \Sigma) = \sup_{y \in \Sigma} d(x, y); \quad (3.20)$$

ρ is continuous, and smooth outside the past focal cut locus of Σ . We will show that ρ is bounded from above,

$$\rho(x) \leq \coth^{-1}(1 + \delta) \quad \text{for all } x \in I^-(\Sigma). \quad (3.21)$$

This implies that every past inextendible timelike curve with future end point on Σ has length $\leq \coth^{-1}(1 + \delta)$.

Suppose to the contrary, there is a point $q \in I^-(\Sigma)$ such that $d(q, \Sigma) = \ell > \coth^{-1}(1 + \delta)$. Let $\gamma : [0, \ell] \rightarrow M$, $t \rightarrow \gamma(t)$, be a past directed unit speed timelike geodesic from $p \in \Sigma$ to q that realizes the distance from q to Σ . γ meets Σ orthogonally, and because it maximizes distance to Σ , ρ is smooth on an open set U containing $\gamma \setminus \{q\}$.

For $0 \leq t < \ell$, the slice $\rho = t$ is smooth near the point $\gamma(t)$; let $H(t)$ be the mean curvature, with respect to the future pointing normal $\nabla\rho$, at $\gamma(t)$ of the slice $\rho = t$. $H = H(t)$ obeys the traced Riccati (Raychaudhuri) equation,

$$H' = \text{Ric}(\gamma', \gamma') + |K|^2, \quad (3.22)$$

where $' = d/dt$, and K is the second fundamental form of Σ .

Equation (3.22), together with the inequalities $|K|^2 \geq H^2/n$, $\text{Ric}(\gamma', \gamma') \geq -n$ and $H(0) \geq n(1 + \delta)$, implies that $\mathcal{H}(t) := H(t)/n$ satisfies,

$$\mathcal{H}' \geq \mathcal{H}^2 - 1, \quad \mathcal{H}(0) \geq 1 + \delta. \quad (3.23)$$

By an elementary comparison with the unique solution to: $h' = h^2 - 1$, $h(0) = 1 + \delta$, we obtain $\mathcal{H}(t) \geq \coth(a - t)$, where $a = \coth^{-1}(1 + \delta) < \ell$, which implies that $\mathcal{H} = \mathcal{H}(t)$ is unbounded on $[0, a)$, contradicting the fact that \mathcal{H} is smooth on $[0, \ell)$. \square

We now briefly consider the ‘‘borderline’’ case in which \mathcal{J}^+ is of zero Yamabe class. As the example of the de Sitter cusp described in Section 2 shows, the analogue of Theorem 3.1 can fail in this case, in that *some* timelike geodesics may be past complete. But, as the following theorem shows, it can fail only under very special circumstances.

Theorem 3.3 *Let (M^{n+1}, g) be a maximal future asymptotically simple and de Sitter spacetime satisfying the Einstein equations with $\Lambda = n(n - 1)/2$, such that*

- (i) *the strong and dominant energy conditions hold, and*
- (ii) *the fall-off condition, $T_i^i \rightarrow 0$ on approach to \mathcal{J}^+ , holds.*

If \mathcal{J}^+ is of zero Yamabe class, then either every timelike geodesic in (M, g) is past incomplete, or else (M, g) is isometric to the warped product with line element

$$ds^2 = -d\tau^2 + e^{2\tau}\tilde{h}, \quad (3.24)$$

where \tilde{h} a Ricci flat metric on \mathcal{J}^+ . In particular, (M, g) satisfies the vacuum ($T_{ij} = 0$) Einstein equations with cosmological constant $\Lambda = n(n - 1)/2$.

Here “maximal” means that (M, g) is not contained in a larger globally hyperbolic spacetime. Note also that the weak energy condition has been replaced by the dominant energy condition.

Comments on the proof: One again works in the gauge (3.16), where now \tilde{h}_0 is a metric of zero scalar curvature on \mathcal{J}^+ , $\tilde{r}_0 = 0$. In this case (3.19) implies that the inequality $H_t \geq n$ holds to order t^3 (see [5] for an application of this in the Riemannian setting). However, by more refined arguments [1, 2] it is possible to show that $H|_{\Sigma_t} \geq n$ for all t sufficiently small. Theorem 3.3 may then be derived from the following rigid version of Proposition 3.2. (The assumption of maximality in Theorem 3.3 is used to show that the local warped product splitting described below can be made global.)

Proposition 3.4 *Let M^{n+1} be a spacetime satisfying the energy condition*

$$\text{Ric}(X, X) \geq -n$$

for all unit timelike vectors X . Suppose M has a smooth compact Cauchy surface Σ with mean curvature H satisfying $H \geq n$. If there exists at least one past complete timelike geodesic in M , then there exists a neighborhood of Σ in $J^-(\Sigma)$ which is isometric to $(-\epsilon, 0] \times \Sigma$, with warped product metric, $ds^2 = -d\tau^2 + e^{2\tau}h$, where h is the induced metric on Σ .

To prove Proposition 3.4, one introduces Gaussian normal coordinates in a neighborhood U of Σ in $J^-(\Sigma)$, $U = [0, \epsilon) \times \Sigma$, $ds^2 = -du^2 + h_u$. Comparison techniques like those used in the proof of Theorem 3.1 imply that each slice $\Sigma_u = \{u\} \times \Sigma$ has mean curvature $H_u \geq n$. It can be further shown that each Σ_u must be totally umbilic, which leads to the desired warp product splitting. Indeed if some Σ_u were not totally umbilic, then Σ_u could be deformed to a Cauchy surface Σ' having mean curvature $H' > n$. Proposition 3.2 would then imply that every timelike geodesic in M is past incomplete, contrary to assumption. We refer the reader to [2] for details. \square

Thus, Theorems 3.1 and 3.3 show that future asymptotically simple and de Sitter spacetimes obeying appropriate energy conditions are *totally* past timelike geodesically incomplete, provided \mathcal{J}^+ is of negative or zero Yamabe class. The Yamabe class condition may, under certain circumstances, be viewed as a topological condition. Consider for example the 3 + 1 dimensional case, and assume \mathcal{J}^+ is orientable. Then, by well-known results of Gromov and Lawson [14], if \mathcal{J}^+ is a $K(\pi, 1)$ space, or contains a $K(\pi, 1)$ space in its prime decomposition, then \mathcal{J}^+ cannot carry a metric of positive scalar curvature, and hence must be of negative or zero Yamabe class. Hence, in 3 + 1 dimensions, this topological condition may replace the Yamabe class assumption.

Theorems 3.1 and 3.3 say nothing about the occurrence of singularities in the case \mathcal{J}^+ is of positive Yamabe class. However the discussion of the previous paragraph suggests a possible connection between the “size” of the fundamental group of \mathcal{J}^+ and

the occurrence of singularities. This view is supported by the following theorem and subsequent corollary.

Theorem 3.5 *Let M^{n+1} , $n \geq 2$, be a cosmological spacetime obeying the null energy condition (3.9), which is asymptotically de Sitter (to both the past and future). If the Cauchy surfaces of M have infinite fundamental group, then M cannot be asymptotically simple, either to the past or the future.*

The failure of asymptotic simplicity strongly suggests the development of singularities to both the past and future: there are null geodesics which do not make it to past and/or future infinity. The result is well illustrated by Schwarzschild-de Sitter space, which has Cauchy surface topology $S^1 \times S^2$, as described in section 2. A further example of interest having Cauchy surface topology $P^3 \# P^3$ has recently been considered by McInnes [17]; this example is obtained as a quotient of Schwarzschild-de Sitter space. The proof of Theorem 3.5 is an application of the author's null splitting theorem [9]; we refer the reader to [2, 10] for details.

Recall, if M is future asymptotically simple and de Sitter then the Cauchy surfaces of M are diffeomorphic to \mathcal{J}^+ . Hence, Theorem 3.5 yields the following corollary.

Corollary 3.6 *Let M be a cosmological spacetime obeying the null energy condition (3.9), which is future asymptotically simple and de Sitter. If \mathcal{J}^+ has infinite fundamental group then M cannot be asymptotically de Sitter to the past, i.e., M does not admit a regular past conformal infinity \mathcal{J}^- , compact or otherwise.*

To emphasize for a moment the evolutionary viewpoint, consider the time-dual of the above corollary. Thus, let M be a spacetime obeying the null energy condition which is *past* asymptotically simple and de Sitter, and, for concreteness, take \mathcal{J}^- to have topology $S^1 \times S^2$. According to work of Friedrich [7], one can use the conformal Einstein equations to maximally evolve well understood initial data on \mathcal{J}^- to obtain a past asymptotically simple and de Sitter vacuum solution to the Einstein equations with $\Lambda > 0$, having $S^1 \times S^2$ Cauchy surfaces. In general, one expects singularities to develop to the future. One can imagine black holes forming, with the spacetime approaching something like Schwarzschild-de Sitter space to the far future, and, hence, admitting a regular future conformal infinity. But note that the time-dual of Corollary 3.6 rules out such a scenario. Thus, either the singularity that develops must be more global (leading to a big crunch), or else something more peculiar is happening. In principle, it is possible that the spacetime approaches to the far future something like the Nariai solution [4], which is just a metric product of the 2-sphere with 2-dimensional de Sitter space, and hence is geodesically complete. Since two spatial directions in the Nariai solution remain bounded toward the future, it does not admit a regular future conformal infinity; roughly speaking \mathcal{J}^+ degenerates to a one dimensional manifold in this case. However, one would not expect such a development to be stable. Thus, it is an interesting open problem to obtain a detailed understanding

(analytic or numerical) of the generic behavior of the future Cauchy development in this situation.

In view of Corollary 3.6, one expects there to exist singularities in the past, in the case \mathcal{J}^+ has infinite fundamental group. We conclude with the following theorem, which reinforces this expectation.

Theorem 3.7 *Let M^{n+1} , $2 \leq n \leq 7$, be a future asymptotically simple and de Sitter spacetime, with compact orientable future infinity \mathcal{J}^+ , which obeys the null energy condition (3.9). If \mathcal{J}^+ (which is diffeomorphic to the Cauchy surfaces of M) has positive first Betti number, $b_1(\mathcal{J}^+) > 0$, then M is past null geodesically incomplete.*

Note that if a Cauchy surface Σ contains a wormhole, i.e., has topology of the form $N \# (S^1 \times S^{n-1})$, then $b_1(\Sigma) > 0$. The theorem is somewhat reminiscent of previous results of Gannon [12], which show, in the asymptotically flat setting, how nontrivial spatial topology leads to the occurrence of singularities.

Sketch of the proof: The proof is an application of the Penrose singularity theorem; see, e.g., [15, Theorem 1, p. 263]

Since M is future asymptotically de Sitter and \mathcal{J}^+ is compact, one can find in the far future a smooth compact spacelike Cauchy surface Σ for M , with second fundamental form which is *positive definite* with respect to the *future* pointing normal. This means that Σ is *contracting* in all directions towards the *past*.

By Poincaré duality, and the fact that there is never any co-dimension one torsion, $b_1(\Sigma) > 0$ if and only if $H_{n-1}(\Sigma, \mathbb{Z}) \neq 0$. By well known results of geometric measure theory (see [16, p. 51] for discussion; this is where the dimension assumption is used), every nontrivial class in $H_{n-1}(\Sigma, \mathbb{Z})$ has a least area representative which can be expressed as a sum of smooth, orientable, connected, compact, embedded minimal (mean curvature zero) hypersurfaces in Σ . Let W be such a hypersurface; note W is spacelike and has co-dimension two in M . As described in [8], since W is minimal in Σ , and Σ is contracting in all directions towards the past in M , W must be a past trapped surface in M . (Recall [15, 3, 18] that a past trapped surface is a compact co-dimension two spacelike submanifold W of M with the property that the two congruences of null normal geodesics issuing to the past from W have negative divergence along W .)

Since W and Σ are orientable, W is two-sided in Σ . Moreover, since W represents a nontrivial element of $H_{n-1}(\Sigma, \mathbb{Z})$, W does not separate Σ , for otherwise it would bound in Σ . This implies that there is a loop in Σ with nonzero intersection number with respect to W . There exists a covering space Σ^* of Σ in which this loop gets unraveled. Σ^* has a simple description in terms of cut-and-paste operations: By making a cut along Σ , we obtain a compact manifold Σ' with two boundary components, each isometric to W . Taking \mathbb{Z} copies of Σ' , and gluing these copies end-to-end we obtain the covering space Σ^* of Σ . In this covering, W is covered by \mathbb{Z} copies of itself, each one separating Σ^* ; let W_0 be one such copy. We know by global hyperbolicity that

M is homeomorphic to $\mathbb{R} \times \Sigma$, and hence the fundamental groups of Σ and M are isomorphic. This implies that the covering spaces of M are in one-to-one correspondence with the covering spaces of Σ . In fact, there will exist a covering spacetime M^* of M in which Σ^* is a Cauchy surface for M^* . Thus, M^* is a spacetime obeying the null energy condition, which contains a noncompact Cauchy surface (namely Σ^*) and a past trapped surface (namely W_0). By the Penrose singularity theorem, M^* is past null geodesically incomplete, and hence so is M . \square

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