

GRAVITY AND THE QUANTUM POTENTIAL

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ABSTRACT. We review some material connecting gravity and the quantum potential and provide a few new observations. The main theme is that in utilizing a conformal (Weyl) geometry the metric plus the Weyl vector plus the quantum mass field determine spacetime geometry. There are strong connections to deBroglie-Bohm style ideas in quantum theory.

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1. INTRODUCTION

We sketch here first some results extracted from [18, 100, 103, 104, 105, 107, 108] on relativistic Bohmian mechanics, Weyl geometry, and quantum gravity (cf. also [1, 9, 10, 14, 15, 16, 17, 20, 24, 25, 26, 27, 28, 29, 30, 31, 32, 39, 40, 41, 44, 45, 50, 51, 52, 55, 60, 61, 70, 72, 73, 89, 90, 92, 93, 101, 102, 106, 109, 110]). We use the Bohm-Weyl terminology to refer to a series of papers by Albohasani, Bisabr, Darabi, Golshani, Motavali, Salehi, Sepangi, A. Shojai, and F. Shojai dealing with the subject; it could perhaps be called the Tehran approach or named after some group of authors. However the idea of linking Weyl geometry and deBroglie-Bohm type quantum theory also does appear elsewhere as indicated in this paper and was perhaps first noticed in [67] (cf. also [78]). The "Tehran" version is consolidated and summarized in [108] by A. and F. Shojai.

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2. SKETCH OF DEBROGLIE-BOHM-WEYL THEORY

From [30] (and references cited there) we know something about Bohmian mechanics and the quantum potential and we go now to [104] to begin the present discussion. In nonrelativistic deBroglie-Bohm theory the quantum potential is (cf. [30]) **(A1)** $Q = -(\hbar^2/2m)(\nabla^2|\Psi|/|\Psi|)$. The particles trajectory can be derived from Newton's law of motion in which the quantum force $-\nabla Q$ is present in addition to the classical force $-\nabla V$. The enigmatic quantum behavior is attributed here to the quantum force or quantum potential (with Ψ determining a "pilot wave" which guides the particle motion). Setting $\Psi = \sqrt{\rho} \exp[iS/\hbar]$ one has

$$(2.1) \quad \frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V + Q = 0; \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \frac{\nabla S}{m} \right) = 0$$

We follow the standard Bohmian approach here and refer to [10, 24, 26, 30, 32, 44, 45, 50, 51] for the Bertoldi-Faraggi-Matone development; there will be some surprising connections arising later. The first equation in (2.1) is a Hamilton-Jacobi (HJ) equation which is identical to Newton's law and represents an energy condition **(A2)** $E = (|p|^2/2m) + V + Q$ (recall from HJ theory $-(\partial S/\partial t) = E (= H)$ and $\nabla S = p$ (cf. [34]). The second equation represents a continuity equation for a hypothetical ensemble related to the particle in question. For the relativistic extension one could simply try to generalize the relativistic energy equation **(A3)** $\eta_{\mu\nu} P^\mu P^\nu = m^2 c^2$ to the form **(A4)** $\eta_{\mu\nu} P^\mu P^\nu = m^2 c^2 (1 + Q) = \mathcal{M}^2 c^2$ where **(A5)** $Q = (\hbar^2/m^2 c^2)(\square|\Psi|/|\Psi|)$ and

$$(2.2) \quad \mathcal{M}^2 = m^2 \left(1 + \alpha \frac{\square|\Psi|}{|\Psi|} \right); \quad \alpha = \frac{\hbar^2}{m^2 c^2}$$

This could be derived e.g. by setting $\Psi = \sqrt{\rho} \exp(iS/\hbar)$ in the Klein-Gordon (KG) equation and separating the real and imaginary parts, leading to the relativistic HJ equation **(A6)** $\eta_{\mu\nu} \partial^\mu S \partial^\nu S = \mathfrak{M}^2 c^2$ (as in (2.1) - note $P^\mu = -\partial^\mu S$) and the continuity equation **(A7)** $\partial_\mu (\rho \partial^\mu S) = 0$. The problem of \mathcal{M}^2 not being positive definite here (i.e. tachyons) is serious however and in fact **(A4)** is not the correct equation (see e.g. [106, 108]). One must use the covariant derivatives ∇_μ in place of ∂_μ and for spin zero in a curved background there results (cf. [106, 108])

$$(2.3) \quad \nabla_\mu (\rho \nabla^\mu S) = 0; \quad g^{\mu\nu} \nabla_\mu S \nabla_\nu S = \mathfrak{M}^2 c^2;$$

To see this one must require that a correct relativistic equation of motion should not only be Poincaré invariant but also it should have the correct nonrelativistic limit. Thus for a relativistic particle of mass \mathfrak{M} (which is a Lorentz invariant quantity) the action functional is **(A8)** $\mathfrak{A} = \int d\lambda (1/2) \mathfrak{M}(r) (dr_\mu/d\lambda)(dr^\nu/d\lambda)$ where λ is any scalar parameter parametrizing the path $r_\mu(\lambda)$ (it could e.g. be the proper time τ). Varying the path via $r_\mu \rightarrow r'_\mu = r_\mu + \epsilon_\mu$ one gets

$$(2.4) \quad \mathfrak{A} \rightarrow \mathfrak{A}' = \mathfrak{A} + \delta\mathfrak{A} = \mathfrak{A} + \int d\lambda \left[\frac{dr_\mu}{d\lambda} \frac{d\epsilon^\mu}{d\lambda} + \frac{1}{2} \frac{dr_\mu}{d\lambda} \frac{dr^\mu}{d\lambda} \epsilon_\nu \partial^\nu \mathfrak{M} \right]$$

By least action the correct path satisfies **(A10)** $\delta\mathfrak{A} = 0$ with fixed boundaries so the equation of motion is **(A11)** $(d/d\lambda)(\mathfrak{M}u_\mu) = (1/2)u_\nu u^\nu \partial_\mu \mathfrak{M}$ or **(A12)** $\mathfrak{M}(du_\mu/d\lambda) = ((1/2)\eta_{\mu\nu} u_\alpha u^\alpha - u_\mu u_\nu) \partial^\nu \mathfrak{M}$ where $u_\mu = dr_\mu/d\lambda$. Now look at the symmetries of the action functional **(A8)**

via $\lambda \rightarrow \lambda + \delta$. The conserved current is then the Hamiltonian $\mathfrak{H} = -\mathfrak{L} + u_\mu(\partial\mathfrak{L}/\partial u_\mu) = (1/2)\mathfrak{M}u_\mu u^\mu = E$. This can be seen by setting $\delta\mathfrak{A} = 0$ where

$$(2.5) \quad 0 = \delta\mathfrak{A} = \mathfrak{A}' - \mathfrak{A} = \int d\lambda \left[\frac{1}{2}u_\mu u^\mu u^\nu \partial_\nu \mathfrak{M} + \mathfrak{M}u_\mu \frac{du^\mu}{d\lambda} \right] \delta$$

which means that the integrand is zero, i.e. **(A14)** $(d/d\lambda)[(1/2)\mathfrak{M}u_\mu u^\mu] = 0$. Since the proper time is defined as $c^2 d\tau^2 = dr_\mu dr^\mu$ this leads to **(A15)** $(d\tau/d\lambda) = \sqrt{(2E/\mathfrak{M}c^2)}$ and the equation of motion becomes **(A16)** $\mathfrak{M}(dv_\mu/d\tau) = (1/2)(c^2\eta_{\mu\nu} - v_\mu v_\nu)\partial^\nu \mathfrak{M}$ where $v_\mu = dr_\mu/d\tau$. The nonrelativistic limit can be derived by letting the particles velocity be ignorable with respect to light velocity. In this limit the proper time is identical to the time coordinate $\tau = t$ and the result is that the $\mu = 0$ component is satisfied identically via **(A17)** $\mathfrak{M}d^2r/dt^2 = -(1/2)c^2\nabla\mathfrak{M}$ ($r \sim \vec{r}$). One can write then **(A18)** $m(d^2r/dt^2) = -\nabla[(mc^2/2)\log(\mathfrak{M}/\mu)]$ where μ is an arbitrary mass scale. In order to have the correct limit the term in parenthesis on the right side should be equal to the quantum potential so **(A19)** $\mathfrak{M} = \mu \exp[-(\hbar^2/m^2c^2)(\nabla^2|\Psi|/|\Psi|)]$. Thus the relativistic quantum mass field (manifestly invariant) is **(A20)** $\mathfrak{M} = \mu \exp[(\hbar^2/2m)(\square|\Psi|/|\Psi|)]$ and setting $\mu = m$ we get **(A21)** $\mathfrak{M} = m \exp[(\hbar^2/m^2c^2)(\square|\Psi|/|\Psi|)]$. If one starts with the standard relativistic theory and goes to the nonrelativistic limit one does not get the correct nonrelativistic equations; this is a result of an improper decomposition of the wave function into its phase and norm in the KG equation (cf. also [10] for related procedures). One notes here also that **(A21)** leads to a positive definite mass squared. Also from [106] this can be extended to a many particle version and to a curved spacetime. In summary, for a particle in a curved background one has (cf. [108] which we follow for the rest of this section)

$$(2.6) \quad \nabla_\mu(\rho\nabla^\mu S) = 0; \quad g^{\mu\nu}\nabla_\mu S\nabla_\nu S = \mathfrak{M}^2c^2; \quad \mathfrak{M}^2 = m^2e^\Omega; \quad \Omega = \frac{\hbar^2}{m^2c^2} \frac{\square_g|\Psi|}{|\Psi|}$$

Since, following deBroglie, the quantum HJ equation in (2.6) can be written in the form **(A22)** $(m^2/\mathfrak{M}^2)g^{\mu\nu}\nabla_\mu S\nabla_\nu S = m^2c^2$ the quantum effects are identical to a change of spacetime metric **(A23)** $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = (\mathfrak{M}^2/m^2)g_{\mu\nu}$ which is a conformal transformation. Therefore **(A22)** becomes an equation **(A24)** $\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu S\tilde{\nabla}_\nu S = m^2c^2$ where $\tilde{\nabla}_\mu$ represents covariant differentiation with respect to the metric $\tilde{g}_{\mu\nu}$. The continuity equation is then **(A25)** $\tilde{g}_{\mu\nu}\tilde{\nabla}_\mu(\rho\tilde{\nabla}_\nu S) = 0$. The important conclusion here is that the presence of the quantum potential is equivalent to a curved spacetime with its metric given by **(A23)**. This is a geometrization of the quantum aspects of matter and it seems that there is a dual aspect to the role of geometry in physics. The spacetime geometry sometimes looks like “gravity” and sometimes reveals quantum behavior. The curvature due to the quantum potential may have a large influence on the classical contribution to the curvature of spacetime. The particle trajectory can now be derived from the guidance relation via differentiation of (2.6) leading to the Newton equations of motion

$$(2.7) \quad \mathfrak{M}\frac{d^2x^\mu}{d\tau^2} + \mathfrak{M}\Gamma_{\nu\kappa}^\mu u^\nu u^\kappa = (c^2g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu \mathfrak{M}$$

Using the conformal transformation above (2.7) reduces to the standard geodesic equation.

Now a general “canonical” relativistic system consisting of gravity and classical matter

(no quantum effects) is determined by the action

$$(2.8) \quad \mathcal{A} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \mathcal{R} + \int d^4x \sqrt{-g} \frac{\hbar^2}{2m} \left(\frac{\rho}{\hbar^2} \mathcal{D}_\mu S \mathcal{D}^\mu S - \frac{m^2}{\hbar^2} \rho \right)$$

where $\kappa = 8\pi G$ and $c = 1$ for convenience. It was seen above that via deBroglie the introduction of a quantum potential is equivalent to introducing a conformal factor $\Omega^2 = \mathfrak{M}^2/m^2$ in the metric. Hence in order to introduce quantum effects of matter into the action (2.8) one uses this conformal transformation to get $(1 + Q \sim \exp(Q))$

$$(2.9) \quad \mathfrak{A} = \frac{1}{2\kappa} \int d^4x \sqrt{-\bar{g}} (\bar{\mathcal{R}} \Omega^2 - 6 \bar{\nabla}_\mu \Omega \bar{\nabla}^\mu \Omega) + \\ + \int d^4x \sqrt{-\bar{g}} \left(\frac{\rho}{m} \Omega^2 \bar{\nabla}_\mu S \bar{\nabla}^\mu S - m \rho \Omega^4 \right) + \int d^4x \sqrt{-\bar{g}} \lambda \left[\Omega^2 - \left(1 + \frac{\hbar^2}{m^2} \frac{\square \sqrt{\rho}}{\sqrt{\rho}} \right) \right]$$

where a bar over any quantity means that it corresponds to the nonquantum regime. Here only the first two terms of the expansion of $\mathfrak{M}^2 = m^2 \exp(\Omega)$ in (2.6) have been used, namely $\mathfrak{M}^2 \sim m^2(1 + \Omega)$. No physical change is involved in considering all the terms. λ is a Lagrange multiplier introduced to identify the conformal factor with its Bohmian value. One uses here $\bar{g}_{\mu\nu}$ to raise or lower indices and to evaluate the covariant derivatives; the physical metric (containing the quantum effects of matter) is $g_{\mu\nu} = \Omega^2 \bar{g}_{\mu\nu}$. By variation of the action with respect to $\bar{g}_{\mu\nu}$, Ω , ρ , S , and λ one arrives at the following quantum equations of motion:

(1) The equation of motion for Ω

$$(2.10) \quad \bar{\mathcal{R}} \Omega + 6 \square \Omega + \frac{2\kappa}{m} \rho \Omega (\bar{\nabla}_\mu S \bar{\nabla}^\mu S - 2m^2 \Omega^2) + 2\kappa \lambda \Omega = 0$$

(2) The continuity equation for particles (**A26**) $\bar{\nabla}_\mu (\rho \Omega^2 \bar{\nabla}^\mu S) = 0$

(3) The equations of motion for particles (here $a' \equiv \bar{a}$)

$$(2.11) \quad (\bar{\nabla}_\mu S \bar{\nabla}^\mu S - m^2 \Omega^2) \Omega^2 \sqrt{\rho} + \frac{\hbar^2}{2m} \left[\square' \left(\frac{\lambda}{\sqrt{\rho}} \right) - \lambda \frac{\square' \sqrt{\rho}}{\rho} \right] = 0$$

(4) The modified Einstein equations for $\bar{g}_{\mu\nu}$

$$(2.12) \quad \Omega^2 \left[\bar{\mathcal{R}}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{\mathcal{R}} \right] - [\bar{g}_{\mu\nu} \square' - \bar{\nabla}_\mu \bar{\nabla}_\nu] \Omega^2 - 6 \bar{\nabla}_\mu \Omega \bar{\nabla}_\nu \Omega + 3 \bar{g}_{\mu\nu} \bar{\nabla}_\alpha \Omega \bar{\nabla}^\alpha \Omega + \\ + \frac{2\kappa}{m} \rho \Omega^2 \bar{\nabla}_\mu S \bar{\nabla}_\nu S - \frac{\kappa}{m} \rho \Omega^2 \bar{g}_{\mu\nu} \bar{\nabla}_\alpha S \bar{\nabla}^\alpha S + \kappa m \rho \Omega^4 \bar{g}_{\mu\nu} + \\ + \frac{\kappa \hbar^2}{m^2} \left[\bar{\nabla}_\mu \sqrt{\rho} \bar{\nabla}_\nu \left(\frac{\lambda}{\sqrt{\rho}} \right) + \bar{\nabla}_\nu \sqrt{\rho} \bar{\nabla}_\mu \left(\frac{\lambda}{\sqrt{\rho}} \right) \right] - \frac{\kappa \hbar^2}{m^2} \bar{g}_{\mu\nu} \bar{\nabla}_\alpha \left[\lambda \frac{\bar{\nabla}^\alpha \sqrt{\rho}}{\sqrt{\rho}} \right] = 0$$

(5) The constraint equation (**A27**) $\Omega^2 = 1 + (\hbar^2/m^2)[(\square \sqrt{\rho})/\sqrt{\rho}]$

Thus the back reaction effects of the quantum factor on the background metric are contained in these highly coupled equations. A simpler form of (2.10) can be obtained by taking the trace of (2.12) and using (2.10) which produces (**A28**) $\lambda = (\hbar^2/m^2) \bar{\nabla}_\mu [\lambda (\bar{\nabla}^\mu \sqrt{\rho})/\sqrt{\rho}]$. A

solution of this via perturbation methods using the small parameter $\alpha = \hbar^2/m^2$ yields the trivial solution $\lambda = 0$ so the above equations reduce to

$$(2.13) \quad \bar{\nabla}_\mu(\rho\Omega^2\bar{\nabla}^\mu S) = 0; \quad \bar{\nabla}_\mu S\bar{\nabla}^\mu S = m^2\Omega^2; \quad \mathfrak{G}_{\mu\nu} = -\kappa\mathfrak{T}_{\mu\nu}^{(m)} - \kappa\mathfrak{T}_{\mu\nu}^{(\Omega)}$$

where $\mathfrak{T}_{\mu\nu}^{(m)}$ is the matter energy-momentum (EM) tensor and

$$(2.14) \quad \kappa\mathfrak{T}_{\mu\nu}^{(\Omega)} = \frac{[g_{\mu\nu}\square - \nabla_\mu\nabla_\nu]\Omega^2}{\Omega^2} + 6\frac{\nabla_\mu\Omega\nabla_\nu\Omega}{\omega^2} - 2g_{\mu\nu}\frac{\nabla_\alpha\Omega\nabla^\alpha\Omega}{\Omega^2}$$

with **(A29)** $\Omega^2 = 1 + \alpha(\square\sqrt{\rho}/\sqrt{\rho})$. Note that the second relation in (2.13) is the Bohmian equation of motion and written in terms of $g_{\mu\nu}$ it becomes $\nabla_\mu S\nabla^\mu S = m^2c^2$.

In the preceding one has tacitly assumed that there is an ensemble of quantum particles so what about a single particle? One translates now the quantum potential into purely geometrical terms without reference to matter parameters so that the original form of the quantum potential can only be deduced after using the field equations. Thus the theory will work for a single particle or an ensemble (cf. also Remark 3.3). Thus first ignore gravity and look at the geometrical properties of the conformal factor given via

$$(2.15) \quad g_{\mu\nu} = e^{4\Sigma}\eta_{\mu\nu}; \quad e^{4\Sigma} = \frac{\mathfrak{M}^2}{m^2} = \exp\left(\alpha\frac{\square_\eta\sqrt{\rho}}{\sqrt{\rho}}\right) = \exp\left(\alpha\frac{\square_\eta\sqrt{|\mathfrak{T}|}}{\sqrt{|\mathfrak{T}|}}\right)$$

where \mathfrak{T} is the trace of the EM tensor and is substituted for ρ (true for dust). The Einstein tensor for this metric is **(A30)** $\mathfrak{G}_{\mu\nu} = 4g_{\mu\nu}\square_\eta\exp(-\Sigma) + 2\exp(-2\Sigma)\partial_\mu\partial_\nu\exp(2\Sigma)$. Hence as an Ansatz one can suppose that in the presence of gravitational effects the field equation would have a form

$$(2.16) \quad \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \kappa\mathfrak{T}_{\mu\nu} + 4g_{\mu\nu}e^\Sigma\square e^{-\Sigma} + 2e^{-2\Sigma}\nabla_\mu\nabla_\nu e^{2\Sigma}$$

This is written in a manner such that in the limit $\mathfrak{T}_{\mu\nu} \rightarrow 0$ one will obtain (2.15). Taking the trace of the last equation one gets **(A31)** $-\mathcal{R} = \kappa\mathfrak{T} - 12\square\Sigma + 24(\nabla\Sigma)^2$ which has the iterative solution **(A32)** $\kappa\mathfrak{T} = -\mathcal{R} + 12\alpha\square[(\square\sqrt{\mathcal{R}})/\sqrt{\mathcal{R}}]$ leading to **(A33)** $\Sigma = \alpha[(\square\sqrt{|\mathfrak{T}|}/\sqrt{|\mathfrak{T}|})] \simeq \alpha[(\square\sqrt{|\mathcal{R}|}/\sqrt{|\mathcal{R}|})]$ to first order in α .

One goes now to the field equations for a toy model. First from the above one sees that \mathfrak{T} can be replaced by \mathcal{R} in the expression for the quantum potential or for the conformal factor of the metric. This is important since the explicit reference to ensemble density is removed and the theory works for a single particle or an ensemble. So from (2.16) for a toy quantum gravity theory one assumes the following field equations

$$(2.17) \quad \mathfrak{G}_{\mu\nu} - \kappa\mathfrak{T}_{\mu\nu} - \mathfrak{Z}_{\mu\nu\alpha\beta}\exp\left(\frac{\alpha}{2}\Phi\right)\nabla^\alpha\nabla^\beta\exp\left(-\frac{\alpha}{2}\Phi\right) = 0$$

where **(A34)** $\mathfrak{Z}_{\mu\nu\alpha\beta} = 2[g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta}]$ and $\Phi = (\square\sqrt{|\mathcal{R}|}/\sqrt{|\mathcal{R}|})$. The number 2 and the minus sign of the second term in **(A34)** are chosen so that the energy equation derived later will be correct. Note that the trace of (2.17) is **(A35)** $\mathcal{R} + \kappa\mathfrak{T} + 6\exp(\alpha\Phi/2)\square\exp(-\alpha\Phi/2) = 0$ and this represents the connection of the Ricci scalar curvature of space time and the trace of the matter EM tensor. If a perturbative solution is admitted one can expand in powers of α to find **(A36)** $\mathcal{R}^{(0)} = -\kappa\mathfrak{T}$ and $\mathcal{R}^{(1)} = -\kappa\mathfrak{T} - 6\exp(\alpha\Phi^0/2)\square\exp(-\alpha\Phi^0/2)$ where

$\Phi^{(0)} = \square\sqrt{|\mathfrak{I}|}/\sqrt{|\mathfrak{I}|}$. The energy relation can be obtained by taking the four divergence of the field equations and since the divergence of the Einstein tensor is zero one obtains

$$(2.18) \quad \kappa\nabla^\nu\mathfrak{T}_{\mu\nu} = \alpha\mathcal{R}_{\mu\nu}\nabla^\nu\Phi - \frac{\alpha^2}{4}\nabla_\mu(\nabla\Phi)^2 + \frac{\alpha^2}{2}\nabla_\mu\Phi\square\Phi$$

For a dust with **(A37)** $\mathfrak{T}_{\mu\nu} = \rho u_\mu u_\nu$ and u_μ the velocity field, the conservation of mass law is **(A38)** $\nabla^\nu(\rho\mathfrak{M}u_\nu) = 0$ so one gets to first order in α **(A39)** $\nabla_\mu\mathfrak{M}/\mathfrak{M} = -(\alpha/2)\nabla_\mu\Phi$ or **(A40)** $\mathfrak{M}^2 = m^2\exp(-\alpha\Phi)$ where m is an integration constant. This is the correct relation of mass and quantum potential.

There is then some discussion about making the conformal factor dynamical via a general scalar tensor action (cf. also [102]) and subsequently one makes both the conformal factor and the quantum potential into dynamical fields and creates a scalar tensor theory with two scalar fields. Thus start with a general action

$$(2.19) \quad \mathfrak{A} = \int d^4x\sqrt{-g} \left[\phi\mathcal{R} - \omega\frac{\nabla_\mu\phi\nabla^\mu\phi}{\phi} - \frac{\nabla_\mu Q\nabla^\mu Q}{\phi} + 2\Lambda\phi + \mathfrak{L}_m \right]$$

The cosmological constant generally has an interaction term with the scalar field and here one uses an ad hoc matter Lagrangian

$$(2.20) \quad \mathfrak{L}_m = \frac{\rho}{m}\phi^a\nabla_\mu S\nabla^\mu S - m\rho\phi^b - \Lambda(1+Q)^c + \alpha\rho(e^{\ell Q} - 1)$$

(only the first two terms $1+Q$ from $\exp(Q)$ are used for simplicity in the third term). Here a, b, c are constants to be fixed later and the last term is chosen (heuristically) in such a manner as to have an interaction between the quantum potential field and the ensemble density (via the equations of motion); further the interaction is chosen so that it vanishes in the classical limit but this is ad hoc. Variation of the above action yields

- (1) The scalar fields equation of motion

$$(2.21) \quad \mathcal{R} + \frac{2\omega}{\phi}\square\phi - \frac{\omega}{\phi^2}\nabla^\mu\phi\nabla_\mu\phi + 2\Lambda + \frac{1}{\phi^2}\nabla^\mu Q\nabla_\mu Q + \frac{a}{m}\rho\phi^{a-1}\nabla^\mu S\nabla_\mu S - mb\rho\phi^{b-1} = 0$$

- (2) The quantum potential equations of motion **(A41)** $(\square Q/\phi) - (\nabla_\mu Q\nabla^\mu\phi/\phi^2) - \Lambda c(1+Q)^{c-1} + \alpha\ell\rho\exp(\ell Q) = 0$

- (3) The generalized Einstein equations

$$(2.22) \quad \mathfrak{G}^{\mu\nu} - \Lambda g^{\mu\nu} = -\frac{1}{\phi}\mathfrak{T}^{\mu\nu} - \frac{1}{\phi}[\nabla^\mu\nabla^\nu - g^{\mu\nu}\square]\phi + \frac{\omega}{\phi^2}\nabla^\mu\phi\nabla^\nu\phi - \frac{\omega}{2\phi^2}g^{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{\phi^2}\nabla^\mu Q\nabla^\nu Q - \frac{1}{2\phi^2}g^{\mu\nu}\nabla^\alpha Q\nabla_\alpha Q$$

- (4) The continuity equation **(A42)** $\nabla_\mu(\rho\phi^a\nabla^\mu S) = 0$

- (5) The quantum Hamilton Jacobi equation **(A43)** $\nabla^\mu S\nabla_\mu S = m^2\phi^{b-a} - \alpha m\phi^{-a}(e^{\ell Q} - 1)$

In (2.21) the scalar curvature and the term $\nabla^\mu S \nabla_\mu S$ can be eliminated using (2.22) and (A43); further on using the matter Lagrangian and the definition of the EM tensor one has

$$(2.23) \quad (2\omega - 3)\square\phi = (a + 1)\rho\alpha(e^{\ell Q} - 1) - 2\Lambda(1 + Q)^c + 2\Lambda\phi - \frac{2}{\phi}\nabla_\mu Q \nabla^\mu Q$$

(where $b = a + 1$). Solving (A41) and (2.23) with a perturbation expansion in α one finds

$$(2.24) \quad Q = Q_0 + \alpha Q_1 + \dots; \quad \phi = 1 + \alpha Q_1 + \dots; \quad \sqrt{\rho} = \sqrt{\rho_0} + \alpha\sqrt{\rho_1} + \dots$$

where the conformal factor is chosen to be unity at zeroth order so that as $\alpha \rightarrow 0$ (A43) goes to the classical HJ equation. Further since by (A43) the quantum mass is $m^2\phi + \dots$ the first order term in ϕ is chosen to be Q_1 (cf. (2.6)). Also we will see that $Q_1 \sim \square\sqrt{\rho}/\sqrt{\rho}$ plus corrections which is in accord with Q as a quantum potential field. In any case after some computation one obtains (A44) $a = 2\omega k$, $b = a + 1$, and $\ell = (1/4)(2\omega k + 1) = (1/4)(a + 1) = b/4$ with $Q_0 = [1/c(2c - 3)]\{[-(2\omega k + 1)/2\Lambda]k\sqrt{\rho_0} - (2c^2 - c + 1)\}$ while ρ_0 can be determined (cf. [108] for details). Thus heuristically the quantum potential can be regarded as a dynamical field and perturbatively one gets the correct dependence of quantum potential upon density, modulo some corrective terms.

One goes next to a number of examples and we only consider here the conformally flat solution (cf. also [104]). Thus take (A45) $g_{\mu\nu} = \exp(2\Sigma)\eta_{\mu\nu}$ where $\Sigma \ll 1$. One obtains from (2.16) (A46) $\mathcal{R}_{\mu\nu} = \eta_{\mu\nu}\square\Sigma + 2\partial_\mu\partial_\nu\Sigma \Rightarrow \mathfrak{G}_{\mu\nu} = 2\partial_\mu\partial_\nu\Sigma - 2\eta_{\mu\nu}\square\Sigma$. One can solve iteratively to get

$$(2.25) \quad \mathcal{R}^{(0)} = -\kappa\mathfrak{I} \Rightarrow \Sigma^{(0)} = -\frac{\kappa}{6}\square^{-1}\mathfrak{I};$$

$$\mathcal{R}^{(1)} = -\kappa\mathfrak{I} + 3\alpha\square\frac{\square\sqrt{|\mathfrak{I}|}}{\sqrt{|\mathfrak{I}|}} \Rightarrow \Sigma^{(1)} = -\frac{\kappa}{6}\square^{-1}\mathfrak{I} + \frac{\alpha}{2}\frac{\square\sqrt{|\mathfrak{I}|}}{\sqrt{|\mathfrak{I}|}}$$

Consequently

$$(2.26) \quad \Sigma = -\frac{\kappa}{6}\square^{-1}\mathfrak{I} + \frac{\alpha}{2}\frac{\square\sqrt{|\mathfrak{I}|}}{\sqrt{|\mathfrak{I}|}} + \dots$$

The first term is pure gravity, the second pure quantum, and the remaining terms involve gravity-quantum interactions. A number of impressive examples are given (cf. also [104]).

One goes now to a generalized equivalence principle. The gravitational effects determine the causal structure of spacetime as long as quantum effects give its conformal structure. This does not mean that quantum effects have nothing to do with the causal structure; they can act on the causal structure through back reaction terms appearing in the metric field equations. The conformal factor of the metric is a function of the quantum potential and the mass of a relativistic particle is a field produced by quantum corrections to the classical mass. One has shown that the presence of the quantum potential is equivalent to a conformal mapping of the metric. Thus in different conformally related frames one feels different quantum masses and different curvatures. In particular there are two frames with one containing the quantum mass field and the classical metric while the other contains the classical mass and the quantum metric. In general frames both the spacetime metric and

the mass field have quantum properties so one can state that different conformal frames are identical pictures of the gravitational and quantum phenomena. We feel different quantum forces in different conformal frames. The question then arises of whether the geometrization of quantum effects implies conformal invariance just as gravitational effects imply general coordinate invariance. One sees here that Weyl geometry provides additional degrees of freedom which can be identified with quantum effects and seems to create a unified geometric framework for understanding both gravitational and quantum forces. Some features here are: (i) Quantum effects appear independent of any preferred length scale. (ii) The quantum mass of a particle is a field. (iii) The gravitational constant is also a field depending on the matter distribution via the quantum potential (cf. [102, 109]). (iv) A local variation of matter field distribution changes the quantum potential acting on the geometry and alters it globally; the nonlocal character is forced by the quantum potential (cf. [103]).

2.1. DIRAC-WEYL ACTION. Next (still following [108]) one goes to Weyl geometry based on the Weyl-Dirac action

$$(2.27) \quad \mathfrak{A} = \int d^4x \sqrt{-g} (F_{\mu\nu} F^{\mu\nu} - \beta^2 {}^W \mathcal{R} + (\sigma + 6) \beta_{;\mu} \beta^{;\mu} + \mathfrak{L}_{matter})$$

Here $F_{\mu\nu}$ is the curl of the Weyl 4-vector ϕ_μ , σ is an arbitrary constant and β is a scalar field of weight -1 . The “;” represent covariant derivatives under general coordinate and conformal transformations (Weyl covariant derivative) defined as (A47) $X_{;\mu} = {}^W \nabla_\mu X - \mathcal{N} \phi_\mu X$ where \mathcal{N} is the Weyl weight of X . The equations of motion are then

$$(2.28) \quad \begin{aligned} \mathfrak{G}^{\mu\nu} = & -\frac{8\pi}{\beta^2} (\mathfrak{T}^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta} (g^{\mu\nu} {}^W \nabla^\alpha {}^W \nabla_\alpha \beta - {}^W \nabla^\mu {}^W \nabla^\nu \beta) + \\ & + \frac{1}{\beta^2} (4 \nabla^\mu \beta \nabla^\nu \beta - g^{\mu\nu} \nabla^\alpha \beta \nabla_\alpha \beta) + \frac{\sigma}{\beta^2} (\beta^{;\mu} \beta^{;\nu} - \frac{1}{2} g^{\mu\nu} \beta^{;\alpha} \beta_{;\alpha}); \\ {}^W \nabla_\mu F^{\mu\nu} = & \frac{1}{2} \sigma (\beta^2 \phi^\mu + \beta \nabla^\mu \beta) + 4\pi J^\mu; \quad \mathcal{R} = -(\sigma + 6) \frac{{}^W \square \beta}{\beta} + \sigma \phi_\alpha \phi^\alpha - \sigma {}^W \nabla^\alpha \phi_\alpha + \frac{\psi}{2\beta} \end{aligned}$$

where (A48) $M^{\mu\nu} = (1/4\pi)[(1/4)g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - F_\alpha^\mu F^{\nu\alpha}]$ and

$$(2.29) \quad 8\pi \mathfrak{T}^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} \mathfrak{L}_{matter}}{\delta g_{\mu\nu}}; \quad 16\pi J^\mu = \frac{\delta \mathfrak{L}_{matter}}{\delta \phi_\mu}; \quad \psi = \frac{\delta \mathfrak{L}_{matter}}{\delta \beta}$$

For the equations of motion of matter and the trace of the EM tensor one uses invariance of the action under coordinate and gauge transformations, leading to

$$(2.30) \quad {}^W \nabla_\nu \mathfrak{T}^{\mu\nu} - \mathfrak{T} \frac{\nabla^\mu \beta}{\beta} = J_\alpha \phi^{\alpha\mu} - \left(\phi^\mu + \frac{\nabla^\mu \beta}{\beta} \right) {}^W \nabla_\alpha J^\alpha; \quad 16\pi \mathfrak{T} - 16\pi {}^W \nabla_\mu J^\mu - \beta \psi = 0$$

The first relation is a geometrical identity (Bianchi identity) and the second shows the mutual dependence of the field equations. Note that in the Weyl-Dirac theory the Weyl vector does not couple to spinors so ϕ_μ cannot be interpreted as the EM potential; the Weyl vector is used as part of the spacetime geometry and the auxillary field (gauge field) β represents the quantum mass field. The gravity fields $g_{\mu\nu}$ and ϕ_μ and the quantum mass field determine the spacetime geometry. Now one constructs a Bohmian quantum gravity which is conformally invariant in the framework of Weyl geometry. If the model has mass this must be a field (since mass has non-zero Weyl weight). The Weyl-Dirac action is a

general Weyl invariant action as above and for simplicity now assume the matter Lagrangian does not depend on the Weyl vector so that $J_\mu = 0$. The equations of motion are then

$$(2.31) \quad \begin{aligned} \mathfrak{G}^{\mu\nu} = & -\frac{8\pi}{\beta^2}(\mathfrak{T}^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta}(g^{\mu\nu}W\nabla^\alpha W\nabla_\alpha\beta - W\nabla^\mu W\nabla^\nu\beta) + \\ & + \frac{1}{\beta^2}(4\nabla^\mu\beta\nabla^\nu\beta - g^{\mu\nu}\nabla^\alpha\beta\nabla_\alpha\beta) + \frac{\sigma}{\beta^2}\left(\beta^{;\mu}\beta^{;\nu} - \frac{1}{2}g^{\mu\nu}\beta^{;\alpha}\beta_{;\alpha}\right); \\ {}^W\nabla_\nu F^{\mu\nu} = & \frac{1}{2}\sigma(\beta^2\phi^\mu + \beta\nabla^\mu\beta); \quad \mathcal{R} = -(\sigma + 6)\frac{{}^W\Box\beta}{\beta} + \sigma\phi_\alpha\phi^\alpha - \sigma{}^W\nabla^\alpha\phi_\alpha + \frac{\psi}{2\beta} \end{aligned}$$

The symmetry conditions are **(A49)** ${}^W\nabla_\nu\mathfrak{T}^{\mu\nu} - \mathfrak{T}(\nabla^\mu\beta/\beta) = 0$ and $16\pi\mathfrak{T} - \beta\psi = 0$ (recall $\mathfrak{T} = \mathfrak{T}_{\mu\nu}^{\mu\nu}$). One notes that from (2.31) results **(A50)** ${}^W\nabla_\mu(\beta^2\phi^\mu + \beta\nabla^\mu\beta) = 0$ (since $F^{\mu\nu} = 0$ - cf. **(B10b)**) so ϕ_μ is not independent of β . To see how this is related to the Bohmian quantum theory one introduces a quantum mass field and shows it is proportional to the Dirac field. Thus using (2.31) and **(A49)** one has

$$(2.32) \quad \Box\beta + \frac{1}{6}\beta\mathcal{R} = \frac{4\pi}{3}\frac{\mathfrak{T}}{\beta} + \sigma\beta\phi_\alpha\phi^\alpha + 2(\sigma - 6)\phi^\gamma\nabla_\gamma\beta + \frac{\sigma}{\beta}\nabla^\mu\beta\nabla_\mu\beta$$

This can be solved iteratively via **(A51)** $\beta^2 = (8\pi\mathfrak{T}/\mathcal{R}) - \{1/[(\mathcal{R}/6) - \sigma\phi_\alpha\phi^\alpha]\}\beta\Box\beta + \dots$. Now assuming $\mathfrak{T}^{\mu\nu} = \rho u^\mu u^\nu$ (dust with $\mathfrak{T} = \rho$) we multiply **(A49)** by u_μ and sum to get **(A52a)** ${}^W\nabla_\nu(\rho u^\nu) - \rho(u_\mu\nabla^\mu\beta/\beta) = 0$. Then put **(A49)** into **(A52a)** which yields **(A52b)** $u^\nu{}^W\nabla_\nu u^\mu = (1/\beta)(g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu\beta$. To see this write (assuming $g^{\mu\nu}\nabla_\nu\beta = \nabla^\mu\beta$)

$$(2.33) \quad \begin{aligned} {}^W\nabla_\nu(\rho u^\mu u^\nu) = & u^\mu{}^W\nabla_\nu\rho u^\mu + \rho u^\nu{}^W\nabla_\nu u^\mu \Rightarrow \\ \Rightarrow u^\mu \left(\frac{u_\mu\nabla^\mu\beta}{\beta} \right) + & u^\nu{}^W\nabla_\nu u^\mu - \frac{\nabla^\mu\beta}{\beta} = 0 \Rightarrow u^\nu{}^W\nabla_\nu u^\mu = (1 - u^\mu u_\mu)\frac{\nabla^\mu\beta}{\beta} = \\ & (g^{\mu\nu} - u^\mu u_\nu g^{\mu\nu})\frac{\nabla_\nu\beta}{\beta} = (g^{\mu\nu} - u^\mu u^\nu)\frac{\nabla_\nu\beta}{\beta} \end{aligned}$$

which is **(A52b)**. Then from **(A51)**

$$(2.34) \quad \beta^{2(1)} = \frac{8\pi\mathfrak{T}}{\mathcal{R}}; \quad \beta^{2(2)} = \frac{8\pi\mathfrak{T}}{\mathcal{R}} \left(1 - \frac{1}{(\mathcal{R}/6) - \sigma\phi_\alpha\phi^\alpha} \frac{\Box\sqrt{\mathfrak{T}}}{\sqrt{\mathfrak{T}}} \right); \dots$$

Comparing with (2.7) and (2.2) shows that we have the correct equations for the Bohmian theory provided one identifies

$$(2.35) \quad \beta \sim \mathfrak{M}; \quad \frac{8\pi\mathfrak{T}}{\mathcal{R}} \sim m^2; \quad \frac{1}{\sigma\phi_\alpha\phi^\alpha - (\mathcal{R}/6)} \sim \alpha$$

Thus β is the Bohmian quantum mass field and the coupling constant α (which depends on \hbar) is also a field, related to geometrical properties of spacetime. One notes that the quantum effects and the length scale of the spacetime are related. To see this suppose one is in a gauge in which the Dirac field is constant; apply a gauge transformation to change this to a general spacetime dependent function, i.e. **(A53)** $\beta = \beta_0 \rightarrow \beta(x) = \beta_0 \exp(-\Xi(x))$ via $\phi_\mu \rightarrow \phi_\mu + \partial_\mu\Xi$. Thus the gauge in which the quantum mass is constant (and the quantum force is zero) and the gauge in which the quantum mass is spacetime dependent are related to one another via a scale change. In particular ϕ_μ in the two gauges differ by $-\nabla_\mu(\beta/\beta_0)$

and since ϕ_μ is a part of Weyl geometry and the Dirac field represents the quantum mass one concludes that the quantum effects are geometrized (cf. also (2.31) which shows that ϕ_μ is not independent of β so the Weyl vector is determined by the quantum mass and thus the geometrical aspect of the manifold is related to quantum effects).

3. BACKGROUND

We give now some background for the last section based on [1, 3, 4, 5, 13, 18, 19, 21, 22, 23, 37, 43, 46, 47, 49, 64, 66, 72, 73, 80, 83, 84, 87, 89, 90, 91, 92, 93, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 113, 118, 120].

3.1. WEYL GEOMETRY AND ELECTROMAGNETISM. First we give some background on Weyl geometry and Brans-Dicke theory following [2]; for differential geometry we use the tensor notation of [2] and refer to e.g. [12, 48, 59, 77, 78, 114, 117] for other notation (see also [116] for an interesting variation). One thinks of a differential manifold $M = \{U_i, \phi_i\}$ with $\phi : U_i \rightarrow \mathbf{R}^4$ and metric $g \sim g_{ij}dx^i dx^j$ satisfying $g(\partial_k, \partial_\ell) = g_{k\ell} = \langle \partial_k, \partial_\ell \rangle = g_{\ell k}$. This is for the bare essentials; one can also imagine tangent vectors $X_i \sim \partial_i$ and dual cotangent vectors $\theta^i \sim dx^i$, etc. Given a coordinate change $\tilde{x}^i = \tilde{x}^i(x^j)$ a vector ξ^i transforming via **(B1)** $\tilde{\xi}^i = \sum \partial_i \tilde{x}^j \xi^j$ is called contravariant (e.g. $d\tilde{x}^i = \sum \partial_j \tilde{x}^i dx^j$). On the other hand $\partial\phi/\partial\tilde{x}^i = \sum (\partial\phi/\partial x^j)(\partial x^j/\partial\tilde{x}^i)$ leads to the idea of covariant vectors $A_j \sim \partial\phi/\partial x^j$ transforming via **(B2)** $\tilde{A}_i = \sum (\partial x^j/\partial\tilde{x}^i) A_j$ (i.e. $\partial/\partial\tilde{x}^i \sim (\partial x^j/\partial\tilde{x}^i)\partial/\partial x^j$). Now define connection coefficients or Christoffel symbols via (strictly one writes $T^\gamma_\alpha = g_{\alpha\beta}T^{\gamma\beta}$ and $T_\alpha^\gamma = g_{\alpha\beta}T^{\beta\gamma}$ which are generally different; we use that notation here but it is not used in subsequent sections since it is unnecessary)

$$(3.1) \quad \Gamma^r_{ki} = - \left\{ \begin{matrix} r \\ k \ i \end{matrix} \right\} = -\frac{1}{2} \sum (\partial_i g_{k\ell} + \partial_k g_{\ell i} - \partial_\ell g_{ik}) g^{\ell r} = \Gamma^r_{ik}$$

(note this differs by a minus sign from some other authors). Note also that (3.1) follows from equations

$$(3.2) \quad \partial_\ell g_{ik} + g_{rk} \Gamma^r_{i\ell} + g_{ir} \Gamma^r_{\ell k} = 0$$

and cyclic permutation; the basic definition of Γ^i_{mj} is found in the transplantation law **(B3)** $d\xi^i = \Gamma^i_{mj} dx^m \xi^j$. Next for tensors $T^\alpha_{\beta\gamma}$ define derivatives **(B4)** $T^\alpha_{\beta\gamma|k} = \partial_k T^\alpha_{\beta\gamma}$ and

$$(3.3) \quad T^\alpha_{\beta\gamma||\ell} = \partial_\ell T^\alpha_{\beta\gamma} - \Gamma^\alpha_{\ell s} T^s_{\beta\gamma} + \Gamma^s_{\ell\beta} T^\alpha_{s\gamma} + \Gamma^s_{\ell\gamma} T^\alpha_{\beta s}$$

In particular covariant derivatives for contravariant and covariant vectors respectively are defined via **(B5)** $\xi^i_{|k} = \partial_k \xi^i - \Gamma^i_{k\ell} \xi^\ell = \nabla_k \xi^i$ and $\eta_{m||\ell} = \partial_\ell \eta_m + \Gamma^r_{m\ell} \eta_r = \nabla_\ell \eta_m$. Now to describe Weyl geometry one notes first that for Riemannian geometry **(B3)** holds along with **(B6)** $\ell^2 = \|\xi\|^2 = g_{\alpha\beta} \xi^\alpha \xi^\beta$ and a scalar product formula $\xi^\alpha \eta_\alpha = g_{\alpha\beta} \xi^\alpha \eta^\beta$. Now however one does not demand conservation of lengths and scalar products under affine transplantation **(B3)**. Thus assume **(B7)** $d\ell = (\phi_\beta dx^\beta) \ell$ where the covariant vector ϕ_β plays a role analogous to $\Gamma^\alpha_{\beta\gamma}$. Combining **(B7)** with **(B3)** and **(B6)** one obtains

$$(3.4) \quad \begin{aligned} d\ell^2 &= 2\ell^2(\phi_\beta dx^\beta) = d(g_{\alpha\beta} \xi^\alpha \xi^\beta) = \\ &= g_{\alpha\beta|\gamma} \xi^\alpha \xi^\beta dx^\gamma + g_{\alpha\beta} \Gamma^\alpha_{\rho\gamma} \xi^\rho \xi^\beta dx^\gamma + g_{\alpha\beta} \Gamma^\beta_{\rho\gamma} \xi^\alpha \xi^\rho dx^\gamma \end{aligned}$$

Rearranging etc. and using **(B6)** again gives **(B8)** $(g_{\alpha\beta|\gamma} - 2g_{\alpha\beta}\phi_\gamma) + g_{\sigma\beta}\Gamma_{\alpha\gamma}^\sigma + g_{\sigma\alpha}\Gamma_{\beta\gamma}^\sigma = 0$ leading to

$$(3.5) \quad \Gamma_{\beta\gamma}^\alpha = - \left\{ \begin{array}{c} \alpha \\ \beta \ \gamma \end{array} \right\} + g^{\sigma\alpha}[g_{\sigma\beta}\phi_\gamma + g_{\sigma\gamma}\phi_\beta - g_{\beta\gamma}\phi_\sigma]$$

Thus we can prescribe the metric $g_{\alpha\beta}$ and the covariant vector field ϕ_γ and determine by (3.5) the field of connection coefficients $\Gamma_{\beta\gamma}^\alpha$ which admits the affine transplantation law **(B3)**. If one takes $\phi_\gamma = 0$ the Weyl geometry reduces to Riemannian geometry. This leads one to consider new metric tensors via **(B9)** $\hat{g}_{\alpha\beta} = f(x^\lambda)g_{\alpha\beta}$ and it turns out that $(1/2)\partial\log(f)/\partial x^\lambda$ plays the role of ϕ_λ in **(B7)**. Here **(B9)** is called a gauge transformation and the ordinary connections $\left\{ \begin{array}{c} \alpha \\ \beta \ \gamma \end{array} \right\}$ constructed from $g_{\alpha\beta}$ are equal to the more general connections $\hat{\Gamma}_{\beta\gamma}^\alpha$ constructed according to (3.5) from $\hat{g}_{\alpha\beta}$ and $\hat{\phi}_\lambda = (1/2)\partial\log(f)/\partial x^\lambda$. The generalized differential geometry is conformal in that the ratio

$$(3.6) \quad \frac{\xi^\alpha\eta_\alpha}{\|\xi\|\|\eta\|} = \frac{g_{\alpha\beta}\xi^\alpha\eta^\beta}{[(g_{\alpha\beta}\xi^\alpha\xi^\beta)(g_{\alpha\beta}\eta^\alpha\eta^\beta)]^{1/2}}$$

does not change under the gauge transformation **(B9)**. Again if one has a Weyl geometry characterized by $g_{\alpha\beta}$ and ϕ_α with connections determined by (3.5) one may replace the geometric quantities by use of a scalar field f with **(B10a)** $\hat{g}_{\alpha\beta} = f(x^\lambda)g_{\alpha\beta}$, $\hat{\phi}_\alpha = \phi_\alpha + (1/2)(\log(f))_{|\alpha}$ and $\hat{\Gamma}_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha$ without changing the intrinsic geometric properties of vector fields; the only change is that of local lengths of a vector via $\hat{\ell}^2 = f(x^\lambda)\ell^2$. Note that one can reduce $\hat{\phi}_\alpha$ to the zero vector field if and only if ϕ_α is a gradient field, namely $F_{\alpha\beta} = \phi_{\alpha|\beta} - \phi_{\beta|\alpha} = 0$ (i.e. $\phi_\alpha = (1/2)\partial_a\log(f) \equiv \partial_\beta\phi_\alpha = \partial_\alpha\phi_\beta$). In this case one has length preservation after transplantation around an arbitrary closed curve and the vanishing of $F_{\alpha\beta}$ guarantees a choice of metric in which the Weyl geometry becomes Riemannian; thus $F_{\alpha\beta}$ is an intrinsic geometric quantity for Weyl geometry (note $F_{\alpha\beta} = -F_{\beta\alpha}$ and **(B10b)** $\{F_{\alpha\beta|\gamma}\} = 0$ where $\{F_{\mu\nu|\lambda}\} = F_{\mu\nu|\lambda} + F_{\lambda\mu|\nu} + F_{\nu\lambda|\mu}$). Similarly the concept of covariant differentiation depends only on the idea of vector transplantation. Indeed one can define **(B11)** $\xi_{||\beta}^\alpha = \xi_{|\beta}^\alpha - \Gamma_{\beta\gamma}^\alpha\xi^\gamma$. In Riemann geometry the curvature tensor is **(B12)** $\xi_{||\beta|\gamma}^\alpha - \xi_{||\gamma|\beta}^\alpha = R_{\eta\beta\gamma}^\alpha\xi^\eta$, Hence here we can write **(B13)** $R_{\beta\gamma\delta}^\alpha = -\Gamma_{\beta\gamma|\delta}^\alpha + \Gamma_{\beta\delta|\gamma}^\alpha + \Gamma_{\tau\delta}^\alpha\Gamma_{\beta\gamma}^\tau - \Gamma_{\tau\gamma}^\alpha\Gamma_{\beta\delta}^\tau$. Using (3.6) one then can express this in terms of $g_{\alpha\beta}$ and ϕ_α but this is complicated. Equations for $R_{\beta\delta} = R_{\beta\alpha\delta}^\alpha$ and $R = g^{\beta\delta}R_{\beta\delta}$ are however given in [2]. One notes that in Weyl geometry if a vector ξ^α is given, independent of the metric, then $\xi_\alpha = g_{\alpha\beta}\xi^\beta$ will depend on the metric and under a gauge transformation one has $\hat{\xi}_\alpha = f(x^\lambda)\xi_\alpha$. Hence the covariant form of a gauge invariant contravariant vector becomes gauge dependent and one says that a tensor is of weight n if, under a gauge transformation **(B14)**, $\hat{T}_{\beta\dots}^{\alpha\dots} = f(x^\lambda)^n T_{\beta\dots}^{\alpha\dots}$. Note ϕ_α plays a singular role in **(B10a)** and has no weight. Similarly **(B15)** $\sqrt{-\hat{g}} = f^2\sqrt{-g}$ (weight 2) and $F^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}$ has weight -2 while **(B16)** $\mathfrak{F}^{\alpha\beta} = F^{\alpha\beta}\sqrt{-g}$ has weight 0 and is gauge invariant. Similarly $F_{\alpha\beta}F^{\alpha\beta}\sqrt{-g}$ is gauge invariant. Now for Weyl's theory of electromagnetism one wants to interpret ϕ_α as an EM potential and one has automatically the Maxwell equations **(B17)** $\{F_{\alpha\beta|\gamma}\} = 0$ along with a gauge invariant complementary

set **(B18)** $\mathfrak{F}_{|\beta}^{\alpha\beta} = \mathfrak{s}^\alpha$ (source equations). These equations are gauge invariant as a natural consequence of the geometric interpretation of the EM field. For the interaction between the EM and gravitational fields one sets up some field equations as indicated in [2] and the interaction between the metric quantities and the EM fields is exhibited there.

REMARK 3.1. As indicated earlier in [2] R^i_{jk} is defined with a minus sign compared with e.g. [77, 117]. There is also a difference in definition of the Ricci tensor which is taken to be $G^{\beta\delta} = R^{\beta\delta} - (1/2)g^{\beta\delta}R$ in [2] with $R = R_{\delta}^{\delta}$ so that **(B19)** $G_{\mu\gamma} = g_{\mu\beta}g_{\gamma\delta}G^{\beta\delta} = R_{\mu\gamma} - (1/2)g_{\mu\gamma}R$ with $G_{\eta}^{\gamma} = R_{\eta}^{\gamma} - 2R \Rightarrow G_{\eta}^{\eta} = -R$ (recall $n = 4$). In [77] the Ricci tensor is simply **(B20)** $R_{\beta\mu} = R_{\beta\mu\alpha}^{\alpha}$ where $R_{\beta\mu\nu}^{\alpha}$ is the Riemann curvature tensor and $R = R_{\eta}^{\eta}$ again. This is similar to [117] where the Ricci tensor is defined as $\rho_{jl} = R^i_{jil}$. To clarify all this we note that **(B21)** $R_{\eta\gamma} = R_{\eta\alpha\gamma}^{\alpha} = g^{\alpha\beta}R_{\beta\eta\alpha\gamma} = -g^{\alpha\beta}R_{\beta\eta\gamma\alpha} = -R_{\eta\gamma\alpha}^{\alpha}$ which confirms the minus sign difference. \blacksquare

3.2. CONFORMAL GRAVITY THEORY. We extract here first from [84] with some embellishments (cf. also [83, 97]). The development in [83, 84, 85, 86] is quite exhaustive and we try to capture the spirit here (although probably providing too many details for a proper survey). However we want to make the treatment extensive enough to stimulate comparison with the deBroglie-Bohm-Weyl (dBBW) theory and to exhibit the relations between Riemannian and Weyl geometry. Although the Jordan frame (JF) and the Einstein frame (EF) formulations of a scalar tensor theory provide mathematically equivalent descriptions of the same physics the physical equivalence is still under discussion (cf. [47] for a discussion of this). There is apparently not even agreement about which should be the physical frame and this is especially true if one allows quantum effects to influence the metric (cf. Section 2). The JF Lagrangian for Brans-Dicke (BD) type theories is **(B22)** $L_{BD} = (\sqrt{-g}/16\pi)(\phi R - (\omega/\phi)(\nabla\phi)^2)$ where R is the Ricci scalar of the JF metric g , ϕ is the BD scalar field and ω is the BD coupling constant (a free parameter). Under the rescaling **(B23)** $\hat{g}_{ab} = \phi g_{ab}$ and the scalar field redefinition $\hat{\phi} = \log(\phi)$ the JF Lagrangian for BD type theory is mapped into the EF Lagrangian for BD type theory, namely **(B24)** $L_E = (\sqrt{-g}/16\pi)(\hat{R} - (\omega + (3/2))(\hat{\nabla}\hat{\phi})^2)$ where \hat{R} is the curvature scalar in terms of the EF metric \hat{g} . Inserting matter involves minimal coupling to the metric in JF theory via **(B25)** $L_{JF} = (\sqrt{-g}/16\pi)(\phi R - (\omega/\phi)(\nabla\phi)^2) + L_{matter}$ and this is the JF formulation of BD theory. For the EF one couples matter minimally to the metric via **(B26)** $L_E = (\sqrt{-g}/16\pi)(\hat{R} - (\omega + (3/2))(\hat{\nabla}\hat{\phi})^2) + L_{matter}$. Here the scalar field $\hat{\phi}$ is minimally coupled to curvature so the dimensional gravitational constant G is a real constant. Due to the minimal coupling between ordinary matter and the spacetime metric the rest mass of any test particle m is also constant over the manifold. This leads to a real dimensionless gravitational coupling constant Gm^2 (for $\hbar = c = 1$) unlike BD theory where $Gm^2 \sim \phi^{-1}$. The equations derivable from **(B26)** are $(\hat{G}_{ab} = \hat{R}_{ab} - (1/2)\hat{g}_{ab}\hat{R})$

$$(3.7) \quad \hat{G}_{ab} = 8\pi\hat{T}_{ab} + \left(\omega + \frac{3}{2}\right)(\hat{\nabla}_a\hat{\phi}\hat{\nabla}_b\hat{\phi}) - \frac{1}{2}\hat{g}_{ab}(\hat{\nabla}\hat{\phi})^2; \quad \square\hat{\phi} = 0; \quad \hat{\nabla}_n\hat{T}^{na} = 0$$

Here $\hat{T}_{ab} = (2/\sqrt{-g})[\partial(\sqrt{-g}L_{matter})/\partial\hat{g}^{ab}]$. The theory given by (3.7) is just the Einstein theory of general relativity with an additional matter source of gravity. For $\hat{\phi} = c$ or $\omega =$

$-3/2$ one recovers the standard theory and this is linked with Riemannian geometry because the test particles follow the geodesics of \hat{g} via **(B27)** $d^2x^a/d\hat{s}^2 = -\hat{\Gamma}_{mn}^a(dx^m/d\hat{s})(dx^n/d\hat{s})$ where $\hat{\Gamma}_{bc}^a = (1/2)\hat{g}^{an}(\hat{g}_{bn|c} + \hat{g}_{cn|b} - \hat{g}_{bc|n})$. Recall that Riemannian geometry is based on the parallel transport law **(B28)** $d\xi^a = -\hat{\gamma}_{mn}^a\xi^m dx^n$ with $d\hat{g}(\xi, \xi) = 0$; this formulation leads to $\hat{\gamma}_{bc}^a = \hat{\Gamma}_{bc}^a$ (cf. Section 3.1). Under the conformal transformation **(B23)** the Lagrangian **(B25)** is mapped into the EF Lagrangian for BD theory, namely

$$(3.8) \quad L_{BD} = \frac{\sqrt{-\hat{g}}}{16\pi} \left(\hat{R} - \left(\omega + \frac{3}{2} \right) (\hat{\nabla}\hat{\phi})^2 \right) + e^{-2\hat{\phi}} L_{matter}$$

while **(B26)** is mapped into the JF Lagrangian

$$(3.9) \quad L_{JF} = \frac{\sqrt{-g}}{16\pi} (\phi R + \phi^{-1}(\nabla\phi)^2) + \phi^2 L_{matter}$$

At the same time under **(B22)** the parallel transport law **(B28)** is mapped into **(B29)** $d\xi^a = -\gamma_{mn}^a\xi^m dx^n$ where $\gamma_{bc}^a = \Gamma_{bc}^a + (1/2)\phi^{-1}(\nabla_b\phi\delta_c^b + \nabla_c\phi\delta_b^a - \nabla^a\phi g_{bc})$ are the affine connections of a Weyl type manifold. Weyl type geometry is given by the law **(B24)** along with **(B30)** $dg(\xi, \xi) = \phi^{-1}dx^n\nabla_n\phi g(\xi, \xi)$ which is equivalent to **(B28)** with respect to the conformal transformation **(B22)**. This means that the JF formulation of GR should be linked with a Weyl type geometry with units of measure varying length over the manifold according to **(B30)**. In the JF GR the gravitational constant G varies like ϕ^{-1} while the rest masses of material particles m vary like $\phi^{1/2}$ (i.e. $Gm^2 = c$ is preserved). One has now two equivalent geometrical representations of the same physical theory (according to one point of view) and we see no reason to argue with this (see the discussion below). The field equations of the JF theory are now

$$(3.10) \quad G_{ab} = \frac{8\pi}{\phi} T_{ab} + \frac{\omega}{\phi^2} \left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}g^{nm}\nabla_n\phi\nabla_m\phi \right) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\square\phi)$$

along with $\square\phi = 0$ where $T_{ab} = (2/\sqrt{-g})\partial(\sqrt{-g}\phi^2 L_{matter})/\partial g^{ab}$ is the stress energy tensor for ordinary matter in the JF. The energy is not conserved since ϕ exchanges energy with the metric and matter fields and the corresponding dynamic equation is **(B31)** $\nabla_n T^{na} = (1/2)\phi^{-1}\nabla^a\phi T$. The equations of motion of an uncharged spinless mass point acted upon by the JF metric field g and by ϕ is

$$(3.11) \quad \frac{d^2x^a}{ds^2} = -\Gamma_{mn}^a \frac{dx^m}{ds} \frac{dx^n}{ds} - \frac{1}{2}\phi^{-1}\nabla_n\phi \left(\frac{dx^n}{ds} \frac{dx^a}{ds} - g^{an} \right)$$

and this does not coincide with the geodesic equation of the JF metric. One can also provide a new connection leading to a more canonical form of the scalar field EM tensor in the JF (cf. [97]) and this is done in another way in [84] by rewriting (3.10) in the form

$$(3.12) \quad \gamma G_{ab} = \frac{8\pi}{\phi} T_{ab} + \frac{(\omega + (3/2))}{\phi^2} (\nabla_a\phi\nabla_b\phi - (1/2)g_{ab}g^{nm}\nabla_n\phi\nabla_m\phi)$$

Again there may be energy questions but quantum input seems to render these moot (cf. Section 2). Here one is writing (3.10) in terms of affine magnitudes in the Weyl type manifold so that the JF Weyl manifold connections γ_{bc}^a do not coincide with the Christoffel symbols of the JF metric Γ_{bc}^a ; then γG_{ab} is given in terms of the γ_{bc}^a instead of the Γ_{bc}^a .

We go next to [83, 85] and consider string connections as well (cf. also [68]). Thus (from [83]) first treat **(B32)** $\hat{g}_{ab} = \Omega^2(x)g_{ab}$ as a transformation of units *and* as a conformal transformation of theory (to be further clarified via [85]). Note e.g. that $\hat{m} = \Omega^{-1}(x)m$ is not constant and consider the actions **(B33)** $S_{BD} = S = \int d^4x \sqrt{-g}(\phi R - (\omega/\phi)(\nabla\phi)^2 + 16\pi L_M)$ where $L_M \sim L_{matter}$ where R is the curvature scalar, ϕ is the BD scalar field (the dilaton), ω is the BD coupling constant, and L_M is the Lagrangian of the matter fields that are minimally coupled to the metric. Under the change of variable $\phi \rightarrow exp(\psi)$ BD theory can be written in the string frame **(B34)** $S_S = S_1 = \int d^4x \sqrt{-\hat{g}}e^\psi(R - \omega(\nabla\psi)^2 + 16\pi e^{-\psi}L_M)$. Dicke used the conformal transformation **(B32)** with **(B35)** $\Omega^2 = exp(\psi)$ to rewrite the action S_1 (or S_{BD}) in the EF, i.e. in a frame where the dilaton is minimally coupled to the curvature, namely **(B36)** $S_E = S_2 = \int d^4x \sqrt{-\hat{g}}(\hat{R} - (\omega + (3/2))(\hat{\nabla}\psi)^2 + 16\pi e^{-2\psi}L_M)$. where \hat{R} is the curvature scalar in terms of \hat{g}_{ab} and the matter fields are now non-minimally coupled to the dilaton ψ . Another effective theory of gravity of BD type was proposed in [68] where there was minimal coupling of the matter fields in the EF, namely **(B37)** $S_3 = \int d^4x \sqrt{-g}(R - \alpha(\nabla\psi)^2 + 16\pi L_M)$ where $\alpha = \omega + (3/2)$. Evidently **(B37)** is just the canonical action of GR with an extra scalar (dilaton) and when $\alpha = 0$ or $\psi = c$ one recovers GR in the Einstein formulation. Now under the conformal transformation **(B32)** - **(B35)** the action **(B37)** can be written in a string frame, namely **(B38)** $S_4 = \int d^4x \sqrt{-\hat{g}}e^{-\psi}(\hat{R} - (\alpha - (3/2))(\hat{\nabla}\psi)^2 + 16\pi e^{-\psi}L_M)$ (different from S_1 in **(B34)**). The theory derivable from **(B38)** is then called conformal GR or string-frame GR and in [83] one selects from the S_i those which provide a physically meaningful formulation of the laws of gravity (i.e. a theory of gravity which is invariant under transformations of the units of measure). This is also elaborated below following [85]. Now according to GR (as in **(B37)** with $\alpha = 0$ or $\psi = c$) the structure of physical spacetime corresponds to that of a Riemannian manifold and in general in theories with minimal coupling of the matter to the metric are naturally linked with Riemannian manifolds. In fact in theories with matter of the form **(B39)** $S_M = 16\pi \int d^4x \sqrt{-g}L_M$ the timelike matter particles follow free motion paths (geodesics) which are solutions of

$$(3.13) \quad \frac{d^2x^a}{ds^2} + \left\{ \begin{array}{c} a \\ b \ c \end{array} \right\} \frac{dx^m}{dx} \frac{dx^n}{ds} = 0$$

where the Christoffel symbols are $(1/2)g^{an}g_{bn|c} + g_{cn|b} - g_{bc|n}$ based on the metric g_{ab} (cf. (3.1) and (3.2)). This situation involves the invariance of vector lengths under parallel transport, meaning that the units of measure of the geometry are point independent. This situation holds then for both string frame BD theory derivable from S_1 and EF GR derivable from S_3 where the underlying manifold is Riemannian in nature. Under conformal rescaling **(B32)** S_1 and S_3 are mapped into their conformal versions S_2 and S_4 respectively and at the same time manifolds of Riemannian structure are mapped into conformally Riemannian manifolds. Therefore in theories derivable from S_2 and S_4 we have conformally Riemannian manifolds or WIST spaces. In particular given **(B40)** $S_M = 16\pi \int d^4x \sqrt{-\hat{g}}e^{-2\psi}L_M$ the equations of motion for test particles will be

$$(3.14) \quad \frac{d^2x^a}{d\hat{s}^2} + \left\{ \begin{array}{c} a \\ m \ n \end{array} \right\} \frac{dx^m}{d\hat{s}} \frac{dx^n}{d\hat{s}} - \frac{\psi|_n}{2} \left(\frac{dx^n}{d\hat{s}} \frac{dx^a}{d\hat{s}} - \hat{g}^{na} \right) = 0$$

and these are conformal to (3.13). In this WIST geometry units of measure may change locally.

REMARK 3.2. This section is aimed at providing linkage points for connection to BW theory with standard conformal gravity and trajectory equations such as (3.12), (3.13), and (3.14) should be useful in this direction (see also [108]) for related examples. ■

Now one looks at the group of transformations of units for length, time, and mass. Instead of **(B32)** one considers the more general **(B32)** with **(B41)** $\Omega^2(x) = \exp[\sigma\psi(x)]$ where σ is a constant. This can again be interpreted as a one parameter point dependent transformation of lengths and it is interesting here to write **(B42)** $\hat{\psi} = (1 - \sigma)\psi$ under which the basic requirements of a WIST geometry are preserved and (3.14) is invariant under such transformations. However this is not true for the underlying Riemannian geometry (e.g. (3.13) is not preserved). It can be checked that the purely gravitational part of the actions S_1 and S_4 are invariant under **(B41)**, **(B42)**, and **(B43)** $\hat{\alpha} = [\alpha/(1 - \sigma)^2]$ when $\sigma \neq 1$. The purely gravitational part of the actions S_2 and S_3 is however not invariant nor is **(B39)**. On the other hand the matter action **(B40)** is invariant and can be written as **(B44)** $S_M = 16\pi \int d^4x \sqrt{-g} \exp[2(\sigma - 1)\psi]$ when $\sigma \neq 1$. Therefore if one checks the actions S_i ($i = 1, \dots, 4$) relative to invariance under **(B41)**, **(B42)**, and **(B43)** the only survivor is S_4 , namely the conformal formulation of GR (or string-frame GR). One notes also that the set of transformations indicated is an Abelian group and composition with parameters σ_1 and σ_2 involves $\sigma_3 = \sigma_1 + \sigma_2 - \sigma_1\sigma_2$ (cf. [46]). For reasons then spelled out more fully in [83] this group is referred to as the group of point dependent transformations of the units of length, time, and mass; the situation $\sigma = 1$ is not a member of this group; it is just a transformation allowing a jump from one formulation of the theory to the conformal version. The action S_4 is then claimed to be the only physically meaningful formulation of the laws of gravity in this context. This has several implications (to be discussed below); in particular string theory may find an entry into quantum gravity via this approach since quantum interaction will arise via the deBroglie-Bohm quantum potential.

We go next to [85] where the arguments above are developed further in an attempt to clarify remarks in [84, 86] and establish physical equivalence among conformally related metrics. A main point is to refute the following argument. In canonical GR the matter couples minimally to the metric that determines metrical relations on a Riemannian space-time, say \hat{g} (note the switch $g \leftrightarrow \hat{g}$ here). In this case matter particles follow the geodesics of the metric \hat{g} in Riemannian geometry and their masses are constant over the spacetime manifold (i.e. it is the metric which matter feels so it is the physical metric). Under the conformal rescaling **(B32)** the matter fields become non-minimally coupled to the conformal metric g and matter particles do not follow the geodesics of this last metric. Further it is not the metric that determines metrical relations on the manifold. Thus although canonical GR and its conformal image may be physically equivalent theories, nevertheless the physical metric is that which determines metrical relations on a Riemannian spacetime and the conformal metric is not the physical metric. It is shown that this conclusion is wrong. Indeed under the conformal rescaling not only the Lagrangian is mapped into its conformal image but the spacetime geometry itself is mapped into a conformal geometry. In this last

geometry metrical relations involve both the conformal metric g and the factor Ω^2 generating the transformation **(B32)**. Hence in the conformal Lagrangian the matter fields should feel both the metric g and the scalar function Ω ; i.e. the matter particles do not follow the geodesics of the conformal metric alone. The result is that under **(B32)** the physical metric of the untransformed geometry is effectively mapped into the physical metric of the conformal geometry. Another point involves the one parameter group of transformations of units and one shows that the only consistent formulation of the laws of gravity (among those investigated in the paper) is the conformal representation of general relativity.

Thus, with some repetition, one looks at the effect of a conformal transformation **(B32)** on the laws of gravity and on the geometry. The Lagrangian for canonical GR (with a scalar field) is **(B44)** $\hat{L}_{GR} = \sqrt{-\hat{g}}(\hat{R} - \alpha(\hat{\nabla}\hat{\phi})^2) + 16\pi\sqrt{-\hat{g}}L_M$ where \hat{R} is the Ricci scalar, $(\hat{\nabla}\hat{\phi})^2 = \hat{g}^{mn}\hat{\phi}_{|m}\hat{\phi}_{|n}$, and $\alpha \geq 0$ (again note the switch $g \leftrightarrow \hat{g}$). When $\phi = c$ or $\alpha = 0$ this is the Einstein theory. Under **(B32)** with $\Omega^2 = \exp(\hat{\phi})$ this becomes **(B45)** $L_{GR} = \sqrt{-g}(R - (\alpha - (3/2))(\nabla\hat{\phi})^2) + 16\pi\sqrt{-g}\exp[2\hat{\phi}]L_M$. This can be given the usual BD form after a change of variable $\hat{\phi} \rightarrow \phi = \exp(\hat{\phi})$, namely **(B46)** $L_{GR} = \sqrt{-g}(\phi R - (\alpha - (3/2))[(\nabla\phi)^2/\phi] + 16\pi\sqrt{-g}\phi^2L_M$. The effective gravitational constant \hat{G} (set equal to 1 in **(B44)**) is real and since the matter particles follow the geodesics of \hat{g} the inertial mass \hat{m} is constant. Thus the dimensionless coupling constant $\hat{G}\hat{m}^2$ ($c = \hbar = 1$) is constant while in conformal GR Gm^2 is also constant with $G \sim \exp(-\hat{\phi}) \sim \phi^{-1}$ and **(B47)** $m = \exp[(1/2)\hat{\phi}]\hat{m}$. On the other hand in BD theory $Gm^2 \sim \phi^{-1}$. As before one can now consider two kinds of Lagrangians for pure gravity **(B48)** $L_1 = \sqrt{-g}(R - \alpha(\nabla\phi)^2)$ and **(B49)** $L_2 = \sqrt{-g}(\phi R - (\alpha - (3/2))[(\nabla\phi)^2/\phi])$ with respect to their transformation properties under rescalings of the units. In particular one considers **(B50)** $\tilde{g} = \phi^\sigma g_{ab}$. Under **(B50)** L_1 goes to **(B51)** $\tilde{L}_1 = \sqrt{-\tilde{g}}[\phi^\sigma \tilde{R} + ((3\sigma - (3/2)\sigma^2)\phi^{-2-\sigma} - \alpha\phi^\sigma)(\tilde{\nabla}\phi)^2]$ so the laws of gravity described by L_1 change under **(B50)**. In particular in the conformal (tilde) frame the effective gravitational constant depends on ϕ due to the nonminimal coupling between the scalar field ϕ and the curvature. On the other hand L_2 is mapped into

$$(3.15) \quad \tilde{L}_2 = \sqrt{-\tilde{g}} \left[\phi^{1-\sigma} \tilde{R} - \frac{(\alpha - (3/2) - 3\sigma + (3/2)\sigma^2)}{(1-\sigma)^2} \phi^{\sigma-1} (\tilde{\nabla}\phi^{1-\sigma})^2 \right]$$

Hence introducing a new scalar field **(B52)** $\tilde{\phi} = \phi^{1-\sigma}$ and defining a new parameter **(B53)** $\tilde{\alpha} = [\alpha + 3\sigma(\sigma - 2)]/(1-\sigma)^2$ one can write

$$(3.16) \quad \tilde{L}_2 = \sqrt{-\tilde{g}} \left[\tilde{\phi} \tilde{R} - \left(\alpha - \frac{3}{2} \right) \frac{(\tilde{\nabla}\tilde{\phi})^2}{\tilde{\phi}} \right]$$

Thus the Lagrangian L_2 is invariant in form under the conformal transformation **(B50)**, the scalar field redefinition **(B52)**, and the parameter transformation **(B53)**. Such transformations are of the form indicated above with composition $\sigma_3 = \sigma_1 + \sigma_2 - \sigma_1\sigma_2$ (cf. [46]) and the identity corresponds to $\sigma = 0$ with inverse of σ being $\tilde{\sigma} = -\sigma/(1-\sigma)$ ($\sigma = 1$ is excluded as before - it is not a units transformation). Since any consistent of spacetime must be invariant under the one parameter group of units transformation (length, time, and mass) one concludes that theories for pure gravity described by L_1 are not consistent while those

based on L_2 type Lagrangians are consistent. Hence e.g. canonical GR and the EF formulation of BD theory are not consistent formulations of the laws of gravity. Consider now separately matter Lagrangians **(B54)** $\sqrt{-g}\phi^2 L_M$ and **(B55)** $\sqrt{-g}L_M$. Here **(B55)** shows minimal coupling of matter to the metric while **(B54)** has nonminimal coupling. Under **(B50)** **(B54)** goes to **(B56)** $\sqrt{-g}\phi^2 L_M = \sqrt{-\tilde{g}}\phi^{2-2\sigma} L_M$ and hence considering **(B52)** one completes the demonstration that **(B54)** is invariant in form under our one parameter group of units transformations. Unfortunately it is straightforward that **(B55)** with minimal coupling is not invariant under this group and hence BD theory (in JF formulation) based on **(B58)** $L_{BD} = L_2 + 16\pi\sqrt{-g}L_M$ is not yet a consistent theory of spacetime. The only surviving theory is the conformal GR based on **(B46)**, i.e. **(B58)** $L_{GR} = L_2 + 16\pi\phi^2 L_M$ which does provide a consistent formulation of the laws of gravity (this is BD plus nonminimal coupling). One notes that Riemannian geometry is not invariant under **(B50)** and **(B52)** so Riemannian geometry is not a consistent formulation for the interpretation of the laws of gravity whereas Weyl geometry works. Finally going to [18] one looks at the introduction of fields $\hat{\phi} = 1 + Q$ where Q is the quantum potential in an attempt to introduce the quantum force into equations of the form (3.14); this is a step in the direction of consolidating BW theory with more conventional treatments but much more is needed.

REMARK 3.3 One notes that the use of $\psi\psi^*$ automatically suggests or involves an ensemble if (or its square root) it is to be interpreted as a probability density. Thus the idea that a particle has only a probability of being at or near x seems to mean that some paths take it there but others don't and this is consistent with Feynman's use of path integrals for example. This seems also to say that there is no such thing as a particle, only a collection of versions or cloud connected to the particle idea. Bohmian theory on the other hand for a fixed energy gives a one parameter family of trajectories associated to ψ (see here [29] for details). This is because the trajectory arises from a third order differential while fixing the solution ψ of the second order stationary Schrödinger equation involves only two "boundary" conditions. As was shown in [29] this automatically generates a Heisenberg inequality $\Delta x \Delta p \geq \hbar$; i.e. the uncertainty is built in when using the wave function ψ and amazingly can be expressed by the operator theoretical framework of quantum mechanics. Thus a one parameter family of paths can be associated with the use of $\psi\psi^*$ and this generates the cloud or ensemble automatically associated with the use of ψ . ■

REMARK 3.4. In connection with [30] where differential calculi on fractals is mentioned it seems promising to consider q -calculus with q related to scale, fractal dimension, and/or power laws. This would involve a discretization but not a grid (cf. [33] for details). ■

4. CONFORMAL STRUCTURE

We extract and summarize here from various sources concerning conformal geometry, QM, Bohmian theory, etc. First we go to [6, 38, 95, 115, 116] for further sketches of Weyl geometry and will relate this to Sections 2.1 and 3.1 later (cf. also [7, 8, 11, 35, 36, 42, 59, 62, 63, 69, 71, 79, 81, 82, 92, 94, 111, 112, 114]).

4.1. DIRAC ON WEYL GEOMETRY. Historically of course [38] takes priority and it is worthwhile to reflect on the comments of a master craftsman. Thus there are two

papers on a new classical theory of the electron but since this material is not essential to our needs here we omit it. In the third paper of [38] the Dirac-Weyl action is developed (cf. also Section 2.1) and we sketch this here in some detail. The main point is to think of EM fields as a property of spacetime rather than something occurring in a gravity formed spacetime. This seems to be in the spirit of considering a microstructure of the vacuum (or an ether) and we find it attractive. The solution proposed by Weyl involved a length change **(D1)** $\delta\ell = \ell\kappa_\mu\delta x^\mu$ under parallel transport $x^\mu \rightarrow x^\mu + \delta x^\mu$. The κ_μ are field quantities occurring along with the $g_{\mu\nu}$ in a fundamental role. Suppose ℓ gets changed to $\ell' = \ell\lambda(x)$ and $\ell + \delta\ell$ becomes **(D2)** $\ell' + \delta\ell' = (\ell + \delta\ell)\lambda(x + \delta x) = (\ell + \delta\ell)\lambda(x) + \ell\lambda_{,\mu}\delta x^\mu$ with neglect of second order terms (here $\lambda_{,\mu} \equiv \partial\lambda/\partial x^\mu$). Then **(D3)** $\delta\ell' = \lambda\delta\ell + \ell\lambda_{,\mu}\delta x^\mu = \lambda(\kappa_\mu + \phi_{,\mu})\delta x^\mu$ where $\phi = \log(\lambda)$. Hence **(D4)** $\delta\ell' = \ell'\kappa'_\mu\delta x^\mu$ with $\kappa'_\mu = \kappa_\mu + \phi_{,\mu}$. If the vector is transported by parallel displacement around a small closed loop the total change in length is **(D5)** $\delta\ell = \ell F_{\mu\nu}\delta S^{\mu\nu}$ where $F_{\mu\nu} = \kappa_{\mu,\nu} - \kappa_{\nu,\mu}$ and $\delta S^{\mu\nu}$ is the element of area enclosed by the small loop. this change is unaffected by **(D4)**. It will be seen that the field quantities κ_μ can be taken to be EM potentials, subject to the transformations **(D4)**, which correspond to no change in the geometry but a change only in the choice of artificial standards of length. The derived quantities $F_{\mu\nu}$ have a geometrical meaning independent of the length standard and correspond to the EM fields. Thus the Weyl geometry provides exactly what is needed for describing both gravitational and EM fields in geometric terms. There was at first some apparent conflict with atomic standards and the theory was rejected, leaving only the idea of gauge transformation for length standard changes.

Dirac's approach however serves to help resurrect the Weyl theory; since we feel that this theory is not perhaps sufficiently appreciated a sketch is given here (cf. however [12]). Dirac first goes into a discussion of large numbers, e.g. e^2/GMm (proton and electron masses), e^2/mc^2 (age of universe), etc. and the Einsteinian theory requires that G be constant which seems in contradiction to $G \sim t^{-1}$ where t represents the epoch time, assumed to be increasing. Dirac reconciles this by assuming the large numbers hypothesis (all dimensionless large numbers are connected) and stipulating that the Einstein equations refer to an interval ds_E which is different from the interval ds_A measured by atomic clocks. Then the objections to Weyl's theory vanish and it is assumed to refer to ds_E . In this spirit then one deals with transformations of the metric gauge under which any length such as ds is multiplied by a factor $\lambda(x)$ depending on its position x , i.e. $ds' = \lambda ds$ and a localized quantity Y may get transformed according to $Y' = \lambda^n Y$, in which case Y is said to be of power n and is called a co-tensor. If $n = 0$ then Y is called an in-tensor and it is invariant under gauge transformations. The equation **(D6)** $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ shows that $g_{\mu\nu}$ is a co-tensor of power 2, since the dx^μ are not affected by a gauge transformation. Hence $g^{\mu\nu}$ is a co-tensor of power -2 and one writes $\sqrt{}$ for $\sqrt{-g}$. One writes $T_{;\mu}$ for the covariant derivative ($\nabla_\mu T$ would be better). and one notes that the covariant derivative of a co-tensor is not generally a co-tensor. However there is a modified covariant derivative $T_{*\mu}$ which is a co-tensor. Consider first a scalar S of power n ; then $S_{;\mu} = S_{,\mu} \equiv S_\mu$; under a change of gauge it transforms to **(D7)** $S'_\mu = (\lambda^n S)_{,\mu} = \lambda^n S_{,\mu} + n\lambda^{n-1}\lambda_{,\mu}S = \lambda^n[S_\mu + n(\kappa'_\mu - \kappa_\mu)S]$ (via **(D4)**). Thus **(D8)** $(S_\mu - n\kappa_\mu S)' = \lambda^n(S_\mu - n\kappa_\mu S)$ so $S_\mu - n\kappa_\mu S$ is a covector of power n and is defined to be the co-covariant derivative of S , i.e. **(D9)** $S_{*\mu} = S_\mu - n\kappa_\mu S$. To obtain the

co-covariant derivative of co-vectors and co-tensors we need a modified Christoffel symbol **(D10)** $*\Gamma_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha - g_\mu^\alpha \kappa_\nu - g_\nu^\alpha \kappa_\mu + g_{\mu\nu} \kappa^\alpha$ (the notation $\Gamma_{\mu\nu}^\alpha$ for the correct $\Gamma_{\mu\nu}^\alpha$ is also used in [38] - cf. Section 3.1). This is known to be invariant under gauge transformations. Let now A_μ be a co-vector of power n and form **(D11)** $A_{\mu,\nu} - *\Gamma_{\mu\nu}^\alpha A_\alpha$ which is evidently a tensor since it differs from the covariant derivative $A_{\mu;\nu}$ by a tensor and under gauge transformations one has (cf. **(D4)** where $\phi_{,\mu} = \kappa'_\mu - \kappa_\mu$)

$$(4.1) \quad (A_{\mu,\nu} - *\Gamma_{\mu\nu}^\alpha A_\alpha)' = \lambda^n A_{\mu,\nu} + n\lambda^{n-1} \lambda_\nu A_\mu - *\Gamma_{\mu\nu}^\alpha \lambda^n A_\alpha = \\ = \lambda^n [A_{\mu,\nu} + n(\kappa'_\nu - \kappa_\nu) A_\mu - *\Gamma_{\mu\nu}^\alpha A_\alpha]$$

Thus **(D12)** $(A_{\mu,\nu} - n\kappa_\nu A_\mu - *\Gamma_{\mu\nu}^\alpha A_\alpha)' = \lambda^n [A_{\mu,\nu} - n\kappa_\nu A_\mu - *\Gamma_{\mu\nu}^\alpha A_\alpha]$ so take **(D13)** $A_{\mu*\nu} = A_{\mu,\nu} - n\kappa_\nu A_\mu - *\Gamma_{\mu\nu}^\alpha A_\alpha$ as the co-covariant derivative of A_α . this can be written via **(D10)** as **(D14)** $A_{\mu*\nu} = A_{\mu;\nu} - (n-1)\kappa_\nu A_\mu + \kappa_\mu A_\nu - g_{\mu\nu} \kappa^\alpha A_\alpha$. Similarly for a vector B^μ of power n one has **(D15)** $B_{*\nu}^\mu = B_{;\nu}^\mu - (n+1)\kappa_\nu B^\mu + \kappa^\mu B_\nu - g_\nu^\mu \kappa_\alpha B^\alpha$. For a co-tensor with various suffixes up and down one can form the co-covariant derivative via the same rules; one notes that the co-covariant derivative always has the same power as the original. Next observe **(D16)** $(TU)_{*\sigma} = T_{*\sigma} U + TU_{*\sigma}$ while **(D17)** $g_{\mu\nu*\sigma} = 0$ and $G_{*\sigma}^{\mu\nu} = 0$ so one can raise and lower suffixes freely in a co-tensor before carrying out co-covariant differentiation. Thus one can raise the μ in **(D14)** giving **(D15)** with A^μ replacing B^μ and $n-2$ in place of n. The potentials κ_μ do not form a co-vector because of the wrong transformation laws **(D4)** but the $F_{\mu\nu}$ defined by **(D4)** are unaffected by gauge transformations so they form an in-tensor. One obtains the co-covariant divergence of a co-vector B^μ by putting $\nu = \mu$ in **(D15)** to get **(D16)** $B_{*\mu}^\mu = B_{;\mu}^\mu - (n+4)\kappa_\mu B^\mu$ (for $n = -4$ this is the ordinary covariant divergence).

We list some formulas for second co-covariant derivatives now with a sketch of derivation. Thus for a scalar of power n **(D17)** $S_{*\mu*\nu} = S_{*\mu;\nu} - (n-1)\kappa_\nu S_{*\mu} + \kappa_\mu S_{*\nu} - g_{\mu\nu} \kappa^\sigma S_{*\sigma}$. Putting $S_{*\mu} = S_{\mu} - n\kappa_\mu S$ on gets

$$(4.2) \quad S_{*\mu*\nu} = S_{\mu;\nu} - n\kappa_{\mu;\nu} - n\kappa_\mu S_\nu - n\kappa_\nu (S_\mu - n\kappa_\mu S) + \kappa_\nu S_{*\mu} + \kappa_\mu S_{*\nu} - g_{\mu\nu} \kappa^\sigma S_\sigma$$

Now $S_{\mu;\nu} = S_{\nu;\mu}$ so **(D18)** $S_{*\mu*\nu} - S_{*\nu*\mu} = -n(\kappa_{\mu;\nu} - \kappa_{\nu;\mu})S = -nF_{\mu\nu}S$. This is tedious but instructive and we continue. Let A_μ be a co-vector of power n so

$$(4.3) \quad A_{\mu*\nu*\sigma} = A_{\mu*\nu;\sigma} - n\kappa_\sigma A_{\mu*\nu} + (g_\mu^\rho \kappa_\sigma + g_\sigma^\rho \kappa_\mu - g_{\mu\sigma} \kappa^\rho) A_{\rho*\nu} + (g_\nu^\rho \kappa_\sigma + g_\sigma^\rho \kappa_\nu - g_{\sigma\nu} \kappa^\rho) A_{\mu*\rho}$$

A lengthy calculation then yields **(D19)** $A_{\mu*\nu*\sigma} - A_{\mu*\sigma*\nu} = *\mathfrak{B}_{\mu\nu\sigma\rho} A^\rho - (n-1)F_{\nu\sigma} A_\mu$ where

$$(4.4) \quad *\mathfrak{B}_{\mu\nu\sigma\rho} = B_{\mu\nu\sigma\rho} + g_{\rho\nu}(\kappa_{\mu;\sigma} + \kappa_\mu \kappa_\sigma) + g_{\mu\sigma}(\kappa_{\rho;\nu} + \kappa_\rho \kappa_\nu) - g_{\rho\sigma}(\kappa_{\mu;\nu} + \kappa_\mu \kappa_\nu) - \\ - g_{\mu\nu}(\kappa_{\rho;\sigma} + \kappa_\rho \kappa_\sigma) + (g_{\rho\sigma} g_{\mu\nu} - g_{\rho\nu} g_{\mu\sigma}) \kappa^\alpha \kappa_\alpha$$

One can consider $*\mathfrak{B}$ as a generalized Riemann-Christoffel tensor but it does not have the usual symmetry properties for such a tensor; however one can write **(D20)** $*\mathfrak{B}_{\mu\nu\sigma\rho} = *B_{\mu\nu\sigma\rho} + (1/2)(g_{\rho\nu} F_{\mu\sigma} + g_{\mu\sigma} F_{\rho\nu} - g_{\rho\sigma} F_{\mu\nu} - g_{\mu\nu} F_{\rho\sigma})$ and then $*B_{\mu\nu\sigma\rho}$ has all the usual symmetries, namely

$$(4.5) \quad *B_{\mu\nu\sigma\rho} = -*B_{\mu\sigma\nu\rho} = -*B_{\rho\nu\sigma\mu} = *B_{\nu\mu\rho\sigma}; *B_{\mu\nu\sigma\rho} + *B_{\mu\sigma\rho\nu} + *B_{\mu\rho\nu\sigma} = 0$$

Thus is appropriate to call $*B_{\mu\nu\sigma\rho}$ the Riemann-Christoffel (RC) tensor for Weyl space; it is a co-tensor of power 2. The contracted RC tensor is **(D21)** $*R_{\mu\nu} = *B_{\mu\nu\sigma}^\sigma = R_{\mu\nu} -$

$\kappa_{\mu;\nu} - \kappa_{\nu;\mu} - g_{\mu\nu}\kappa_{;\sigma}^{\sigma} - 2\kappa_{\mu}\kappa_{\nu} + 2g_{\mu\nu}\kappa^{\sigma}\kappa_{\sigma}$ and is an in-tensor. A further contraction gives the total curvature **(D22)** $*R = *R_{\sigma}^{\sigma} = R - 6\kappa_{;\sigma}^{\sigma} + 6\kappa_{\sigma}^{\sigma}$ which is a co-scalar of power -2.

One gets field equations from an action principle with an in-invariant action, hence one of the form **(D23)** $I = \int \Omega \sqrt{d^4x}$ where Ω must be a co-scalar of power -4 to compensate $\sqrt{}$ having power 4. The usual contribution to Ω from the EM field is $(1/4)F_{\mu\nu}F^{\mu\nu}$ (of power -4 since it can be written as $F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$ with F factors of power zero and g factors of power -2). One also needs a gravitational term and the standard $-R$ could be $*R$ but this has power -2 and will not do. Weyl proposed $(*R)^2$ which has the correct power but seems too complicated to be satisfactory. Here one takes $*R = 0$ as a constraint and puts the constraint into the Lagrangian via $\gamma*R$ with γ a co-scalar field of power -2 in the form of a Lagrange multiplier. This leads to a scalar-tensor theory of gravitation and one can insert other terms involving γ . For convenience one takes $\gamma = -\beta^2$ with β as the basic field variable (co-scalar of power -1) and adds terms $k\beta^{*\sigma}\beta_{*\sigma}$ (co-scalar of power -4); terms $c\beta^4$ can also be added to get **(D24)** $I = \int [(1/4)F_{\mu\nu}F^{\mu\nu} - \beta^{2*}R + k\beta^{*\mu}\beta_{*\mu} + c\beta^4] \sqrt{d^4x}$ as a vacuum action. Now $\beta^{*\mu}\beta_{*\mu} = (\beta^{\mu} + \beta\kappa^{\mu})(\beta_{\mu} + \beta\kappa_{\mu})$ and using **(D22)** one obtains

$$(4.6) \quad -\beta^{2*}R + k\beta^{*\mu}\beta_{*\mu} = -\beta^2R + k\beta^{\mu}\beta_{\mu} + (k-6)\beta^2\kappa^{\mu}\kappa_{\mu} + 6(\beta^2\kappa^{\mu})_{;\mu} + (2k-12)\beta\kappa^{\mu}\beta_{\mu}$$

The term involving $(\beta^2\kappa^{\mu})_{;\mu}$ can be discarded since its contribution to the action density is a perfect differential, namely **(D25)** $(\beta^2\kappa^{\mu})_{;\mu}\sqrt{} = (\beta^2\kappa^{\mu}\sqrt{})_{;\mu}$ and for the simplest vacuum equations one chooses $k=6$ so that **(D24)** becomes **(D26)** $I = \int [(1/4)F_{\mu\nu}F^{\mu\nu} - \beta^2R + 6\beta^{\mu}\beta_{\mu} + c\beta^4] \sqrt{d^4x}$. Thus I no longer involves the κ_{μ} directly but only via $F_{\mu\nu}$ and I is invariant under transformations $\kappa_{\mu} \rightarrow \kappa_{\mu} + \phi_{,\mu}$ so the equations of motion that follow from the action principle will be unaffected by such transformations (i.e. they have no physical significance). Now consider three kinds of transformation:

- (1) Any transformation of coordinates.
- (2) Any transformation of the metric gauge combined with the appropriate transformation of potentials $\kappa_{\mu} \rightarrow \kappa_{\mu} + \phi_{,\mu}$.
- (3) In the vacuum one may make a transformation of potentials as above without changing the metric gauge or alternatively one may transform the metric gauge without changing the potentials. This works only where there is no matter.

For the field equations one makes small variations in all the field quantities $g_{\mu\nu}$, κ_{μ} , and β , calculates the change in I and sets it equal to zero. Thus write **(D27)** $\delta I = \int [(1/2)P^{\mu\nu}\delta g_{\mu\nu} + Q^{\mu}\delta\kappa_{\mu} + S\delta\beta] \sqrt{d^4x}$ and drop the $c\beta^4\sqrt{}$ term since it is probably only of interest for cosmological purposes. One has **(D28)** $\delta[(1/4)F_{\mu\nu}F^{\mu\nu}\sqrt{}] = (1/2)E^{\mu\nu}\sqrt{}\delta g_{\mu\nu} - J^{\mu}\sqrt{}\delta\kappa_{\mu}$ with neglect of a perfect differential. Here $E^{\mu\nu}$ is the EM stress tensor **(D29)** $E^{\mu\nu} = (1/4)g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu\alpha}F_{\alpha}^{\nu}$ and J^{μ} is the charge current vector **(D30)** $F^{\mu} = F_{;\nu}^{\mu\nu} = \sqrt{-1}(F^{\mu\nu}\sqrt{})_{;\nu}$. Considerable calculation and neglect of perfect differentials leads finally to

$$(4.7) \quad P^{\mu\nu} = E^{\mu\nu} + \beta^2[2R^{\mu\nu} - g^{\mu\nu}R] - 4g^{\mu\nu}\beta\beta_{;\rho}^{\rho} + 4\beta\beta^{\mu;\nu} + 2g^{\mu\nu}\beta^{\sigma}\beta_{\sigma} - 8\beta^{\mu}\beta^{\nu};$$

$$Q^{\mu} = -J^{\mu}; \quad S = -2\beta R - 12\beta_{;\mu}^{\mu}$$

and the field equations for the vacuum are **(D31)** $P^{\mu\nu} = 0$, $Q^{\mu} = 0$, and $S = 0$. These are not all independent since **(D32)** $P_{\sigma}^{\sigma} = -2\beta^2R - 12\beta\beta_{;\sigma}^{\sigma} = \beta S$ so the S equation is a consequence of the P equations. If one omits the EM term from the action it becomes the

same as the Brans-Dicke action except that the latter allows an arbitrary value for k ; with $k \neq 6$ the vacuum equations are independent so the BD theory has one more vacuum field equation, namely $\square(\beta^2) = 0$.

Now the action integral is invariant under transformations of the coordinate system and transformations of gauge; each of these leads to a conservation law connecting the quantities $P^{\mu\nu}$, Q^μ , S defined via **(D27)**. For coordinate transformations $x^\mu \rightarrow x^\mu + b^\mu$ one gets

$$(4.8) \quad -\delta g_{\mu\nu} = g_{\mu\sigma} b_{,\nu}^\sigma + g_{\nu\sigma} b_{,\mu}^\sigma + g_{\mu\nu,\sigma} b^\sigma; \quad -\delta\beta = \beta_\sigma b^\sigma; \quad -\delta\kappa_\mu = \kappa_\sigma b_{,\mu}^\sigma + \kappa_{\mu,\sigma} b^\sigma$$

Putting these variations in **(D27)** yields

$$(4.9) \quad \delta I = - \int [(1/2)P^{\mu\nu}(g_{\mu\sigma} b_{,\nu}^\sigma + g_{\nu\sigma} b_{,\mu}^\sigma + g_{\mu\nu,\sigma} b^\sigma) + Q^\mu(\kappa_\sigma b_{,\mu}^\sigma + \kappa_{\mu,\sigma} b^\sigma) + S\beta_\sigma b^\sigma] \sqrt{d^4x} = \\ = \int [(P_\sigma^\mu \sqrt{\cdot})_{,\mu} - (1/2)P^{\mu\nu} g_{\mu\nu,\sigma} \sqrt{\cdot} + (Q^\mu \kappa_\sigma \sqrt{\cdot})_{,\mu} - Q^\mu \kappa_{\mu,\sigma} \sqrt{\cdot} - S\beta_\sigma \sqrt{\cdot}] b^\sigma d^4x$$

This δI vanishes for arbitrary b^σ so one puts the coefficient of b^σ equal to zero; using **(D33)** $(P_\sigma^\mu \sqrt{\cdot})_{,\mu} - (1/2)P^{\mu\nu} g_{\mu\nu,\sigma} \sqrt{\cdot} = P_{\sigma;\mu}^\mu \sqrt{\cdot}$ and **(D34)** $(Q^\mu \kappa_\sigma \sqrt{\cdot})_{,\mu} = \kappa_\sigma Q_{;\mu}^\mu \sqrt{\cdot} + \kappa_{\sigma,\mu} Q^\mu \sqrt{\cdot}$ this reduces to **(D35)** $P_{\sigma;\mu}^\mu + \kappa_\sigma Q_{;\mu}^\mu + F_{\sigma\mu} Q^\mu - S\beta_\sigma = 0$. Next consider a small transformation in gauge **(D36)** $\delta g_{\mu\nu} = 2\lambda g_{\mu\nu}$, $\delta\beta = -\lambda\beta$, and $\delta\kappa_\mu = [\log(1 + \lambda)]_{,\mu} = \lambda_{,\mu}$. Putting this in **(D27)** yields

$$(4.10) \quad \delta I = \int [P^{\mu\nu} \lambda g_{\mu\nu} + Q^\mu \lambda_{,\mu} - S\lambda\beta] \sqrt{d^4x} = \int [P_\mu^\mu \sqrt{\cdot} - (Q^\mu \sqrt{\cdot})_{,\mu} - S\beta \sqrt{\cdot}] \lambda d^4x$$

Putting the coefficient of λ equal to zero gives **(D27)** $P_\mu^\mu - Q_{;\mu}^\mu - S\beta = 0$; here **(D35)** and **(D37)** are the conservation laws. For the vacuum one sees that **(D37)** is the same as **(D32)** since $Q_{;\mu}^\mu = 0$ from (4.7); also **(D35)** reduces to **(D38)** $P_{\sigma;\mu}^\mu + F_{\sigma\mu} Q^\mu - \beta^{-1} \beta_\sigma P_\mu^\mu = 0$ which may be considered as a generalization of the Bianchi identities. The conservation laws **(D35)** and **(D37)** hold more generally than for the vacuum, namely whenever the action integral can be constructed from the field variables $g_{\mu\nu}$, κ_μ , β alone.

Now let the coordinates of a particle be z^μ , functions of the proper time s measured along its world line. Put $dz^\mu/ds = v^\mu$ for velocity so $v_\mu v^\mu = 1$ and v^μ is a co-vector of power -1. One adds to the action the further terms **(D39)** $I_1 = -m \int \beta ds$ and $I_2 = e \int \beta^{-1} \beta_{*\mu} v^\mu ds$ (m and e being constants). Then these terms are in-invariants with **(D40)** $I_2 = e \int (\beta^{-1} \beta_\mu + \kappa_\mu) v^\mu ds = e \int [(d/ds)(\log(\beta)) + \kappa_\mu v^\mu] ds$ and the first term contributes nothing to the action principle. Thus $I_2 = e \int \kappa_\mu v^\mu ds$ which is unchanged when $\kappa_\mu \rightarrow \kappa_\mu + \phi_{,\mu}$ since the extra term is $e \int (d\phi/ds) ds$. Thus for a particle with action $I_1 + I_2$ the transformations (3) above are still possible. Now some calculation yields

$$(4.11) \quad m[g_{\mu\sigma} d(\beta v^\mu)/ds + \beta \Gamma_{\sigma\mu\nu} v^\mu v^\nu - \beta_\sigma] = -e v^\mu F_{\mu\sigma} \equiv m[d(\beta v^\mu)/ds + \Gamma_{\rho\sigma}^\mu v^\rho v^\sigma - \beta^\mu] = e F^{\mu\nu} v_\nu$$

This is the equation of motion for a particle of mass m and charge e ; if $e = 0$ it could be called an in-geodesic. If one works with the Einstein gauge then the case $e = 0$ gives the usual geodesic equation. Next one considers the influence the of particle on the field and this is done by generating a dust of particles and a continuous fluid leading to an equation

$$(4.12) \quad \rho[(\beta v^\mu)_{,\nu} + \Gamma_{\alpha\sigma}^\mu v^\alpha v^\sigma - \beta^\mu] = \sigma^{\mu\nu}$$

where ρ and σ refer to mass and charge density respectively.

4.2. THE SCHRÖDINGER EQUATION IN WEYL SPACE. We go now to Santamato [95] and derive the SE from classical mechanics in Weyl space (cf. also [6, 36, 96]). The idea is to relate the quantum force (arising from the quantum potential) to geometrical properties of spacetime; the Klein-Gordon (KG) equation is also treated in this spirit. One wants to show how geometry acts as a guidance field for matter (as in general relativity). Initial positions are assumed random (as in the Madelung approach) and thus the theory is really describing the motion of an ensemble. Thus assume that the particle motion is given by some random process $q^i(t, \omega)$ in a manifold M (where ω is the sample space tag) whose probability density $\rho(q, t)$ exists and is properly normalizable. Assume that the process $q^i(t, \omega)$ is the solution of differential equations **(D41)** $\dot{q}^i(t, \omega) = (dq^i/dt)(t, \omega) = v^i(q(t, \omega), t)$ with random initial conditions $q^i(t_0, \omega) = q_0^i(\omega)$. Once the joint distribution of the random variables $q_0^i(\omega)$ is given the process $q^i(t, \omega)$ is uniquely determined by **(D41)**. One knows that in this situation **(D42)** $\partial_t \rho + \partial_i(\rho v^i) = 0$ with initial Cauchy data $\rho(q, t) = \rho_0(q)$. The natural origin of v^i arises via a least action principle based on a Lagrangian $L(q, \dot{q}, t)$ with

$$(4.13) \quad L^*(q, \dot{q}, t) = L(q, \dot{q}, t) - \Phi(q, \dot{q}, t); \quad \Phi = \frac{dS}{dt} = \partial_t S + \dot{q}^i \partial_i S$$

Then $v^i(q, t)$ arises by minimizing **(D43)** $I(t_0, t_1) = E[\int_{t_0}^{t_1} L^*(q(t, \omega), \dot{q}(t, \omega), t) dt]$ where t_0, t_1 are arbitrary and E denotes the expectation (cf. [30, 74, 75, 76] for stochastic ideas). The minimum is to be achieved over the class of all random motions $q^i(t, \omega)$ obeying **(D41)** with arbitrarily varied velocity field $v^i(q, t)$ but having common initial values. One proves first

$$(4.14) \quad \partial_t S + H(q, \nabla S, t) = 0; \quad v^i(q, t) = \frac{\partial H}{\partial p_i}(q, \nabla S(q, t), t)$$

Thus the value of I in **(D43)** along the random curve $q^i(t, q_0(\omega))$ is **(D44)** $I(t_1, t_0, \omega) = \int_{t_0}^{t_1} L^*(q(t, q_0(\omega)), \dot{q}(t, q_0(\omega)), t) dt$. Let $\mu(q_0)$ denote the joint probability density of the random variables $q_0^i(\omega)$ and then the expectation value of the random integral is

$$(4.15) \quad I(t_1, t_0) = E[I(t_1, t_0, \omega)] = \int_{\mathbf{R}^n} \int_{t_0}^{t_1} \mu(q_0) L^*(q(t, q_0), \dot{q}(t, q_0), t) d^n q_0 dt$$

Standard variational methods give then

$$(4.16) \quad \delta I = \int_{\mathbf{R}^n} d^n q_0 \mu(q_0) \left[\frac{\partial L^*}{\partial \dot{q}^i}(q(t_1, q_0), \partial_t q(t_1, q_0), t) \delta q^i(t_1, q_0) - \int_{t_0}^{t_1} dt \left(\frac{\partial}{\partial t} \frac{\partial L^*}{\partial \dot{q}^i}(q(t, q_0), \partial_t q(t, q_0), t) - \frac{\partial L^*}{\partial q^i}(q(t, q_0), \partial_t q(t, q_0), t) \right) \delta q^i(t, q_0) \right]$$

where one uses the fact that $\mu(q_0)$ is independent of time and $\delta q^i(t_0, q_0) = 0$ (recall common initial data is assumed). Therefore **(D45)** $(\partial L^*/\partial \dot{q}^i)(q(t, q_0), \partial_t q(t, q_0), t) = 0$ and

$$(4.17) \quad \frac{\partial}{\partial t} \frac{\partial L^*}{\partial \dot{q}^i}(q(t, q_0), \partial_t q(t, q_0), t) - \frac{\partial L^*}{\partial q^i}(q(t, q_0), \partial_t q(t, q_0), t) = 0$$

are the necessary conditions for obtaining a minimum of I . Conditions (4.17) are the usual Euler-Lagrange equations whereas **(D45)** is a consequence of the fact that in the most

general case one must retain varied motions with $\delta q^i(t_1, q_0)$ different from zero at the final time t_1 . Note that since L^* differs from L by a total time derivative one can safely replace L^* by L in (4.17) and putting (4.13) into **(D45)** one obtains the classical equations **(D46)** $p_i = (\partial L / \partial \dot{q}^i)(q(t, q_0), \dot{q}(t, q_0), t) = \partial_i S(q(t, q_0), t)$. It is known now that if **(D47)** $\det[\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j] \neq 0$ then the second equation in (4.14) is a consequence of the gradient condition **(D46)** and of the definition of the Hamiltonian function $H(q, p, t) = p_i \dot{q}^i - L$. Moreover (4.17) and **(D46)** entrain the HJ equation in (4.14). In order to show that the average action integral (4.15) actually gives a minimum one needs $\delta^2 I > 0$ but this is not necessary for Lagrangians whose Hamiltonian H has the form

$$(4.18) \quad H_C(q, p, t) = \frac{1}{2m} g^{ik} (p_i - A_i)(p_k - A_k) + V$$

with arbitrary fields A_i and V (particle of mass m in an EM field A) which is the form for nonrelativistic applications; given positive definite g_{ik} such Hamiltonians involve sufficiency conditions **(D48)** $\det[\partial^2 L / \partial \dot{q}^i \partial \dot{q}^k] = mg > 0$. Finally (4.17) with L^* replaced by L shows that along particle trajectories the EL equations are satisfied, i.e. the particle undergoes a classical motion with probability one. Notice here that in (4.14) no explicit mention of generalized momenta is made; one is dealing with a random motion entirely based on position. Moreover the minimum principle **(D43)** defines a 1-1 correspondence between solutions $S(q, t)$ in (4.14) and minimizing random motions $q^i(t, \omega)$. Provided v^i is given via (4.14) the particle undergoes a classical motion with probability one. Thus once the Lagrangian L or equivalently the Hamiltonian H is given **(D42)** and (4.14) uniquely determine the stochastic process $q^i(t, \omega)$. Now suppose that some geometric structure is given on M so that the notion of scalar curvature $R(q, t)$ of M is meaningful. Then we assume (ad hoc) that the actual Lagrangian is **(D49)** $L(q, \dot{q}, t) = L_C(q, \dot{q}, t) + \gamma(\hbar^2/m)R(q, t)$ where **(D50)** $\gamma = (1/6)(n-2)/(n-1)$ with $n = \dim(M)$. Since both L_C and R are independent of \hbar we have $L \rightarrow L_C$ as $\hbar \rightarrow 0$.

Now for a Riemannian geometry **(D51)** $ds^2 = g_{ik}(q)dq^i dq^k$ it is standard that in a transplantation $q^i \rightarrow q^i + \delta q^i$ one has **(D52)** $\delta A^i = \Gamma_{k\ell}^i A^\ell dq^k$. Here however it is assumed that for $\ell = (g_{ik} A^i A^k)^{1/2}$ one has **(D53)** $\delta \ell = \ell \phi_k dq^k$ where the ϕ_k are covariant components of an arbitrary vector of M (Weyl geometry). For a different perspective we review the material on Weyl geometry in [95]. Thus the actual affine connections $\Gamma_{k\ell}^i$ can be found by comparing **(D53)** with $\delta \ell^2 = \delta(g_{ik} A^i A^k)$ and using **(D52)**. A little linear algebra gives then

$$(4.19) \quad \Gamma_{k\ell}^i = - \left\{ \begin{array}{c} i \\ k \ell \end{array} \right\} + g^{im} (g_{mk} \phi_\ell + g_{m\ell} \phi_k - g_{k\ell} \phi_m)$$

(again in [95] the notation $\Gamma_{k\ell}^i$ is used in place of $\Gamma_{k\ell}^i$ - cf. Section 3.1). Thus we may prescribe the metric tensor g_{ik} and ϕ_i and determine via (4.19) the connection coefficients. Note that $\Gamma_{k\ell}^i = \Gamma_{\ell k}^i$ and for $\phi_i = 0$ one has Riemannian geometry. Covariant derivatives are defined for contravariant A^k via **(D54)** $A_{,i}^k = \partial_i A^k - \Gamma^{k\ell}_i A^\ell$ and for covariant A_k via **(D55)** $A_{k,i} = \partial_i A_k + \Gamma_{ki}^\ell A_\ell$ (where $S_{,i} = \partial_i S$). Note Ricci's lemma no

longer holds (i.e. $g_{ik,\ell} \neq 0$) so covariant differentiation and operations of raising or lowering indices do not commute. The curvature tensor $R_{k\ell m}^i$ in Weyl geometry is introduced via **(D56)** $A_{,k,\ell}^i - A_{,\ell,k}^i = F_{mk\ell}^i A^m$ from which arises the standard formula of Riemannian geometry **(D57)** $R_{mk\ell}^i = -\partial_\ell \Gamma_{mk}^i + \partial_k \Gamma_{m\ell}^i + \Gamma_{n\ell}^i \Gamma_{mk}^n - \Gamma_{nk}^i \Gamma_{m\ell}^n$ where(4.19) must be used in place of the Christoffel symbols. The tensor $R_{mk\ell}^i$ obeys the same symmetry relations as the curvature tensor of Riemann geometry as well as the Bianchi identity. The Ricci symmetric tensor R_{ik} and the scalar curvature R are defined by the same formulas also, viz. $R_{ik} = R_{i\ell k}^\ell$ and $R = g^{ik} R_{ik}$. For completeness one derives here **(D58)** $R = \dot{R} + (n-1)[(n-2)\phi_i \phi^i - 2(1/\sqrt{g})\partial_i(\sqrt{g}\phi^i)]$ where \dot{R} is the Riemannian curvature built by the Christoffel symbols. Thus from (4.19) one obtains

$$(4.20) \quad g^{k\ell} \Gamma_{k\ell}^i = -g^{k\ell} \left\{ \begin{matrix} i \\ k \ell \end{matrix} \right\} - (n-2)\phi^i; \quad \Gamma_{k\ell}^i = - \left\{ \begin{matrix} i \\ k \ell \end{matrix} \right\} + n\phi_k$$

Since the form of a scalar is independent of the coordinate system used one may compute R in a geodesic system where the Christoffel symbols and all $\partial_\ell g_{ik}$ vanish; then (4.19) reduces to **(D59)** $\Gamma_{k\ell}^i = \phi_k \kappa_\ell^i + \phi_\ell \delta_k^i - g_{k\ell} \phi^i$. Hence **(D60)** $R = -g^{km} \partial_m \Gamma_{k\ell}^i + \partial_i (g^{k\ell} \Gamma_{k\ell}^i) + g^{\ell m} \Gamma_{n\ell}^i \Gamma_{mi}^n - g^{m\ell} \Gamma_{n\ell}^i \Gamma_{m\ell}^n$. Further from **(D59)** one has **(D61)** $g^{\ell m} \Gamma_{n\ell}^i \Gamma_{mi}^n = -(n-2)(\phi_k \phi^k)$ at the point in consideration. Putting all this in **(D60)** one arrives at **(D62)** $R = \dot{R} + (n-1)(n-2)(\phi_k \phi^k) - 2(n-1)\partial_k \phi^k$ which becomes **(D58)** in covariant form. Now the geometry is to be derived from physical principles so the ϕ_i cannot be arbitrary but must be obtained by the same averaged least action principle **(D43)** giving the motion of the particle. The minimum in **(D43)** is to be evaluated now with respect to the class of all Weyl geometries having arbitrarily varied gauge vectors but fixed metric tensor. Note that once **(D49)** is inserted in (4.13) the only term in **(D43)** containing the gauge vector is the curvature term. Then observing that $\gamma > 0$ when $n \geq 3$ the minimum principle **(D43)** may be reduced to the simpler form **(D63)** $E[R(q(t, \omega), t)] = \min$ where only the gauge vectors ϕ_i are varied. Using **(D58)** this is easily done. First a little argument shows that $\hat{\rho}(q, t) = \rho(q, t)/\sqrt{g}$ transforms as a scalar in a coordinate change and this will be called the scalar probability density of the random motion of the particle. Starting from **(D42)** a manifestly covariant equation for $\hat{\rho}$ is found to be **(D65)** $\partial_t \hat{\rho} + (1/\sqrt{g})\partial_i(\sqrt{g}v^i \hat{\rho}) = 0$. Now return to the minimum problem **(D63)**; from **(D58)** and **(D64)** one obtains

$$(4.21) \quad E[R(q(t, \omega), t)] = E[\dot{R}(q(t, \omega), t)] + (n-1) \int_M [(n-2)\phi_i \phi^i - 2(1/\sqrt{g})\partial_i(\sqrt{g}\phi^i)] \hat{\rho}(q, t) \sqrt{g} d^n q$$

Assuming fields go to 0 rapidly enough on ∂M and integrating by parts one gets then

$$(4.22) \quad E[R] = E[\dot{R}] - \frac{n-1}{n-2} E[g^{ik} \partial_i(\log(\hat{\rho})) \partial_k(\log(\hat{\rho}))] + \frac{n-1}{n-2} E\{g^{ik} [(n-2)\phi_i + \partial_i(\log(\hat{\rho}))][(n-2)\phi_k + \partial_k(\log(\hat{\rho}))]\}$$

Since the first two terms on the right are independent of the gauge vector and g^{ik} is positive definite $E[R]$ will be a minimum when **(D66)** $\phi_i(q, t) = -[1/(n-2)]\partial_i[\log(\hat{\rho})(q, t)]$. This shows that the geometric properties of space are indeed affected by the presence of the particle and in turn the alteration of geometry acts on the particle through the quantum

force $f_i = \gamma(\hbar^2/m)\partial_i R$ which according to **(D58)** depends on the gauge vector and its derivatives. It is this peculiar feedback between the geometry of space and the motion of the particle which produces quantum effects.

In this spirit one goes next to a geometrical derivation of the SE. Thus inserting **(D66)** into **(D58)** one gets **(D67)** $R = \dot{R} + (1/2\gamma\sqrt{\rho})[1/\sqrt{g}]\partial_i(\sqrt{g}g^{ik}\partial_k\sqrt{\rho})$ where the value **(D50)** for γ has been used. On the other hand the HJ equation (4.13) can be written as **(D68)** $\partial_t S + H_C(q, \nabla S, t) - \gamma(\hbar^2/m)R = 0$ where **(D49)** has been used. When **(D67)** is introduced into **(D68)** the HJ equation **(D76)** and the continuity equation **(D65)**, with velocity field given by (4.14), form a set of two nonlinear PDE which are coupled by the curvature of space. Therefore self consistent random motions of the particle (i.e. random motions compatible with **(D60)**) are obtained by solving **(D65)** and **(D68)** simultaneously. For every pair of solutions $S(q, t, \hat{\rho}(q, t))$ one gets a possible random motion for the particle whose invariant probability density is $\hat{\rho}$. The present approach is so different from traditional QM that a proof of equivalence is needed and this is only done for Hamiltonians of the form (4.18) (which is not very restrictive). The HJ equation **(D69)** corresponding to (4.18) is

$$(4.23) \quad \partial_t S + \frac{1}{2m}g^{ik}(\partial_i S - A_i)(\partial_k S - A_k) + V - \gamma\frac{\hbar^2}{m}R = 0$$

with R given by **(D67)**. Moreover using (4.14) as well as (4.18) the continuity equation **(D65)** becomes **(D69)** $\partial_t \hat{\rho} + (1/m\sqrt{g})\partial_i[\hat{\rho}\sqrt{g}g^{ik}(\partial_k S - A_k)] = 0$. Owing to **(D67)** (4.23) and **(D69)** form a set of two nonlinear PDE which must be solved for the unknown functions S and $\hat{\rho}$. Now a straightforward calculations shows that, setting **(D70)** $\psi(q, t) = \sqrt{\hat{\rho}(q, t)\exp[(i/\hbar)S(q, t)]}$, the quantity ψ obeys a linear PDE (corrected from [95])

$$(4.24) \quad i\hbar\partial_t\psi = \frac{1}{2m}\left\{\left[\frac{i\hbar\partial_i\sqrt{g}}{\sqrt{g}} + A_i\right]g^{ik}(i\hbar\partial_k + A_k)\right\}\psi + \left[V - \gamma\frac{\hbar^2}{m}\dot{R}\right]\psi = 0$$

where only the Riemannian curvature \dot{R} is present (any explicit reference to the gauge vector ϕ_i having disappeared). (4.24) is of course the SE in curvilinear coordinates whose invariance under point transformations is well known. Moreover **(D70)** shows that $|\psi|^2 = \hat{\rho}(q, t)$ is the invariant probability density of finding the particle in the volume element $d^n q$ at time t. Then following Nelson's arguments that the SE together with the density formula contains QM the present theory is physically equivalent to traditional nonrelativistic QM. One sees also from **(D70)** and (4.24) that the time independent SE is obtained via **(D71)** $S = S_0(q) - Et$ with constant E and $\hat{\rho}(q)$. In this case the scalar curvature of space becomes time independent; since starting data at t_0 is meaningless one replaces **(D65)** with a condition $\int_M \hat{\rho}(q)\sqrt{g}d^n q = 1$.

REMARK 4.1. We recall (cf. [30]) that in the nonrelativistic context the quantum potential has the form $Q = -(\hbar^2/2m)(\partial^2\sqrt{\rho}/\sqrt{\rho})$ ($\rho \sim \hat{\rho}$ here) and in more dimensions this corresponds to $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho})$. In Section 5.2 we have a SE involving $\psi = \sqrt{\rho}\exp[(i/\hbar)S]$ with corresponding HJ equation (4.23) which corresponds to the flat space 1-D **(D72)** $S_t + (s')^2/2m + V + Q = 0$ with continuity equation $\partial_t \rho + \partial(\rho S'/m) = 0$ (take $A_k = 0$ here). The continuity equation in **(D69)** corresponds to $\partial_t \rho + (1/m\sqrt{g})\partial_i[\rho\sqrt{g}g^{ik}(\partial_k S)] = 0$. For $A_k = 0$ (4.23) becomes **(D73)** $\partial_t S + (1/2m)g^{ik}\partial_i S\partial_k S + V - \gamma(\hbar^2/m)R = 0$. This leads to

an identification **(D74)** $Q \sim -\gamma(\hbar^2/m)R$ where R is the Ricci scalar in the Weyl geometry (related to the Riemannian curvature built on standard Christoffel symbols via **(D58)**). Here $\gamma = (1/6)[(n-2)(n-2)]$ as in **(D50)** which for $n = 3$ becomes $\gamma = 1/12$; further by **(D66)** the Weyl field $\phi_i = -\partial_i \log(\rho)$. Consequently (see also Remark 4.3).

PROPOSITION 4.1. For the SE (4.24) in Weyl space the quantum potential is $Q = -(\hbar^2/12m)R$ where R is the Weyl-Ricci scalar curvature. For Riemannian flat space $\dot{R} = 0$ this becomes via **(D67)**

$$(4.25) \quad R = \frac{1}{2\gamma\sqrt{\rho}}\partial_i g^{ik}\partial_k\sqrt{\rho} \sim \frac{1}{2\gamma}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \Rightarrow Q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$$

as is should and the SE (4.24) reduces to the standard SE $i\hbar\partial_t\psi = -(\hbar^2/2m)\Delta\psi + V\psi$ ($A_k = 0$). \blacksquare

REMARK 4.2. We recall next from [30] that the Fisher information connection to the SE involves a classical ensemble with particle mass m moving under a potential V **(D75)** $S_t + \frac{1}{2m}(S')^2 + V = 0$; $P_t + \frac{1}{m}\partial(P S')' = 0$ where S is a momentum potential; note that no quantum potential is present but this will be added on in the form of a term $(1/2m)\int dt(\Delta N)^2$ in the Lagrangian which measures the strength of fluctuations. This can then be specified in terms of the probability density P leading to a SE (cf. [54, 56, 57, 58, 88]). One can also approach this via (1-dimension for simplicity)

$$(4.26) \quad S_t + \frac{1}{2m}(S')^2 + V + \frac{\lambda}{m}\left(\frac{(P')^2}{P^2} - \frac{2P''}{P}\right) = 0$$

Note that $Q = -(\hbar^2/2m)(R''/R)$ becomes for $R = P^{1/2}$ **(D76)** $Q = -(2\hbar^2/2m)[(2P''/P) - (P'/P)^2]$. Thus the addition of the Fisher information serves to quantize the classical system. One also defines an information entropy (IE) via **(D77)** $\mathfrak{S} = -\int \rho \log(\rho) d^3x$ ($\rho = |\psi|^2$) leading to

$$(4.27) \quad \frac{\partial\mathfrak{S}}{\partial t} = \int (1 + \log(\rho))\partial(v\rho) \sim \int \frac{(\rho')^2}{\rho}$$

modulo constants involving $D \sim \hbar/2m$. \mathfrak{S} is typically not conserved and $\partial_t\rho = -\nabla \cdot (v\rho)$ ($u = D\nabla\log(\rho)$ with $v = -u$) corresponds to standard Brownian motion with $d\mathfrak{S}/dt \geq 0$. Then high IE production corresponds to rapid flattening of the probability density. Note here also that $\mathfrak{F} \sim -(2/D^2)\int \rho Q dx = \int dx[(\rho')^2/\rho]$ is a functional form of Fisher information. This leads one to conjecture that for the SE (4.24) in Weyl space there is a direct connection between the Ricci-Weyl curvature and Fischer information in the form of the quantum potential; this in turn suggests a connection between information entropy and curvature. \blacksquare

REMARK 4.3. The formulation above from [95] was modified in [96] to a derivation of the Klein-Gordon (KG) equation via an average action principle with the restrictions of Weyl geometry released. The spacetime geometry was then obtained from the action principle to obtain Weyl connections with a gauge field ϕ_μ . The Riemann scalar curvature \dot{R} is then related to the Weyl scalar curvature R via an equation **(D78)** $R = \dot{R} - 3[(1/2)g^{\mu\nu}\phi_\mu\phi_\nu + (1/\sqrt{-g})\partial_\mu(\sqrt{-g}g^{\mu\nu}\phi_\nu)]$. Explicit reference to the underlying Weyl

structure disappears in the resulting SE (as in (4.24)). The HJ equation in [96] has the form (for $A_\mu = 0$ and $V = 0$) **(D79)** $g^{\mu\nu}\partial_\mu S\partial_\nu S = m^2 - (R/6)$ so in some sense (recall here $\hbar = c = 1$) **(D80)** $m^2 - (R/6) \sim \mathfrak{M}^2$ (via (2.3)) where $\mathfrak{M}^2 = m^2 \exp(Q)$ and $Q = (\hbar^2/m^2 c^2)(\square\sqrt{\rho}/\sqrt{\rho}) \sim (\square\sqrt{\rho}/m^2\sqrt{\rho})$ via (2.6) (for signature $(-, +, +, +)$). Thus for $\exp(Q) \sim 1 + Q$ one has **(D81)** $m^2 - (R/6) \sim m^2(1 + Q) \Rightarrow (R/6) \sim -Qm^2 \sim -(\square\sqrt{\rho}/\sqrt{\rho})$. This agrees also with [36] where the whole matter is analyzed incisively. ■

REFERENCES

- [1] M. Abolhasani and M. Golshani, gr-qc 9709005
- [2] R. Adler, M. Bazin, and M. Schiffer, Introduction to general relativity, McGraw Hill, 1965
- [3] E. Alfinito, R. Manda, and G. Vitiello, gr-qc 9904027
- [4] O. Arias, T. Gonzalez, Y. Leyva, and I. Quiros, gr-qc 0307016
- [5] O. Arias and I. Quiros, gr-qc 0212006
- [6] J. Audretsch, Phys. Rev. D, 27 (1083), 2872-2884
- [7] D. Barabash and Y. Shtanov, hep-th 9807291
- [8] O. Barabash and Y. Shtanov, astro-ph 9904144
- [9] J. Bell, Speakable and unspeakable in quantum mechanics, Cambridge Univ. Press, 1987
- [10] G. Bertoldi, A. Faraggi, and M. Matone, hep-th 9909201
- [11] Y. Bisabr, gr-qc 0302102
- [12] M. Blagojević, Gravitation and gauge symmetry, IOP Press, 2002
- [13] A. Blaut and J. Kowalski-Glikman, gr-qc 9710136, 9706076, 9710039, 950601, 9509040, and 9607004
- [14] D. Bohm, B. Hiley, and P. Kaloyerou, Phys. Rept. 144 (1987), 323-375
- [15] D. Bohm and B. Hiley, The undivided universe, Routledge, Chapman and Hall, 1993
- [16] D. Bohm and B. Hiley, Phys. Repts., 144 (1987), 323-348
- [17] D. Bohm, Phys. Rev. 95 (1952), 166-179, 180-193
- [18] R. Bonal, I. Quiros, and R. Cardenas, gr-qc 0010010
- [19] C. Brans, gr-qc 9705069
- [20] L. deBroglie, Electrons et photons, Solvay Conf., Paris, pp. 105-141
- [21] K. Bronnikov, gr-qc 0110125 and 0204001
- [22] S. Capozziello, A. Feoli, G. Lambiase, and G. Papini, gr-qc 0007029
- [23] R. Cardenas, t. Gonzalez, O. Martin, and I. Quiros, astro-ph 0210108
- [24] R. Carroll, Quantum theory, deformation, and integrability, North-Holland, 2000
- [25] R. Carroll, Proc. Conf. Symmetry, Kiev, 2003, to appear
- [26] R. Carroll, Canadian Jour. Phys., 77 (1999), 319-325
- [27] R. Carroll, Direct and inverse problems of mathematical physics, Kluwer, 2000, pp. 39-52
- [28] R. Carroll, Generalized analytic functions, Kluwer, 1998, pp. 299-311
- [29] R. Carroll, quant-ph 0309023 and 0309159
- [30] R. Carroll, quant-ph 0401082 and 0403156
- [31] R. Carroll, On the quantum potential, book in preparation
- [32] R. Carroll, Nucl. Phys. B, 502 (1997), 561-593; Springer Lect. Notes Physics, 502, 1998, pp. 33-56
- [33] R. Carroll, Calculus revisited, Kluwer, 2002
- [34] R. Carroll, Mathematical physics, North-Holland, 1988
- [35] C. Castro, hep-th 9512044; physics 0010072
- [36] C. Castro, Found. Phys. Lett., 4 (1991), 81; Found. Phys., 22 (1992), 569
- [37] S. Chakraborty, N. Chakraborty, and N. Debnath, gr-qc 0306040
- [38] P. Dirac, Proc. Royal Soc. London A, 209 (1951), 291-296, 212 (1952), 330-339, 333 (1973), 403-418
- [39] D. Dürr, S. Goldstein, and N. Zanghi, quant-ph 9511016, 0308039, and 0308038
- [40] D. Dürr, S. Goldstein, R. Tumulka, and N. Zanghi, quant-ph 0208072; 0303156; 0303056; 0311127
- [41] D. Dürr, S. Goldstein, and S. Zanghi, Jour. Stat. Phys., 67 (1992), 843-907
- [42] V. Dzhunushaliev and H. Schmidt, gr-qc 9908049
- [43] A. Edery and M. Paranjape, astro-ph 9808345
- [44] A. Faraggi and M. Matone, Phys. Rev. Lett., 78 (1997), 163-166
- [45] A. Faraggi and M. Matone, Inter. Jour. Mod. Phys. A, 15 (2000), 1869-2017
- [46] V. Faraoni, Phys. Lett. A, 245 (1998), 26 (gr-qc 9805057)
- [47] V. Faraoni, E. Gunzig, and P. Nardone, gr-qc 9811047
- [48] F. deFelice and C. Clarke, Relativity on curved manifolds, Cambridge Univ. Press, 1995
- [49] A. Feoli, W. Wood, and G. Papini, gr-qc 9805035
- [50] E. Floyd, Inter. Jour. Mod. Phys. A, 14 (1999), 1111-1124; 15 (2000), 1363-1378; Found. Phys. Lett., 13 (2000), 235-251; quant-ph 0009070, 0302128 and 0307090

- [51] E. Floyd, *Phys. Rev. D*, 29 (1984) 1842-1844; 26 (1982), 1339-1347; 34 (1986), 3246-3249; 25 (1982), 1547-1551; *Jour. Math. Phys.*, 20 (1979), 83-85; 17 (1976), 880-884; *Phys. Lett. A*, 214 (1996), 259-265; *Inter. Jour. Theor. Phys.*, 27 (1998), 273-281
- [52] E. Floyd, *quant-ph* 0206114
- [53] T. Fülöp and S. Katz, *quant-ph* 9806067
- [54] P. Garbaczewski, *cond-mat* 0211362 and 0301044
- [55] S. Goldstein, J. Taylor, R. Tumulka, and N. Zanghi, *quant-ph* 0405039
- [56] M. Hall, *quant-ph* 9806013, 9903045, 9912055, 0103072, 0107149, and 0302007
- [57] M. Hall and M. Reginatto, *quant-ph* 0102069 and 0201084
- [58] M. Hall, K. Kumar, and M. Reginatto, *quant-ph* 0103041
- [59] S. Hawking and G. Ellis, *The large scale structure of spacetime*, Cambridge Univ. Press, 1973
- [60] P. Holland, *The quantum theory of motion*, Cambridge Univ. Press, 1997
- [61] P. Holland, *Foundations of Physics*, 38 (1998), 881-911; *Nuovo Cimento B*, 116 (2001), 1043 and 1143; *Phys. Lett. A*, 128 (1988), 9; *quant-ph* 0305175
- [62] A. Iorio, L. O’Raifeartaigh, I. Sachs, and C. Wiesendanger, *hep-th* 9607110
- [63] B. Kelleher, *Class. Quant. Grav.*, 21 (2004), 483-495
- [64] S. Kim, *qr-qc* 9703065
- [65] J. Kowalski-Glikman, *gr-qc* 9511014 and 9805015
- [66] C. Levit and J. Sarfatti, *physics* 9704029
- [67] F. London, *Zeit. f. Phys.*, 42 (1927), 375
- [68] G. Magnano, *gr-qc* 9511027
- [69] P. Mannheim, *gr-qc* 9810087
- [70] M. Matone, *hep-th* 0005274 and 0212260
- [71] J. Miritzis, *gr-qc* 0402039
- [72] H. Motavali, H. Salehi, and M. Golshani, *hep-th* 0011062
- [73] H. Motavali and M. Golshani, *hep-th* 0011064
- [74] M. Nagasawa, *Schrödinger equations and diffusion theory*, Birkhäuser, 1993
- [75] M. Nagasawa, *Stochastic processes in quantum physics*, Birkhäuser, 2000
- [76] E. Nelson, *Quantum fluctuations*, Princeton Univ. Press, 1985; *Dynamical theory of Brownian motion*, Princeton Univ. Press, 1967
- [77] H. Ohanian and R. Ruffini, *Gravity and spacetime*, Norton, 1994
- [78] L. O’Raifeartaigh, *The dawning of gauge theory*, Princeton Univ. Press, 1997
- [79] H. deOliveira, J. Salim, and S. Sautu, *Class. Quant. Grav.*, 14 (1997), 2833-2844
- [80] G. Preparata, S. Rovelli, and S. Xue, *gr-qc* 9806044
- [81] L. Querella, *gr-qc* 9806005
- [82] L. Querella, *gr-qc* 9992044
- [83] I. Quiros, *gr-qc* 0004014, 0011056, and 9904004; *hep-th* 0009169 and 0010146
- [84] I. Quiros, R. Bonal, and R. Cardenas, *gr-qc* 9908075
- [85] I. Quiros, R. Cardenas, and R. Bonal, *gr-qc* 0002071
- [86] I. Quiros, R. Bonal, and R. Cardenas, *gr-qc* 9905071
- [87] J. Rankin, *gr-qc* 9404023 and 9408029
- [88] M. Reginatto, *quant-ph* 9909065
- [89] H. Salehi and H. Sepangi, *hep-th* 9810207
- [90] H. Salehi, H. Motavali, and M. Golshani, *hep-th* 0011063
- [91] H. Salehi, H. Sepangi, and F. Darabi, *gr-qc* 0002058
- [92] H. Salehi and Y. Bisabr, *hep-th* 0301208 and 0001095
- [93] H. Salehi, *hep-th* 9912115
- [94] J. Salim and S. Sautu, *Class. Quant. Grav.*, 13 (1996), 353-360
- [95] E. Santamoto, *Phys. Rev. D*, 29 (1984), 216-222
- [96] E. Santamoto, *Phys. Rev. D* 32 (1985), 2615-2621; *Jour. Math. Phys.*, 25 (1984), 2477-2480
- [97] D. Santiago and A. Silbergleit, *gr-qc* 9904003
- [98] I. Schmelzer, *gr-qc* 9811073, 0001101, and 0205035

- [99] M. Serva, *Ann. Inst. H. Poincaré, Phys. Theor.*, 49 (1988), 415-432
- [100] F. Shojai and M. Golshani, *Inter. Jour. Mod. Phys. A*, 13 (1998), 2135-2144 (gr-qc 9903047)
- [101] F. Shojai and M. Golshani, *Int. Jour. Mod. Phys. A*, 13 (1998), 677-693
- [102] F. Shojai, A. Shojai, and M. Golshani, *Mod. Phys. Lett. A*, 13 (1998), 2725-2729 and 2915-2922
- [103] A. Shojai, F. Shojai, and M. Golshani, *Mod. Phys. Lett. A*, 13 (1998), 2965-2969 (gr-qc 9903048 and 9903049)
- [104] A. Shojai, *Inter. Jour. Mod. Phys. A*, 15 (2000), 1757-1771 (gr-qc 0010013)
- [105] F. Shojai and A. Shojai, gr-qc 0105102 and 0109052
- [106] F. Shojai and A. Shojai, *Physica Scripta*, 64 (2001), 413 (quant-ph 0109025); gr-qc 0311076
- [107] F. Shojai and A. Shojai, *Gravitation and Cosmology*, 9 (2003), 163 (gr-qc 0306099); gr-qc 0311076
- [108] F. Shojai and A. Shojai, gr-qc 0404102
- [109] F. Shojai and A. Shojai, *Inter. Jour. Mod. Phys. A*, 15 (2000), 1859-1868 (gr-qc 0010012)
- [110] A. Shojai and M. Golshani, quant-ph 9812019, 9612023, and 9612021
- [111] Y. Shtanov, quant-ph 9705024; gr-qc 9503005
- [112] E. Squires, quant-ph 9508014
- [113] S. Tiwari, quant-ph 0109048
- [114] R. Wald, *General relativity*, Univ. Chicago Press, 1984
- [115] J. Wheeler, *Phys. Rev. D*, 41 (1990), 431
- [116] J. Wheeler, hep-th 9706214, 0002068, and 0305017; gr-qc 9411030
- [117] T. Willmore, *Riemannian geometry*, Oxford Univ. Press, 1993
- [118] W. Wood and G. Papini, gr-qc 9612042
- [119] J. Zambrini, *Phys. Rev. A*, 38 (1987), 3631-3649; 33 (1986), 1532-1548; *Jour. Math. Phys.*, 27 (1986), 2307-2330
- [120] A. Zee, hep-th 0309032