An alternative quantization of
$H = xp$

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Abstract

Let $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real β the Fourier coefficients

$$u_\nu := \frac{1}{\sqrt{2\pi}} \oint u(x) e^{-i\nu x} dx$$

enable the definitions of the norms (e.g. [ILi] 11.1.5, [BrK])

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty} |\nu|^{2\beta} |u_\nu|^2 \; .$$

We consider the one dimensional Pseudo-Differential (convolution) model operators $S_i \; i = -1, 0. -1$ of Symm, Riesz and Calderon-Zygmund type ([EsG])

$$(S_i u)(x) := \oint s_i(x-y) u(y) dy$$

With

$$s_{-1}(x) := -\frac{1}{\sqrt{2\pi}} \ln\left|2\sin\frac{x}{2}\right| \qquad s_0(x) := -\frac{1}{\sqrt{2\pi}} \frac{1}{2} \cot\frac{x}{2} \qquad s_1(x) := -\frac{1}{\sqrt{2\pi}} \sin^{-2}\frac{x}{2} \; .$$

The operators are bounded and self-adjoint with respect to the energy inner norm

$$(u,u)_{\alpha+i/2} = (S_i u, u)_\alpha \, .$$

and linked by the Dirac function:

$$\delta(x) := \frac{1}{2\pi} \int_0^{2\pi} e^{ikx} dk = \frac{1}{\pi} \int_0^{\pi} \cos(kx) dk = \frac{1}{2} \operatorname{sgn}'(x) \in H_{-n/2-\varepsilon} \; .$$

We propose to alternatively model the location and momentum of a quantum ([BeM]) by

$$quantum = x \in H_{-1/2-\varepsilon} \to location(x) = S_0[x] =: \tilde{Q}[x] \to momentum(x) = S_1[x] = \tilde{P}[x]$$

As a consequence it holds

$$([\tilde{P}\tilde{Q}]x, y)_{-1/2-\varepsilon} = (x,y)_{-1/2-\varepsilon} \, .$$

§1 Introduction and main result

In order to avoid technical difficulties we restrict ourself to the one-dimensional periodical case. Let $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real β the Fourier coefficients

$$\hat{u}(\nu) := u_\nu := \frac{1}{\sqrt{2\pi}} \oint u(x) e^{i\nu x} dx =: \oint u(x) \psi_n(x) dx$$

enable the definitions of the norms (see e.g. [LiI] Remark 11.1.5, [BrK])

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty} |\nu|^{2\beta} |u_\nu|^2 \quad .$$

Then H is the space of L_2-periodic function in R.

A quantum in quantum mechanics is modeled is an element of a Hilbert space H_0. The transformation of non-quantum PDE into this framework is called quantification. Thereby e.g. the transformation of the momentum (which is modelled as the derivative of a function u) becomes an operator P with related domain $D(P) \subset H_0, D(P) \neq H_0$ in the Hilbert space H_0. As a consequence the commutator with the location operator Q with domain $D(Q) = H_0$ does not vanish. This mathematical fact leads to Heissenberg uncertainty relation between quantum momentum and a quantum location states.

We propose an alternative model of the quantum location and momentum operators, which, at the same time, provides an alternative quantification technique.

The Hilbert transformation $H = S_0$ (§2) defines an isomorphism onto the Hilbert spaces H_β. It is proposed to model the location of a quantum as its Hilbert transformation (which is the one-dimensional version of the rotation invariant (singular integral) Riesz operators for $n > 1$), i.e.

$$quantum = x \rightarrow location(x) = \tilde{Q}[x] := S_0[x].$$

Alternatively we propose to model the momentum of a quantum as the Calderon-Zygmund singular integral (Pseudo-differential) operator $Z := S_1$ (§2) with same properties of the derivative operator, but now defined on the same Hilbert space as the location operator, i.e.

$$quantum = x \rightarrow momentum(x) = \tilde{P}[x] := S_1[x].$$

As a first consequence the domain of location and momentum operator is now identical:

$$D(\tilde{P}) = D(\tilde{Q}) = H_\beta .$$

The operators S_i are related to each other (§2) in combination with the Dirac function:

$$\delta(x) := \frac{1}{2\pi}\int_0^{2\pi} e^{ikx}dk = \frac{1}{\pi}\int_0^{\pi}\cos(kx)dk = \frac{1}{2}\mathrm{sgn}'(x) \in H_{-n/2-\varepsilon} \subset H_{-1} \ .$$

The constant Fourier term of a Hilbert transformed function is vanishing. As a consequence in a properly chosen distributional Hilbert space $x \in H_{-\alpha}$, $\alpha = \beta + n/2 > 0$ (depending from the space dimension) one could achieve that

$$(\delta, x)_{\beta-n/2} = 0 \ ,$$

which then goes along with (in a weak $H_\beta -$sense for $x = S_0 y$ with vanishing constant Fourier term)

$$(\tilde{Q}\tilde{P} - \tilde{P}\tilde{Q})[x] = 0 \ .$$

In [BrK1] we prose an alternative ground state energy model which fits into same concept.

We summaries the result of §2 in

Theorem: Let $v \in H_0$ with vanishing constant Fourier terms $\hat{v}(0) = 0$. Then the commutator operators $\bar{S}_i$ ($i = -1,0,1$) are related to the inner product of the Hilbert spaces $H_{\beta-n/2}$ by

$$(\bar{S}_i[u], v)_{\beta-n/2} = c(S_i[u], v)_\beta + c(\delta, v)_\beta = c(S_i[u], v)_\beta = (u, v)_{\beta-n/2}$$

i.e. the commutators operators $\bar{S}_i$ define isomorphism ($n = 1$) onto $H_{\beta-1/2}$.

§2 Hilbert Scales and Pseudo-Differential Operators

Definition and remark: Let the kernel functions s_i ($i=-1,0,1$) with its corresponding Fourier transforms be given by:

i) $$s_{-1}(x):=\frac{1}{\sqrt{2\pi}}\ln\frac{1}{2\sin\frac{x}{2}} \quad , \quad \hat{s}_{-1}(\nu)=\frac{1}{\sqrt{2\pi}}\oint s_{-1}(x)e^{i\nu x}dx=\frac{1}{2\nu}\operatorname{sgn}(\nu)$$

ii) $$s_0(x):=-\frac{1}{\sqrt{2\pi}}\frac{1}{2}\cot\frac{x}{2} \quad , \quad \hat{s}_0(\nu)=\frac{1}{\sqrt{2\pi}}\oint s_0(x)e^{i\nu x}dx=i\frac{1}{2}\operatorname{sgn}(\nu)$$

iii) $$s_1(x):=-\frac{1}{\sqrt{2\pi}}\frac{1}{\sin^2\frac{x}{2}}=-s_0'(x) \quad , \quad \hat{s}_1(\nu)=\frac{1}{\sqrt{2\pi}}\oint s_1(x)e^{i\nu x}dx=2\nu\operatorname{sgn}(\nu) \quad .$$

Remark: The principle value of the not locally integrable function $1/x$ is the distribution g defined by ([PeB])

$$(g,\varphi):=\lim\int_{x\geq\varepsilon}\varphi(x)\frac{dx}{x}=\int_{-\infty}^{\infty}\log|x|\varphi'(x)dx \qquad \text{for each } \varphi\in C_c^\infty \ .$$

The corresponding Fourier transform is given by

$$\left[P.v.(\frac{1}{\pi x})\right]^{\wedge}(\nu)=-i\operatorname{sgn}(\nu) \quad .$$

Remark: From [StE1] IV 6.3, we note that a periodic function on R in the form

$$u(x)=\sum_{-\infty}^{\infty}u_\nu e^{i\nu x} \quad \text{with} \quad |u_\nu|\leq\frac{1}{\nu}$$

is an element of the function space of bounded mean oscillation, i.e. $u\in BMO(R)$.

Suppose

$$\sum_{-\infty}^{\infty}v_\nu e^{i\nu x}\in BMO(R) \quad , \quad v_\nu\geq 0$$

Then, whenever $|u_\nu|\leq v_\nu$

$$\sum_{-\infty}^{\infty}u_\nu e^{i\nu x}\in BMO(R) \quad .$$

Remark: For the Dirac function it holds

$$\delta(x) := \frac{1}{2\pi}\int_0^{2\pi} e^{ikx}dk = \frac{1}{\pi}\int_0^{\pi}\cos(kx)dk = \frac{1}{2}\mathrm{sgn}'(x) \in H_{-n/2-\varepsilon} \subset H_{-1}$$.

For $\bar{u}(x) := xu(x)$ it holds $\hat{\bar{u}}(\nu) = i \cdot \hat{u}'(\nu)$ and therefore

i) $\nu\hat{s}'_{-1}(\nu) = -\hat{s}_{-1}(\nu) + \delta(\nu)$

ii) $\hat{s}'_0(\nu) = i\delta(\nu)$

iii) $\frac{1}{2\nu}\hat{s}'_1(\nu) = 2\hat{s}_{-1}(\nu) + \frac{1}{2}\delta(\nu)$.

Definition: The kernel functions s_i ($i = -1,0,1$) with its symbols $\hat{s}_i$ define Pseudo-Differential model operators of Symm, Riesz and Calderon-Zygmund type (e.g. [LiI] (1.2.31)-(1.2.33), [IiI1]):

S_{-1} : $(S_{-1}u)(x) := \oint s_{-1}(x-y)u(y)dy$

S_0 : $(S_0 u)(x) := \oint s_0(x-y)u(y)dy$

S_1 : $(S_1 u)(x) := \oint s_1(x-y)u(y)dy$

$\bar{S}_i$: $(\bar{S}_i u)(x) := \oint s_i(x-y)yu(y)dy$.

We summaries key properties of those model operators in

Lemma: i) The operators S_i can be applied to define the inner products of the Hilbert spaces $H_{\alpha-i/2}$, i.e. it holds

$$(S_i u, v)_\alpha = (u,v)_{\alpha-i/2}$$

and the mappings $S_i : H_{\alpha+i/2} \to H_\alpha$ are isomorphisms.

ii) The operator S_0 is skew-symmetric in the space $L_2(0,2\pi)$ (e.g. [GaD], [PeB])

$$(S_0 u, v)_\alpha = -(u, S_0 v)_\alpha$$

It maps the space $H := L_2(0,2\pi) - R$ isometric onto itself and it holds $\|S_0 u\|_\alpha = \|u\|_\alpha$ and $S_0{}^2 = -I$. The constant Fourier term vanishes, i.e. $(S_0[u])_0 = 0$ i.e. for $u \in L_2$ it holds

$$(S_0 u)(x) = i\sum_1^\infty \left[u_{-\nu}e^{-i\nu x} - u_\nu e^{i\nu x}\right] \in L_2 .$$

iii) If $u, Hu \in L_2$ then u and $S_0 u$ are orthogonal, i.e.

$$\int_{-\infty}^{\infty} u(y) S_0[u](y) dy = 0 \ .$$

vi) The commutator $\bar{S}_0$ fulfills

$$x S_0[u](x) - S_0[xu](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy$$

respectively

$$x S_0[u](x) - S_0[xu](x) = 0$$

for u odd or $u = Hv$.

v) The The operator S_0 defines also a bijective mapping from the Hölder space $C^{0.\lambda}$ onto itself [MuN], §18, 19.

From the remarks and lemmata above it follows

Corollary: Let $v \in H_\alpha$ with vanishing constant Fourier terms $\hat{v}(0) = 0$. Then the commutators operators $\bar{S}_i$ ($i = -1,0,1$) are related to the inner product of the Hilbert spaces $H_{\beta - i/2}$ by

$$(\bar{S}_i[u], v)_{\beta - i/2} = c(S_i[u], v)_\beta + c(\delta, v)_\beta = c(S_i[u], v)_\beta = (u, v)_{\beta - i/2}$$

i.e. the commutators operators $\bar{S}_i$ define a isomorphism on $H_{\beta - i/2}$.

Appendix

Note 1 From e.g. [GaD] pp.63, [GrI] 1.441, [Ili1], [MuN] chapter 3, §28, we recall

$$\frac{1}{2\pi}\oint_{0\to 2\pi} e^{in\varphi}\ln\frac{1}{2\sin\frac{\varphi-\vartheta}{2}}d\vartheta=\begin{cases}-\frac{1}{2n}e^{in\varphi} & n=1,2,3,\ldots\\ 0 & n=0\\ \frac{1}{2n}e^{in\varphi} & n=-1,-2,..\end{cases}$$

$$\frac{1}{\pi}\oint_{0\to 2\pi} e^{in\varphi}\frac{1}{2}\cot\frac{\varphi-\vartheta}{2}d\vartheta=\begin{cases}-ie^{in\varphi} & n=1,2,3,\ldots\\ 0 & n=0\\ ie^{in\varphi} & n=-1,-2,..\end{cases}$$

$$\frac{1}{\pi}\oint_{0\to 2\pi} e^{in\varphi}\frac{1}{4\sin^2\frac{\varphi-\vartheta}{2}}d\vartheta=\begin{cases}-ne^{in\varphi} & n=1,2,3,\ldots\\ 0 & n=0\\ ne^{in\varphi} & n=-1,-2,..\end{cases}$$

Note 2 From [GrI] 3.814 we recall the formula

$$\int_0^\pi \frac{1-\frac{x}{2}\cot\frac{x}{2}}{\sin^2\frac{x}{2}}dx=\frac{\pi}{2}$$

Note 3

$$S_0[xu](x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yu(y)}{x-y}dy=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(x-z)u(x-z)}{z}dz=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{xu(x-z)}{z}dz-\frac{1}{\pi}\int_{-\infty}^{\infty}u(x-z)dz=xS_0[u](x)-\frac{1}{\pi}\int_{-\infty}^{\infty}u(y)dy$$

Note 4 With respect to the Dirac function to be used to model “wave packages” we note that for negative integer Hilbert scale factor the Calderon-Zygmund operator with symbol $|\nu|$ ([EsG] (3.17), (3.35)) is defined by

$$(\Lambda u)(x)=(\sum_{k=1}^{n}R_kD_ku)(x)=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\sum_{k=1}^{n}p.v.\int_{-\infty}^{\infty}\sum_{k=1}^{n}\frac{x_k-y_k}{|x-y|^{n+1}}\frac{\partial u(y)}{\partial y_k}dy=-\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}p.v.\int_{-\infty}^{\infty}\frac{\Delta_y u(y)}{|x-y|^{n-1}}dy=-(\Delta\Lambda^{-1})u(x)$$

whereby R_k denotes the Riesz operators ([AbH] p. 19, 106, [PeB] example 9.9)

$$R_ku=-i\frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}p.v.\int_{-\infty}^{\infty}\frac{x_k-y_k}{|x-y|^{n+1}}u(y)dy\ .$$

It holds ([EsG] (3.15))

$$\Lambda^{-1}u=\frac{\Gamma(\frac{n-1}{2})}{2\pi^{(n+1)/2}}\int_{-\infty}^{\infty}\frac{u(y)dy}{|x-y|^{n-1}}\quad,\quad n\geq 2\ .$$

Note 5

The Riesz operators fulfill certain properties with respect to commutation with translations and homothesis, as well as a crucial property with respect to the rotation group $SO(n)$, [PeB], [StE], [BrK1] p.13):

If $j \neq j$ then $R_j R_k$ is a singular convolution operator. On the other hand it holds $R_j^2 = -(1/n)I + A_j$ where A_j is a convolution operator. It further holds

$$\|R_j\| = 1 \quad , \; R_j^* = -R_j \quad , \; \sum R_j^2 = -I \quad , \quad \sum \|R_j u\|^2 = \|u\|^2 \;\; , u \in L_2 \; .$$

The crucial property for our purpose is related to rotations ([PeB] example 9.9, 9.10, [StE]): let

$$m := m(x) := (m_1(x), ...m_n(x))$$

be the vector of the Mikhlin multipliers of the Riesz operators and $\rho = \rho_{ik} \in SO(n)$, then

$m(\rho(x)) = \rho(m(x))$, whereby

$$R_k u = -ic_n p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y) dy \quad \text{with} \quad c_n := \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$$

$$m_j(\rho(x)) = \sum \rho_{jk} m_k(x)$$

and

$$m(\rho(x)) = c_n \int_{S^{n-1}} \left(\frac{\pi i}{2} sign(x\rho^{-1}(y)) + \log\left|\frac{1}{x\rho^{-1}(y)}\right|\right) \frac{y}{|y|} d\sigma(y)$$

$$= c_n \int_{S^{n-1}} \left(\frac{\pi i}{2} sign(xy) + \log\left|\frac{1}{xy}\right|\right) \frac{y}{|y|} d\sigma(y) \; .$$

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