

Quantum Gravity and Ground State Energy

A truly infinitesimal Hilbert space geometry

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Abstract

Quantum gravity is a field of theoretical physics that seeks to describe the force of gravity according to the principles of quantum mechanics. Ground state (or vacuum) energy modeling is concerned about necessarily “existing” “empty space” energy in the absence of matter.

A new mathematical gravity model is proposed, which is built on the definition of the inner product of the “new ground state energy” model. The latter one is developed for the model problem of 2π – periodic function space $H_0 := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, equipped with the norm (for real β)

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty} |\nu|^{2\beta} |u_\nu|^2$$

and its related Hilbert scale in combination with the Hilbert (H), the Symm (A) and the Calderón-Zygmund ([BrK0], [EsG], [Lil], [PeB], [StE]) PDO operators. For $u, v \in L_2^*(\Gamma)$ it holds

$$D(S) \cong H_0^\# = L_2^*(\Gamma), \quad R(S) \cong H_{-1}^\# = D(A).$$

In general for $\alpha \leq 0$ one can define

$$((du, dv))_{\alpha+1} := (H[du], H[dv])_\alpha = (Su, Sv)_\alpha = (ASu, ASv)_{\alpha+1} = (u, v)_{\alpha+1}.$$

This ground state energy (inner product) model ensures convergent quantum oscillator energy series. It enables the definition of a truly infinitesimal (non-archimedean) geometry, based on a Hilbert space (inner product) generated metric. The domains and ranges of the related self-adjoint, positive definite (Pseudo Differential) operators with its related eigenpair structures enable the transformation to differential forms. By this, H. Weyl’s today’s “truly” infinitesimal (affine connexions, parallel displacements, differentiable manifolds based) geometry is replaced by a truly infinitesimal (rotation group based) geometry. The rotation group property is a result of corresponding properties of the Riesz operators, which are the generalized Hilbert transform operators for space-time dimension $n > 1$.

An inner geometry considers quantities, which can be expressed by the 1st fundamental form g_{ij} , while the exterior geometry also considers (exterior) tangential spaces to a “manifold”. The Levi-Civita derivatives D_i , whereby its commutator in the form

$$[D_i, D_j] = D_i D_j - D_j D_i =: R_{ij} \neq 0$$

gives the Riemannian curvature tensor, is still an inner geometry based concept, while the definition of an exterior derivative is required to build the (exterior) Riemannian geometry.

We emphasize, that our model still fulfills B. Riemann’s requirements to model the “root cause of the potential metrics” as “acting forces”. Our differentiation is about the fact, that the root cause lies not “exterior” to the geometry (as B. Riemann was assuming), but interior.

The new concept replaces the Standard Model of Elementary Particles, given by $SU(3) \times SU(2) \times U(1)$, as all physical models, which can be expressed by the Hamiltonian formalism (enabled by variational theory) can be reformulated in a weaker inner product variational form than standard L_2 – inner product. A “transfer” to the “standard” representation is given by density arguments. We emphasize, that due to the reduced regularity assumptions to the domain, the Lagrange (kinetic) formalism is no longer equivalent (i.e. no longer valid compared) to the Hamiltonian (field) formalism.

As a consequence, current L_2 – Hilbert space based quantum model is not a represent of a “quantum object”, but “represents” a “quantum” as its “purely” quantum energy = “substance” (Leibniz). The related “ideal/transcendental” world of the “objects” itself are the H_{-1} – Hilbert space (in case of for $\alpha = -1$). The discrete eigenvalues of the L_2 – Hilbert space reflects to mean value and variance quantities of the projections (break down) of the corresponding continuous spectrum of the related “objects” out of $L_2^\perp = (H_{-1} - L_2)$ – Hilbert space.

Quantum gravity and ground state energy

Quantum gravity is a field of theoretical physics that seeks to describe the force of gravity according to the principles of quantum mechanics. Ground state (or vacuum) energy modeling is concerned about necessarily “existing” “empty space” energy in the absence of matter.

Some open questions

Light consists of particles, as the current of electrons increases with the increase of the frequency, but not with the increase of the intensity (the “force” of the light). This phenomenon leads Einstein to the concept of photons with minimal quantum energy. But photons have no mass nevertheless it holds the Einstein equation $E = m \cdot c^2$. In addition the light is an electro-magnetic wave in the sense of Maxwell equations.

Not everything what happens, does have a route cause, but is the result of the human “pattern thinking”. How in this context can the phenomenon of “time” be explained?

The Higgs boson combines the existence of mass together with the action of the weak force. But why it provides especially to the quarks that much mass, is still a mystery.

Concept

A truly infinitesimal Hilbert space geometry is proposed, which builds the framework to integrate of physical laws/PDE (real world), which can be represented by the Hamiltonian formalism, with its corresponding quantum states (transcendental world).

The main conceptual changes to current /mathematical concepts are:

1. The semi-Riemannian manifolds equipped with a metric are replaced by a Hilbert scale with differential form elements
2. H. Weyl’s (almost truly) affine infinitesimal small geometry (for differential forms) is replaced by a truly rotation/torsion infinitesimal small geometry (which consequently also replaces current gauge theory)
3. Current quantization “transformation” techniques from “real” into “quantum” world are replaced by Hilbert scale orthogonal projections from “truly” “quantum” variational equation world” into a “real” variational equation world.

The baseline

B. Riemann, „Über die Hypothesen, welche der Geometrie zu Grunde liegen“

p. 149: „Es muss also entweder das dem Raume zu Grunde liegende Wirkliche eine discrete Mannigfaltigkeit bilden, oder der Grund der Maßverhältnisse außerhalb, in darauf wirkenden bindenden Kräften, gesucht werden.“

“The question of the validity of the hypothesis of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space. In this last question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in a discrete manifoldness, the ground of its metric relations is given in the notion of it, while in a continuous manifoldness, this ground must come from outside. Either therefore the reality which underlies space must from a discrete manifoldness, or we seek the ground of its metric relations outside it, in binding forces which act upon it.”

To model the second option Riemann developed the Riemannian geometry, which is an exterior geometry in contrast to the inner (Euclidean) geometry. For the infinitesimal small the Riemannian geometry is “close” to the Euclidean geometry.

An inner geometry considers quantities, which can be expressed by the 1st fundamental form g_{ij} , while the exterior geometry also considers (exterior) tangential spaces to a “manifold”. The Levi-Civita derivatives D_i , whereby its commutator in the form

$$[D_i, D_j] = D_i D_j - D_j D_i =: R_{ij} \neq 0$$

gives the Riemannian curvature tensor, is still an inner geometry based concept, while the definition of an exterior derivative is required to build the (exterior) Riemannian geometry.

We emphasize, that our model still fulfills B. Riemann’s requirements above, i.e. the “root cause of the potential metrics lie in the acting forces”. The difference is due to the fact, that the root cause lies not “exterior” to the geometry (as B. Riemann was assuming), but interior.

Idea

We propose to replace H. Weyl's infinitesimal small affine geometry by infinitesimal small rotation geometry. At the same time it validates Riemann's conjecture of an Euclidean rotation geometry. The rotating "objects/substances" are differentials, which links back to Leibniz's concepts of monads. At the end the concept of a hyper-real universe beyond (Kant's) physical reality (i.e. physics) becomes (Kant's and Plato's) transcendental "reality" ([PoP]).

The proposed mathematical gravity model is built on the definition of the inner product of the "new ground state energy" model. The key mathematical tools are the (Pseudo Differential) Riesz operators with differential domains.

The concept to apply Riesz operators to differentials goes in line with J. Plemelj’s alternative definition of a potential, building on mass element " $d\mu$ ", alternatively to a mass density, only. The corresponding group, which characterizes the Hilbert space geometry, is the infinitesimal **rotation group**. This also goes along with Riemann's conjecture of an **infinitesimal small Euclidean geometry**. The proposed Hilbert space relates also to the L_2 – Hilbert space, which is the as-is framework of today's quantum mechanics and quantum field theory. Consequently the Hilbert scale theory is the proper quantum gravity modeling framework to combine variational theory of basically all relevant physical models with quantum “mechanics”.

The proposed Hilbert space based model overcomes the still unsolved particle-wave paradox, providing a purely geometrical rationalized "continuum" (H. Weyl), while overcoming the "contacting body" interaction challenge of "quants without extension, but equipped with flavor and spin". This generates a constraint, which lead to the restricting handicap of the affine (only) geometry that only parallelized “quants” (i.e. vectors with a direction) are taken into (modeling) scope. The today's related mathematical concept to handle the "contacting body" issue is about the concept of continuous transformations, built on S. Lie's concept of contact transforms. The proposed model overcomes the related challenges, as the rotating differentials do not require any specific “contact” information, when passing the related coordinate axis.

We note that also the Legendre transform is a contact transform. If the Legendre transform is applicable (ensured by only (!) sufficiently high regularity assumptions to the differential), it is applied to prove the equivalence of the Lagrange and the Hamilton formalism. We emphasize, that it would be sufficient to have a Hamilton formalism, only, to define existing physical laws in the framework of variational theory. It's not necessary that PDEs like the Maxwell equations need to be valid for both representations, the integral form and in the differential form.

A (truly infinitesimal) quantum geometry

In the proposed truly infinitesimal quantum geometry the following characteristics are valid:

- The Riemannian geometry with its related affine connexion concept (translation group between only parallel vectors in scope) is replaced by rotation group properties of a differential forms (Hilbert space) geometry (note: a semi-Riemannian manifold is torsion-free)
- Only the Hamiltonian formalism is valid for the infinitesimal small. This is no longer the case for the Lagrange formalism, as the equivalence of both is no longer given, because the Legendre transformation is no longer valid as transformation from

$$H_{\alpha+1} \rightarrow H_{\alpha} \text{ for } \alpha < 0$$

- The "break down" of from the (transcendental) truly infinitesimal geometry to the (transcendental) physical / "real" quantum mechanics geometry is achieved by orthogonal projection from

$$P_{\alpha,\beta}: H_{\alpha} \rightarrow H_{\beta} \text{ for real } \alpha < 0 \leq \beta.$$

The **Planck constant** becomes a "quantity" of an approximation "error" value between the those two "worlds", i.e. between the transcendental H_{α} -"world" ($\alpha < 0$) and the human being "real" H_0 -world.

- The L_2 -based probability theory is still valid, as long as the energy Hilbert space is a subset of L_2 but directly applied to energy/matter substances and not kinetic to "substances" objects without extension.

Dirac function

We note that for $n=1$ the **Dirac "function"** is contained in the Hilbert space H_{-1} , i.e. in this case "wave package measurements" could be covered by the model. The situation changes for space dimension $n > 1$, as the regularity of the Dirac function depends from the space dimension: the Hilbert scale factor to this is $(-n/2 + \varepsilon)$. We emphasize that until space-time dimension $n=4$ the regularity requirements would be covered by the Hilbert space H_{-2} .

With respect to the Dirac function to be used to model "wave packages" we note that for negative integer Hilbert scale factor the Calderon-Zygmund operator with symbol $|V|$ ([EsG] (3.17), (3.35)) is defined by

$$(\Lambda u)(x) = \left(\sum_{k=1}^n R_k D_k u \right)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^n p.v. \int \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy = -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v. \int \frac{\Delta_y u(y)}{|x-y|^{n-1}} dy = -(\Delta \Lambda^{-1})u(x)$$

It enables a norm with its related Hilbert space, which is less regular than the standard energy norm. By this the physical energy Hilbert space can be chosen as $H_{1/2}$ instead of H_0 , only. As a consequence the Dirac function becomes part of the domain for space-time dimension $n = 1, 2, 3, 4$.

Proposition

In [BrK1], [BrK4] there is a one-dimensional ground state energy model proposed for the quantum oscillator. We propose to extend this model to the 4-dimensional Minkowski space, which is basically about

- the building of a Hilbert space framework for a “truly geometrical world” model [WeH]
- the modeling of vacuum energy as rotating monads (Leibniz).

The quantum oscillator model in [BrK1], [BrK4] is based on $H_{\nu}^{\#}(\Gamma)$ with $\Gamma := S^1(\mathbb{R}^2)$ and $\nu = 0, -1$. We suggest for the “world model” and the related “momentum/energy/matter model” Hilbert spaces (based on the Minkowski hyperbolic geometry and an Archimedean non-ordered field) of the type

$$H_{\nu}^{\#}(\mathbb{R}_1^4) \quad \text{with } \nu = -1, 0$$

with the norm ([BrK1])

$$\|ds\|^2 = \|dx_1\|^2 + \|dx_2\|^2 + \|dx_3\|^2 - \|dx_4\|^2$$

alternatively to k-analytic manifolds and the (only!) metric

$$ds^2 = \sum g_{ik} dx_i dx_k .$$

This concept provides a purely inner geometry (i.e. 1st fundamentals forms, only), building the Hamiltonian formalism only, as the Legendre (contact) transformation is no longer defined for $\nu = -1$.

The concept enables a purely infinitesimal geometry with proper linkages to differential forms. The latter ones build the foundation of nearly all today’s relevant physical models.

In case of $n > 1$ this means, that not only the direction into the dimension axis need to be covered by a proper model.

By chance or by purpose the Hilbert space $H_{-1}^{\#}(\Gamma)$ provides the answer to the question from B. Riemann, about the characterization of those periodical function, which can be represented as Fourier series ([LaD] 2.2), [RiB], see also below).

Conclusion for experimental and theoretical physics

Mathematical physical models need to put phenomena into perspective with appropriately defined concepts (e.g. force). Thereby physical experiments need to be set-up properly to validate consistency between phenomena, concepts and observed experimental results. In variational theory this is reflected by sufficiently high regularity assumptions to the "test space". The only thing what can be measured in quantum phenomena related experiments (i.e. affecting the infinitesimal small) is the phenomenon/observable "energy". From this perspective the physical regularity conditions for the proposed mathematical model requires an operator norm of the operator S , which defines the range of the operator S and which is equivalent to the $H_0 = L_2$ space.

Consequently, the regularity requirements to the domain $D(S)$ of the operator S needs to be a subset of H_{-1} .

Consequences

1. "God does not play dice" is right
2. Planck's comments to his black body radiation observations are still valid
3. Bohr's interpretation (field/matter dualism) is building on wrong assumption of the proper mathematical quantum mechanics (Hilbert space) model.

We emphasize that our "ground state energy" model "defines" a quantum "object" as an element of a Hilbert space H_{-a} . A spontaneous (Higgs-) breakdown can then be modeled as a projection from H_{-a} into H_0 Hilbert space.

We note that the Dirac delta function is an element of H_{-a} for $a > n/2$, whereby n denotes the dimension of the field (see below).

Planck constant: The above also changes the interpretation of a "quantum of action": It's the phenomenon which "happens" resp. "is observed" in the "physical reality world", which is represented by the Hilbert space H_0 . This means that the **Planck constant** becomes the "role" of an approximation "error" value between the "ideal /mathematical" H_{-1} -world and the "real /physical" H_0 -world (which also provides the framework for the probability theory to handle its anticipated uncertainty).

The Hilbert space model

We propose to generalize the one-dimension ground state energy (Hilbert space) model ([BrK1], [BrK4]), which is the momentum space $H_0 := L_2^*(\Gamma)$ with $\Gamma := S^1(\mathbb{R}^2)$, equipped with the norm (for real β)

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty} |\nu|^{2\beta} |u_\nu|^2 .$$

This requires “differentiating/momentum building” of “less regular” “functions than $H_0 := L_2^*(\Gamma)$. This kind of Hilbert space is built by the singular integral operator

$$(Au)(x) := -\int \log 2 \sin \frac{x-y}{2} u(y) dy \quad , \quad D(A) = L_2^*(\Gamma) \quad ,$$

which can be extended to an operator with domain Hilbert space $H_{-1}^\#$. It holds

$$(u, v)_{-1} = (Au, Av)_0 \quad , \quad (u, v)_{-1/2} = (Au, v)_0 \quad , \quad (u', v')_{-1} = (Au', Av')_0 = (Hu, Hv)_0 \quad .$$

whereby

$$(Au')(x) = -\int \log 2 \sin \frac{x-y}{2} du(y) = \int \cot \frac{x-y}{2} u(y) dy = (Hu)(x) \quad .$$

An inner product of an “energy” Hilbert space for differential forms can be defined by

$$((du, dv))_{\alpha+1} := (H(du), H(dv))_\alpha = (Su, Sv)_\alpha = (u, v)_{\alpha+1} \quad \text{for real } \alpha .$$

In order to achieve convergent energy norms for “quantum” element, one can put

$$\alpha = -1$$

which leads to reduced regularity requirements in the form $u, v \in H_0^\#$. Then the related norm equivalences are given by:

$$\|du\|_0 \cong \|uSu\|_{-1} \cong \|ASu\|_0 \cong \|Hu\|_0 \cong \|u\|_0 \quad .$$

The regularity of the Dirac function is depending from the space-time dimension. In case of $n > 2$ the choose of $\alpha = -1$ lead to too less regularity requirements for the related energy space. The definition

$$\alpha = -1/2$$

with corresponding energy Hilbert space $H_{1/2}^\#$ would be possible and more appropriate. Another (not proposed) option could be

$$\alpha = -n/2 .$$

The generalization to higher space-time dimensions lead to the concept of the Riesz transformation. The crucial property of the Riesz transformation is its behavior concerning “rotations”. In combination with the concept of J. Plemelj, which is basically about a Stieltjes integral to model a potential, whereby the density is given by a differential only, this enables the definition of an inner product for differential (k-forms). We note that the curvature operator in a 4 dimensional Riemannian manifold can be interpreted as operator between 2-forms.

The "new ground state energy model" allows, requires and enables the rotation of ("all") infinitesimal small "entity directions/extensions" at a space-time "world" point x. By this, the restriction of Weyl's

affine geometry to only **affine** infinitesimal small "vectors" is no longer required. The characteristic of an affine geometry is the fact, that only parallel distances can be measured against each other; vectors are the mathematical model of translations (resp. parallel displacements) and the geometry is described by the group properties of vectors. An affine geometry with space dimension n is the "same" as its related $(n-1)$ -dimensional projective group.

With respect to F. Klein's algebraic approach to classify a geometry ([KIF]), we note:
"all properties, which do not change by the transformations of a group defines the geometry".

We note, that the **1st fundamental form** is related to **(inner) geometry** concepts like lengths, angles, Christoffel symbols & the Levi-Civita derivative. The corresponding mathematical model concepts are inner products and (dual) Hilbert spaces. The **2nd fundamental form** addresses the (parallel/affine) displacement of tangential (vector) spaces, i.e. it leaves the (inner geometry) Hilbert space framework. The additionally required mathematical concepts are about "hyper areas" and related distance functions. Therefore, not only the terminology changes to **"exterior geometry"**. The gauge theory framework is a consequence to re-build again necessary vector space properties.

"Energy" Operators

Let ∇u denotes the gradient operator applied to a function u and Su be the Calderon-Zygmund operator according to [BrK1]. In variational theory the Dirichlet integral

$$D(u, v) := (\nabla u, \nabla v) = (u, v)_1$$

defines the energy inner product with (Sobolev) domain $H_1 \times H_1$, which defines the regularity requirements to the functions u and v (which model in case of the Navier-Stokes equations the velocity of fluid particles). Sobolev embedding theorems gives the relationship to the Banach spaces of continuous and differentiable functions.

The singular (Calderon-Zygmund Pseudo Differential) operator with domain of (Cartan's) differential forms is proposed to be the non-standard alternative to the "standard", non-bounded (momentum) differential operator. With respect to the energy Hilbert space inner product this means that the Dirichlet (energy) integral is replaced by the corresponding inner product

$$(Su, Sv)_\alpha$$

The Calderon-Zygmund operator is basically an isomorphism from $S : H_{a+1} \rightarrow H_a$ with a real. The requirements from physics determines the setting of the scale factor α .

The (singular) Calderon-Zygmund integrodifferential operator requires less regularity assumption to the domain than H_1 . The Dirichlet integral goes along with modeling energy and momentum, which requires the concept of space and bodies within this space (WeH, III, 22, d). The primary physical concepts and physical laws are the laws of conservation of energy and momentum.

The purely field space framework (which defines a truly "continuum", [WeH]), defined by the new inner product with other than H_1 domain enables a purely radiation concept (Hamiltonian principle) and releases from the concept of a body system. This overcomes current conceptual mathematical issues of the particle-field dualism (paradox).

In case of $n = 1$ and "continuous" regularity assumptions it holds

$$((du, dv)) = (H(du), H(dv))_{-1} = (Su, Sv)_{-1} = (u, v)_0$$

whereby H denotes the Hilbert transform operator. This means, that a quantum in current quantum mechanics, which is modeled as an element of the Hilbert $H_0 = L_2$ has the "mass energy" norm $((du, du))$, which is mathematically spoken a Hilbert space norm of a 1-form.

Mass Elements

We refer to [PIJ]: In his famous book J. Plemelj also provided a physical Interpretation of " $d\mu$ " with respect to the concept of a "mass element" creating a potential not only by the density of the mass, but by the element " $d\mu$ " itself:

$$(H(du))(x) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{du(t)}{x-t}$$

This means, that the quantity of a quantum " $d\mu$ " in the sense of quantum mechanics (as an element of the Hilbert space L_2) corresponds to the norm of the mass element " $d\mu$ " in our new ground state energy model.

Navier Stokes equations

In case of $n=3$ the existence of classical smooth solutions of the non-linear, non-stationary Navier-Stokes equations is a fundamental open mathematical problem. There are partial solvability results existing. Interesting by itself is the fact, that the pressure requires neither initial value nor boundary conditions, but the problem statement is about well posedness of a partial differential equations system. The questions concerning the existence of weak solutions of the non-linear, non-stationary Navier-Stokes equations have been basically answered. Corresponding "extrapolation" to related strong solutions by density arguments are different per category (linear/non-linear, stationary/non-stationary, space dimension), basically due to the structure of the Stokes operator, the Serin gap according to Sobolev embedding theorems and the logarithm convexity of the Sobolev spaces.

We propose to look for an alternative, slightly modified variational representation of the Navier-Stokes equations with $u \in H_\alpha$ ($-1 \leq \alpha < 0$). Then the corresponding domain of the pressure is given by the

Hilbert space $H_{\alpha+1}$, which would enable an appropriate initial value condition for the pressure $p \in H_{\alpha+1}$.

Applying the concept above also to the modeling of the mass balances of incompressible fluids in the context of the Euler equations would lead to an alternative formulation of the Navier-Stokes equations as Pseudo-Differential equations, building on the Calderon-Zygmund (singular) integral operator, based on a Hamiltonian representation

Energy and Matter

"Atoms" contain basically no mass nearly all of the mass is "built" of the quantum fluctuation of the vacuum energy. This vacuum energy fluctuates, but is finite. It presents itself in form of gluons, which are the interconnection particles, which hold together the quarks. The mass of a proton consists nearly exclusively of the energy of the gluons:

"Mass is essentially the manifestation of the vacuum energy".

The energy of the Einstein gravitation field is all time negative. The energy in the universe is constant.

Applying the Riesz operators $R := R_j, j = 1, \dots, n$ to a differential $du = du_j$ leads to the Calderon Zygmund operator $Su := S(u)(x_j)$ ([EsG], [PeB], [StE]). The ("ground state energy") operator norm of $\|R(du)\|$ is equivalent to the operator norm of the **Calderon Zygmund operator** $\|Su\|$.

Let H_a be the Hilbert space with the scale factor a , which is an arbitrary real number. This operator S acts "isomorph" on Hilbert scales in the following way:

$$S: H_{a+1} \rightarrow H_a \quad \text{for any real number } a.$$

It means that the operator S "acts" in same manner as the (momentum) differential operator d/dx , which requires that the domain is a subset of H_1 . The essential and determines difference between the two operators is, that in case of the operator S the "regularity" of the domain $D(S)$ can be arbitrarily chosen along the Hilbert scale. At the same time variational theory enables a representation of well posed PDE in weak form with respect to any Hilbert space H_a , as well.

According to the (normal derivative) concept of [PIJ], which is in line with the proposal of "Ground State Energy" the "function" $v = S\mu$ is the physical model of the "energy/momentum" model of a mass element "du". According to [BrK1] the energy norm of a **rotating** "mass element $d\mu$ " (i.e. Leibniz's =monad / differential) is defined by

$$\left((R(d\mu), R(d\mu)) \right)_{a+1} := (v, v)_a = (S\mu, S\mu)_a = (\mu, \mu)_{a+1}.$$

We note that in case of sufficiently high regularity assumptions of "mass elements" (e.g. that a mass elements is described by its "mass densities du/dx ", only, i.e. du/dx is defined resp. u needs to be at least an element of H_1 , i.e. du/dx is an element of H_0) it means that a related norm quantity, defined by

$$\left((R(d\mu), R(d\mu)) \right)_1 := (v, v)_0 = (\mu', \mu')_0 = (\mu, \mu)_1,$$

is valid. Therefore, in case of an assumed H_1 regularity of "mass elements du", this would be equivalent to the standard "energy" norm of variational theory.

For the relationship to the regularity of the Dirac "function" and related modeling of "quantum wave packages" in the context of Hilbert scale framework of an appropriate quantum gravity we refer to the chapter "Dirac function" below.

The Riesz and Calderón-Zygmund operators

The Calderón-Zygmund integrodifferential operator with symbol $|\nu|$ ([EsG] (3.17), (3.35)) is defined by

$$(\Lambda u)(x) = \left(\sum_{k=1}^n R_k D_k u \right)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^n p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy = -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{\Delta_y u(y)}{|x-y|^{n-1}} dy = -(\Delta \Lambda^{-1})u(x)$$

whereby R_k denotes the Riesz operators ([AbH] p. 19, 106, [PeB] example 9.9)

$$R_k u = -i \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y) dy$$

It holds ([EsG] (3.15))

$$\Lambda^{-1} u = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{(n+1)/2}} \int_{-\infty}^{\infty} \frac{u(y) dy}{|x-y|^{n-1}}, \quad n \geq 2.$$

The Riesz operators fulfill certain properties with respect to commutation with translations and homothesis, as well as a crucial property with respect to the rotation group $SO(n)$, [PeB], [StE], [BrK1] p.13):

If $j \neq k$ then $R_j R_k$ is a singular convolution operator. On the other hand it holds $R_j^2 = -(1/n)I + A_j$ where A_j is a convolution operator. It further holds

$$\|R_j\| = 1, \quad R_j^* = -R_j, \quad \sum R_j^2 = -I, \quad \sum \|R_j u\|^2 = \|u\|^2, \quad u \in L_2.$$

The crucial property for our purpose is related to rotations ([PeB] example 9.9, 9.10, [StE]):
let

$$m := m(x) := (m_1(x), \dots, m_n(x))$$

be the vector of the Mikhlin multipliers of the Riesz operators and $\rho = \rho_{ik} \in SO(n)$, then $m(\rho(x)) = \rho(m(x))$, whereby

$$R_k u = -i c_n p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y) dy \quad \text{with} \quad c_n := \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$$

$$m_j(\rho(x)) = \sum \rho_{jk} m_k(x)$$

and

$$\begin{aligned} m(\rho(x)) &= c_n \int_{S^{n-1}} \left(\frac{\pi i}{2} \text{sign}(x \rho^{-1}(y)) + \log \left| \frac{1}{x \rho^{-1}(y)} \right| \right) \frac{y}{|y|} d\sigma(y) \\ &= c_n \int_{S^{n-1}} \left(\frac{\pi i}{2} \text{sign}(xy) + \log \left| \frac{1}{xy} \right| \right) \frac{y}{|y|} d\sigma(y). \end{aligned}$$

As a consequence, there is a corresponding change from a Riemannian manifold with its "extension quantities" (Grassmann) to a Hilbert space framework of differentials.

We also note that in order to model "extended quantities" in a "continuum" there are differentiable manifolds required in case of a Riemannian manifold ([ScE] 1.1.3).

J. Plemelj and an alternative normal derivative definition

A new mathematical concept to define the normal derivative on the boundary with only "continuous" regularity assumption (only using interior domain values) was given by [PIJ]. J. Plemelj's mathematical concept enables the "existence" of a massless particle in the form of a differential connected to its related potential by a Stieltjes integral potential representation. We recall from [PIJ] I, §8:

"bisher war es ueblich fuer das Potential $V(p)$ die Form

$$V(u)(s) = \int \gamma(s-t)u(t)dt$$

vorauszusetzen, wobei dann $u(t)dt$ die Massendichtigkeit der Belegung genannt wurde. Eine solche Annahme erweist sich aber als eine derart folgenschwere Einschraenkung, dass dadurch dem Potentials $V(p)$ der grosste Teil seiner Leistungsfahigkeit hinweg genommen wird."

$$V(u)(s) = \int \gamma(s-t)du(t) \cdot$$

With respect to the normal derivative we recall from ([PIJ] p. 11):

"Vom Integral $\int \frac{\partial U}{\partial n} ds$ auf einer nichtgeschlossenen Kurve ergibt sich aus der Gleichung (6) eine Eigenschaft von grosser Wichtigkeit. Das Integral hängt nämlich nur von den Endpunkten ab und nicht von der näheren Form der sie verbindenden Integrationskurve in der Weise, dass die Integrale alle gleich einander gleich sind, welche Integrationswege entsprechen, die durch stetige Deformation im Regularitätsgebiete auseinander hervorgehen. **Sind also p und q zwei Punkte im Regularitätsgebiete** und verbindet man sie durch irgendeine Kurve (die Tangenten hat), so ist $\int_p^q \frac{\partial U}{\partial n} ds$ wohl definiert und hat einen von der näheren Form der Kurve nicht abhängigen Wert.

... Das Integral zwischen zwei Punkten p und q

$$\bar{U}(q) = - \int_p^q \frac{\partial U}{\partial n} ds$$

ist, weil von der Kurve unabhängig, eine wohl definierte Funktion der Grenzen p und q und soll in seiner Abhängigkeit von q mit $\bar{U}(q)$ bezeichnet werden."

Quantum gravity and the Heisenberg uncertainty relation

From a mathematical (functional analysis) point of view the key challenge/issue of the Heisenberg uncertainty relation is the fact, that the commutator of the location (multiplication) operator Q and momentum operator P is only a Pseudo Differential Operator (PDO) of order "zero", while conceptually it should be a PDO of order "2". The "constant" PDO of order "zero" is the root cause of the "discrete" quantum energy levels, which jeopardizes a truly quantum field theory.

Let Q and P denote the location and momentum quantum operators and I denote the Identity operator. Then the Heisenberg uncertainty relation is given by

$$[P, Q](u)(x) := (PQ - QP)(u)(x) = c \cdot I(u)(x) ,$$

i.e. the commutator is a Pseudo Differential Operator (PDO) of order "0".

Replacing the momentum operator P by $S = S_1$ (notation according to ([BrK1]), which is PDO of order "1"; the corresponding commutator definition leads to a PDO of order "2" in the form

$$[S, Q](u)(x) := (SQ - QS)(u)(x) =: S_2(u)(x)$$

In case of the one dimensional model problem, as considered in (BrK) this is a singular (convolution) integral operator with kernel function $s_2(x) = x \cdot s_1(x)$, which is a Laplacian-like differential operator of order "2".

For a reduced regularity assumptions (i.e. $u \in H_0$, only) all related variational equations, which require a kind of "energy" inner product, require a variational Hilbert space framework with reduced regularity, i.e. H_{-1} , which leads to an (energy-space) inner product (u, v) . Let $((u, v))$ denote the inner product of H_{-1} . Then the corresponding "variance/commutator" measuring norm is given by

$$((S_2 u, u)) = (u, u).$$

In case of higher regularity assumptions (i.e. $u \in H_1$), which is required to have a kind of continuity according to the Sobolev embedding theorems) the corresponding Hilbert scale shift upwards by +1 would lead to the norm

$$((S_2 u, u)) = (\nabla u, \nabla u)$$

which is the "standard" energy measuring metric.

In other words:

Bohr's probability theory based interpretation of the Heisenberg uncertainty relation is built on the assumption of an underlying L_2 variational Hilbert space framework. Bohr's Interpretation is just a physical explanation of purely mathematical facts & consequences, which are described by the Heisenberg uncertainty relation, which is the same mathematical framework as used in the as-is probability theory with relationships between the concepts of "expected value" and "variance".

The as-is "standard" Hilbert space framework is

Standard: " A variational theory embedded in the "standard" $L_2 = H_0$ Hilbert space and its related inner product (u, v) ".

This means, Bohr's interpretation is a consequence of the existing "as-is" mathematical framework, (which by chance or by nature also fits to the framework of probability theory). At the same time the

same L_2 – Hilbert space framework creates the so-called “Serrin gap” issue, when analyzing the non-linear, non-stationary Navier-Stokes equations within a variational framework.

Moving the “as-is” mathematical Hilbert space framework downwards the Hilbert scale by -1, i.e. applying variational theory in a “to-be” $H(-1)$ (instead of the standard $L_2 = H_0$ Hilbert space framework, leads to

Non-Standard: “ A variational theory embedded in the “non-standard” H_{-1} Hilbert space and its related inner product $((u, v))$ ”.

- It overcomes the “uncertainty issue” (while keeping consistent with the “standard-as-is-case-model”, when projecting into the higher regularity Hilbert space $L_2 = H_0$:

The Riemann (curvature /manifold) and the Riesz (rotation /Hilbert space) operators

Let σ_i be a 1-form and $d\sigma_i$ be a differential; let V^* be the dual space of the vector space V with dimension n and let $\Lambda(V^*) = \sum_1^n \Lambda^k(V^*)$ be the exterior algebra of the vector space V . Then the curvature tensor of a Pseudo-Riemannian manifold can be interpreted as a transformation of the 2-forms in the sense of

$$R : \Lambda^2(M^m) \rightarrow \Lambda^2(M^m)$$

$$R(\sigma_i \wedge \sigma_j) = \frac{1}{2} \sum_1^n R_{\alpha\beta}^i \cdot \sigma_\alpha \wedge \sigma_\beta$$

Especially in case of $n = 4$ this builds an endomorphism $R : \Lambda^2(M^4) \rightarrow \Lambda^2(M^4)$, i.e. it is related to the group $SO(3, R)$, where the domain of R is given by the 4-dimensional Riemannian manifold. We also note that the Hodge operator $*$: $\Lambda^2(M^4) \rightarrow \Lambda^2(M^4)$ acts on $\Lambda^2(M^4)$ in the form of $** = (-1)^k$.

The link to our proposed truly Hilbert space geometry framework could be given by the following analogue definition: let

$$Riesz : R(d\sigma_i, d\sigma_j) := \sum R(d\sigma_\alpha, d\sigma_\beta)$$

be the Riesz energy (inner) form operator with the domain of $R(ds)$ is $V^* = H_{-1}$. The affinity and neighborhood between $R(s)$ and $R(ds)$ is obvious. We mention that for a vector space with dimension $n = 4$ the Riemannian operator $R(s)$ is an endomorphism.

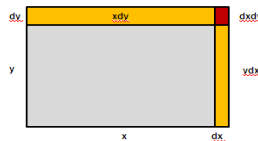
S. Lie's contact transformations and B. Riemann's continuous manifolds

Lie's theory of "contact transformation" builds the foundation of the Lie theory in the context of the manifolds [LiS], [LiS1]. The most popular contact transformation is the Legendre transformation. By this **contact transformation** (!) the equivalence

$$L(x, y) \rightarrow H(x, \frac{dL}{dy})$$

of the Hamiltonian and the Lagrange formalism is proven. A key concept to it is the Leibniz formula: Leibniz formula is already giving non trivial differential calculus in the form

$$d(x + y) = dx + dy \quad d(xy) = xdy + ydx + dx dy = xdy + ydx \cdot$$



In standard analysis the term $dx dy$ is neglected as infinitely small of second order (!). This might be a first opportunity, when extending k-forms into a non-standard framework: The Legendre transformation (Lagrange --> Hamilton) of $f(x, y)$ is defined by

$$g := g(x, y) := y\psi - f = y\psi(x, y) - f(x, y)$$

and

$$d(g) = yd\psi - \frac{\partial f}{\partial x} dx + (d\psi dy) \cdot$$

The product $d\psi dy$ is neglected to be zero in the standard theory as infinitesimal small of second order compared to dx . If one would neglected this and calculate in a non-standard way it would result into

$$d(g) = (y + dy)d\psi - \frac{\partial f}{\partial x} dx \cdot$$

Proof. Putting

$$\psi := \psi(x, y) := \frac{\partial f(x, y)}{\partial y}$$

the differential of $f(x, y)$ gives

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial x} dx + \psi dy$$

As holds

$$\frac{\partial [y\psi(x, y)]}{\partial \psi} = y \quad \text{and} \quad d(\psi y) = \frac{\partial(\psi y)}{\partial \psi} d\psi + \frac{\partial(\psi y)}{\partial y} dy = yd\psi + \psi dy$$

it follows

$$d(g) = d(\psi y) - df = yd\psi + \psi dy - \left[\frac{\partial f}{\partial x} dx + \psi dy \right] = yd\psi - \frac{\partial f}{\partial x} dx + (d\psi dy) \cdot$$

The product $d\psi dy$ is neglected to be zero in the standard theory as infinitesimal small of second order compared to dx .

The terminology of "multiple extended quantities" was used by B. Riemann synonym to a "continuous manifold", conceptually based on two essential attributes: "continuity" and "multiple extensions". Since Helmholtz, Riemann, Poincare and Lie the history of manifolds are the attempt to build a mathematical structure to model the whole (the continuum) and the particular (the part) to put its combination then into relationship to describe motion, action etc. From the paper from E. Scholz below we recall the two conceptual design strategies:

Strategy I: design of an "atomistic" theory of the continuum: to H. Weyl's opinion this contradicts to the essence of the continuum by itself

Strategy II: develop a mathematical framework, which symbolically explores the "relationship between the part and the whole" for the case of the continuum.

The later one leads to the concept of affine connexion based on the concept of a manifold, which were developed during a time period of about 100 years.

The concept of manifolds leads to the concept of co-variant derivatives, affine connexion and Lie algebra to enable analysis and differential geometry, but ([ScE1])...a .." *truly infinitesimal geometry ... should know a transfer principle for length measurements between infinitely close points only.*"

Our alternative definition of the energy (inner product) Dirichlet integral ([BrK1]), which is rotation invariant with respect to infinitely close points, is proposed to build a truly infinitesimal geometry, which then would lead to a "principle of general contra-variance"

Standard Model of Elementary Particles (SMEP): $SU(3) \times SU(2) \times U(1)$

The SMEP is given by $SU(3) \times SU(2) \times U(1)$. Its components are the following interaction dynamics fields:

1. Electromagnetic Interaction Dynamics (EID): $U(1)$
2. Weak Interaction Dynamics (WID): $SU(2) \times U(1)$
3. Strong Interaction Dynamics (SID): $SU(3)$.

It is "just" modeled as the "orthogonal" group stick together by the three "force specific" (gauge) groups. The today's gravity model is modeled as a Riemannian manifold with metric, building on the concept of exterior derivatives and (infinitesimal small) affine geometry. Gauge theory and affine geometry theory both try to overcome to contact body problem in the infinitesimal small. A spin, for example, is an "angular momentum" carried by elementary particle. It has a definite magnitude. Einstein's requirement for a unifying geometrical field theory (particles result as singularity-free solutions of Maxwell and gravity field equations) cannot be fulfilled by both theories. The mathematical framework to enable this (in combination with variational theory) is a Hilbert space. The building of an appropriately defined Hilbert space goes along with the definition of a corresponding inner product, which generates a norm, which then gives the metric of the Hilbert space.

The generalization of the one-dimensional ground state energy model leads to the concept of Pseudo Differential operators (PDO) with domains of differentials: The generalization of the Hilbert transforms leads to the concept of the Riesz operators, which show remarkable properties concerning "rotation".

A correspondingly defined negatively scaled Hilbert space leads to a geometry based field theory, which is independent and therefore does not need to build on EID, WID and SID. It is about a truly and purely (intrinsic) infinitesimal geometry, which enables "Differentials (monads) Interaction Dynamics (DID)" and which is built on the 4 dimensional space-time Minkowski space.

With respect to the today's Standard Model of Elementary Particles (SMEP) this leads to a Non-Standard Model of Elementary Particles (NMEP).

The negatively scaled Hilbert space enables a quantum state statistics (expectations value and variance) for bosons. The consequence is that the related "expectation value and variance" measures are now decoupled from the corresponding probability theory measures. By this the statement "god does not throw dice" is true, but, at the same time, one has to add, that "god doesn't count, measure and gauge", and therefore god doesn't need finite "length" units to measure distances, especially god does not need the Archimedean axiom.

In this context we recall from [ScE] p. 1594:

"...a truly infinitesimal geometry ...should know a transfer principle for length measurements between infinitely close points only".

We note that the alternative to the "Superstring" "theory", the "Loop Quantum Gravity", is built on a Hilbert space K_{diff} , modeling 3D diffeomorphism invariance and transformation properties of spin network states under diffeomorphism ([RoC] 6.4). The Hamiltonian for the fields is built in a standard analysis framework and defined by ([RoC] 6.4.2, 7.3)

$$H := H_{Einstein} + H_{Yang-Mills} + H_{Dirac} + H_{Higgs} .$$

[RoC] 1.2.1: *“The LQG is characterized by the choice of a different algebra of basis field functions, as in Quantum Field Theory (QFT). In conventional QFT this is generally the canonical algebra formed by the positive and negative frequency components of the field modes. The quantization of this algebra leads to the creation and annihilation operators A and A^+ . The characterization of the positive and negative frequencies requires a background space-time. In contrast to this, what characterizes LQG is the choice of a different algebra of basis field functions: a non-canonical algebra based on the holonomies of the gravitational connection. The holonomy (or “Wilson loop”) is the matrix of the parallel transport along a closed curve.”*

Therefore, conceptually also the LQG will struggle with the same handicaps, as H. Weyl's affine geometry.

We quote from [WeH], p. 18: *„Ich bin fest davon überzeugt, dass die Substanz heute ihre Rolle in der Physik ausgespielt hat. Der Anspruch dieses von Aristoteles als einer metaphysischen konzipierten Idee, ., das Wesen der realen Materie auszudrücken – der Anspruch der Materie, die fleischgewordene Substanz zu sein, ist unberechtigt. Die Physik muss sich ebenso der ausgedehnten Substanz entledigen, wie die Psychologie schon längst aufgehört hat, die Gegebenheiten des Bewusstseins als „Modifikationen“ aufzufassen, die einer einheitlichen Seelensubstanz inhärieren.“*

Superstrings and Loop Quantum Gravity

The two existing attempts to build a quantum gravity theory are "SUPERstring SYmmetry" (SUSY) and "Loop Quantum Gravity" (LQG):

- The "Superstring" "theory", [GrM], [KaM] addresses the element of "energy/force" of a "quantum=string" by allowing a string to vibrate in several forms to interact with its environment, i.e. to interact with other strings. It does not address the "particle/string" issue of "contacting "bodies/substance" w/o extension", i.e. an explanation of the interaction of such "string substances" is still unsolved resp. not modeled adequately

- The "Loop Quantum Gravity", [RoC] theory is building on a modification of H. Weyl's affine geometry concept, which is about transformation properties of spin network states under diffeomorphism. Therefore, the "contacting" body issue is still not answered. Nevertheless it builds a Hilbert space framework to enable the linkage to quantum mechanics and quantum field theory.

Our approach builds a Hilbert space framework, which provides an (energy inner product) model to generate "energy" by rotation of "monads". The baseline for this is the "new ground state energy model". It overcomes the current conceptual issue, that different (gauge) groups need to be defined (acting on related to be defined sets (manifolds)) to model the dynamic fields of the three different kind of Nature forces (excluding the "graviton" "quantum", i.e. excluding the gravitation "force"). The rationale for the today's field definition concept, where appropriately defined "groups are acting on sets", is the fact, that the sets in scope are manifolds, which "accept" only parallel/affine infinitesimal small "objects" (parallel displacements, vector and exterior derivative concept). As a consequence, the space-time dimension of manifolds per "force" provides a constraint factor for each of the three field definitions. The consequences are that there needs to be concepts like "flavor" and "spin" to be introduced /added to the model to increase the number of degrees of freedom.) We can be lucky that there are at most four Nature forces to be modeled somehow. Otherwise there would be the need to introduce additional quantum attributes like "taste" or "pattern").

The conceptual new element of the proposed Hilbert space approach compared to SUSY and LQG is about the fact, that it does not need a required concept of "action of a group on a (manifold) set", which links current gauge groups to manifolds. The proposed Hilbert space framework provides an "all-inclusive-package" of appropriate "vector field" properties with a corresponding "world" metric, defined by the related inner product and norm. The Hilbert scale theory provides the appropriate framework to model the (purely) physical requirements properly, including a proper spectral and approximation theory.

M-Theory

We recall from [KaM] p.4, 10 "The powerful techniques of renormalization theory developed in quantum field theory over the past decades have failed to eliminate the infinities of quantum gravity.....the problem has been, however, that even the powerful gauge symmetries of Yang-Mills theory and the general covariance of Einstein's equations are insufficient to yield a finite quantum theory of gravity. At present, the most promising hope for a truly unified and finite description of these two fundamental theories is superstring theory and its latest formulation, M-theory. Roughly speaking the way in which superstring theory solves the riddle of infinities can be visualized by Fig. 1.1, where we calculate the scattering of two point particles by summing over an infinite set of Feynman diagrams with loops. **Unfortunately, the geometry of the superstring and membranes are some of the last features of the model to be developed.** This means that general relativity cannot be a renormalizable theory. ... Because general relativity and quantum mechanics can be derived from a small set of postulates, one or more of these postulates must be wrong. The key must be to drop one or more of these assumptions about Nature on which we have constructed general relativity and quantum mechanics. Over the years several proposals have been made to drop some of our commonsense notions about the universe: continuity, causality, unitarity, locality, point particles."

The Higgs boson

We recall from [HiP]
...."the idea, that the apparently approximate nature of the internal symmetries of elementary-particle physics is the result of asymmetries in the stable solution of exactly symmetric dynamical equations is an attractive one. Within the framework of quantum field theory such a "spontaneous" breakdown of symmetry occurs if a Lagrange-like, fully invariant under the internal symmetry group, has a structure that the physical vacuum is a member of a set of (physically equivalent) states which transform according to a nontrivial representation of the group. That vacuum expectation values of scalar fields, might play such a role in the breaking of symmetries.... in a theory of this type the breakdown of symmetry occurs already at the level of classical field theory...."

We emphasize that our proposed model fits to this statement, while being valid at the same time for the Maxwell equations without any further requirement for additional space-time dimensions to keep consistency between all physical relevant models, which can be represented by the Hamiltonian formalism.

The Higgs boson is about affected / *internal* symmetry groups, conformal invariance, hypothetical vacuum state, massive quantum state and the Higgs boson.

For a quick introduction into the relation between Lie algebras of "internal" symmetry groups, a hypothetical vacuum state usually characterized by a Higgs field, which is a "scalar" field (i.e. not transforming under space-time coordinate change) and a massive quantum state, the Higgs boson we refer to [ScE5].

With respect to the today's "model" of both, the hypothetical and massive quantum state, based on "combined" (finite) transformation groups (gauge theory) we propose the (truly symmetric) analog for our infinitesimal (scale gauge) rotation group (applied to differentials):

massive vacuum state --> the "mass element" dm in the sense of [PIJ]

hypothetical vacuum state: --> the Hilbert transform of the "mass element" dm , i.e. $H[dm]$ as potential of dm .

We note that the Hilbert transform plays a key role in conformal mapping; we especially refer to the method of Theodorsen and Garrick and its relationship to the Cauchy-Riemann differential equations, resp. the relationship between the Riesz operators and the generalized Cauchy-Riemann differential equations.

Non-Standard Analysis and monads

The field of the non-standard numbers ${}^*\mathbb{R}$ is an Archimedean non-ordered field, while the field \mathbb{R} of the real numbers is an Archimedean ordered field.

The field of "Non-standard numbers" has the same cardinality as the field of "Real Numbers". The way, how we are going to measure distances is the same, based on the Archimedean axiom, which is going back to the axiom of Eudoxos, [WeH] p. 41, 45.

Needless to mention, that experimental physics is anyway only requiring rational numbers, while theoretical physics models are calculating with differentials like as with irrational numbers.

One of the probably greatest miss understanding is the assumption, that irrational and transcendental numbers of the field of real numbers are more "real", than the non-standard (non-real) "elements" of monads/ideal points (resp. the later ones are more transcendental than the first ones). The cardinal numbers of both fields are the same. It's just about the differentiator "ordered" versus "non-ordered" field.

How our current understanding and interpretation of the physical/ measurable world would look like, if our children would learn right from the beginning mathematical analysis as described in the language of "ideal" points?

The current "Non-Standard-Analysis" would be a standard one and the other way around. This would mean that our universe would be realized and interpreted as non-standard, as we all were learned at school, but "standard" in the way, how Leibniz would have been defined/interpreted his "differentials" and their actions in the universe. Singularities would become "natural" and consistent to the corresponding physical-mathematical models, big bang would require no $t=0$ information and black hole are just "objects" (no sophisticated phenomena) [RoA],

Probably interesting to mention that today physicists calculate with differentials as "number" objects, but they neglect its physical existence as "particle" objects, while mathematicians calculate with differentials only as "functionals" or within the Cartan differential form calculus, but accept those "objects" as well defined existing "objects" of an e.g. Hilbert space (which is the today's mathematical standard framework for quantum mechanics modeling quantum "objects", ending up with quotes like the following one from N. Bohr: "*If people are not scared about the quantum theory, they haven't understood it*").

Berkeley described Leibniz' differentials as "*ghosts of departed quantities*":

The "truly" infinitesimal geometry of H. Weyl

H. Weyl stated about his concept of a „purely infinitesimal geometry“ [WeH4]), that “*according to his conviction*”, ..it .. “*contains the physical world as a special case*”. There is a three step approach to develop his concept:

1st step: *the continuum in the sense of analysis situs, without any metric-physically speaking, the empty world*

2nd step: *the affine connected continuum; a manifold in which the concept of infinitesimal parallel displacement of vectors is meaningful; in physics, the affine connection appears as the gravitational field; The concept of affine connexion (group properties of vectors / displacements) already enables an analysis of tensor densities*

3rd step: *the metric continuum - physically: the “ether”, whose states are manifested in the phenomena of matter and electricity.”* The affine geometry is equipped with a metric (i.e. the metric is NOT a property of the geometry), which enables a method to *generate* tensor density (e.g. metrical curl density, electromagnetic field density).

In order that the affine geometry is valid in the infinitesimal small, it is necessary and sufficient that the coordinate transformations are continuous differentiable, i.e. it requires a differentiable manifold. The affine connexion goes along with the concept of vectors/displacements, which comply with the group properties of a vector field. The characterizing group of the affine geometry is the infinitesimal **translation group, which acts to the manifold set**. A definition of a purely "inner" product of the manifold itself, which would make the manifold to a Hilbert space, is not possible. The concept of a co-variant derivative enables an exterior product concept, only.

The affine geometry is building on the concept of affine connexion. In order that this concept is defined in the infinitesimal small it is necessary and sufficient that the coordinate transformations are continuous differentiable. This requires a **differentiable** manifold (whereby even the “continuous” contact issue isn’t solved from a physical and philosophical perspective). The original idea of a manifold due to Riemann ([ScE],1.1.2) is a concept, which combines the idea of multiple extended quantities with the definition of a class of "continuous" functions ("variable quantities"). Based on this S. Lie developed the concept of (continuous) contact transformations. Algebraically this is modeled by "*the action of a group on a set*" ([StS]).

The proposed quantum gravity model is a Hilbert space (not only a metric space), where the inner product is defined for differentials, enabled by remarkable properties of the Riesz operators. **The corresponding** (world) **metric** is built on the corresponding inner product/norm $((du, du))$, which now **becomes a property of the Hilbert space geometry**. The Hilbert space framework enables a relationship to the mathematical framework of the quantum mechanics. At the same time the Hilbert space inner product builds the relationship to Riemann's quadratic differential forms ds . In this context we quote from [WeH] §15, p. 86 "*...the laws of transformation of most physical quantities are intimately connected with that of the differentials $dx(i)$ *" p.86 "*Only in the infinitely small we expect to encounter the elementary and uniform laws, hence the world must be comprehended through its behavior in the infinitely small*".

With respect to H. Weyl's vision of a truly infinitesimal geometry, this is still valid and just required from all his arguments and philosophical thoughts [WeH]; the change is from an affine (displacement) geometry to a rotation (Hilbert space based) geometry, analogue to the Euclidean geometry, which expresses the Pythagorean nature of the rotation geometry.

Hilbert scale, Sobolev embedding theorems

The concept leads to a Hilbert space H_{-1} (alternatively to H_0). We note that from a practical side it might be appropriate to choose the Hilbert space index depending from the space Dimension (Sobolev embedding Theorem). In this context we mention, that in the corresponding (weak) variational formulation of the corresponding physical law (u being an element of H_{-a} and the test space elements v being an elements of $H(0)$) the related weak norm is $H_{-1/2}$, i.e. the "strong" norm H_{-1} is weaker than the "weak" norm. We further note that the regularity of the Dirac "function" is in the size of $H_{-n/2}$, in fact by an infinitesimal small positive scale index more regular than $a := -n/2 + \varepsilon$.

We note that in case of a space-time dimension $n=4$, in case physical laws are formulated as variational equation in a $H_{-n/2}$ "continuum" Hilbert space framework, where the test function space is H_0 , the corresponding "weak norm" resp. the "weak norm Hilbert space" is H_{-a} ,

$$a = -\left(\frac{n}{2} + \frac{0+n/2}{2}\right) = -1.$$

We also mention that the embedding of the Hilbert space H_b into H_a is dense and compact for any real a, b with $a < b$. As a consequence, in case of originally, "naturally", infinitesimal small (transcendental) formulated and "valid" "natural" laws in a H_a Hilbert space (from a mathematical point of view), which cannot be verified by physical experience per design, as they are beyond the physical world, any "higher" regularity assumptions, which "projects" into the embedded corresponding Hilbert space H_b , provides an appropriate formulation of an approximating variational equation, which no longer transcendental, but a "physics" law ([VeW] 4.1.6).

In other words: all physical principles (e.g. the Einstein equivalence principle) keep to be valid in the "higher regular" Hilbert space environment, which is required to allow physical experiences. There might be a new (fundamental) physical principle, which is more precise about the spin, i.e. the angular momentum carried by elementary particles (which is in our case the spin (rotation) of the dx(i)-axis) (see also below: "SMEP and Einstein field model).

Continuum, "continuity", Bose-Einstein statistics

Based on the above the Hilbert space H_{-1} provides the mathematical framework for the "continuum". At the same time this Hilbert space provides the answer to the (still) open question of B. Riemann about a characterization of all periodical functions ($n = 1$), which can be represented by its Fourier series (LaD) 2.2, (RiB) W. 237, 238, 244):

Proposition: It is sufficient and necessary that a 2π -periodical function u belongs to the Hilbert space $H_{-1/2}$ to guarantee that it can be represented as its Fourier series.

In this context we also note the wavelet transformations. A function $\psi \in L_2(-\infty, \infty)$ is called a wavelet, if it holds

$$0 < c_\psi := 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \quad \text{i.e.} \quad \|\psi\|_{-1/2}^2 < \infty \quad \text{i.e.} \quad \psi \in H_{-1/2},$$

and the wavelet transformation is defined (for real a, b with $a \neq 0$) by

$$f \rightarrow \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(x) \cdot \psi\left(\frac{x-a}{b}\right) dx.$$

For another open question concerning the characterization of the space-time dimension $n = 4$ (in the context of Huygens's principle and spherical waves) we refer to the below.

In [ZyA] there is a similar kind of "functions" considered as elements of a space H_a -class with $0 < a < 1$: chapter VII, section 11, "Capacity of sets and convergence of Fourier series".

Its main **Theorem 11-3** states:

"let a_n, b_n denote the generalized Fourier coefficients of an element of H_{-1} with respect to the corresponding inner product. Then the trigonometric Fourier series (built by a_n, b_n) is of outer logarithmic capacity 0."

The relationship of this "continuum" (H_{-1} Hilbert space) model to the as-is standard quantum Hilbert space model of today's quantum mechanics and field theory is then "just" an orthogonal projection from the less regular Hilbert space into the standard L_2 Hilbert space. Thus, there is an obvious relationship to today's mathematical definitions of quantum elements and quantum wave packages. The Hilbert space H_{-1} also provides an alternative framework for the quantum state statistics for bosons (now for Hilbert space elements " $d\mu$ "). The expectation value and the variance definitions "just" have to be formulated enabled by the H_{-1} inner product, alternatively to the Standard L_2 inner product. The Hilbert space H_{-1} also provides an adequate framework for the fundamental theorem of thermo dynamics, which is (to the author's best knowledge) the only physical law, which has a mathematical representation by differentials, only:

$$dU = T \cdot dS + P \cdot dN_1,$$

i.e. there is no "mathematically" valid equation existing. At the same time, this representation is the basis for the Bose-Einstein statistics for bosons.

Ramanujan's ingenious formal technique and Riemann's conception of infinity

[BeB] Chapter 6: "*Ramanujan's theory of divergent series emanates from the Euler-Maclaurin summation formula, building on sophisticated constant "c" (1.1) of the series $\text{SUM}(f(k))$. He claims that the constant of a series is like the centre of gravity of a "body" (p. 63). For example the constant of $\text{SUM}(1/k)$ is the Euler constant "Gamma". The difficulties in Ramanujan's definition of a constant for a series have been overcome by Hardy.*" This happened basically by a series of auxiliary constants, similar to those in the above example of Euler's proof of the limit of $\ln(2)$ and $(3/2)*\ln(2)$.

In the third quarterly report ([BeB], 3.7-3.9) "*Ramanujan briefly studies orders of infinity. None of the results were new, but his approach seems novel*". He introduced the conception of *slowly divergent series*. Those relate to the operator A above.

In the first quarterly report [BeB] the relationship to the Bessel function series is handled, which gives the context to the Bessel function section. The today's well known concept of generalized Fourier transforms, built on the orthogonal eigen pairs of a Hilbert space in case of H_{-1} intrinsically, leads to generalized Fourier series representations of the periodical Delta function, the fractional part number function and other related "Fourier series representation" [BeB] chapter 8, entry 17(iv) and 17(v)).

The Hilbert space H_{-1} is the proposed Hilbert space to model the zero point energy, (see "2nd proof, Jan 2011" section). It might be that in the sense of the Riemann characterization question and also in the context of wavelets and embedding theorems of the Dirac function for space-time dimension $n = 4$ the higher regular Hilbert space $H_{-1/2}$ would be more appropriate.

We refer to [NoE]: The theory of invariant variation problems and invariant "infinitesimal transformations" keeps still applicable within the H_{-1} Hilbert space and provides the basis for a proper linkage of "function" elements of H_{-1} with differential forms.

The corresponding (new) Delta function with respect to the H_{-1} inner product might provide opportunities to overcome current required "re-normalization" techniques to "solve" singularity issues in today's QED (R. Feynman).

In the context of the above we also refer to the [HaJ], [DrW]: "*The ground state is the amplitude for the universe to appear from nothing.*" The article above questions the interpretation Hawking/Hartle calculates the amplitude for the universe to appear from nothing".

Einstein's requirements about unified geometrical field theory

Definition: A **gravitational singularity** or **space-time singularity** is a location where the quantities that are used to measure the gravitational field become infinite in a way that does not depend on the coordinate system. These quantities are the scalar invariant curvature of space-time, which includes a measure of the density of matter.

Our "particles/differentials" concept (the confirmation of Leibniz concerning his "monads" concept), as elements of an appropriate Hilbert space H_{-a} , $a > 0$, are per definition an intrinsic part of that "continuum". The "measurable" "realization" (from a physics point of view) of those particles is modeled as potential of those "mass elements" (not as only mass densities, as in standard theory, but in the sense of J. Plemelj). Their role in corresponding variational formulation of any considered physical laws are "just" the weak solutions of those variational equations. The corresponding mathematical (functional analytical) concept is about "orthogonal projection" with respect to the corresponding inner product. This is valid especially for the Maxwell and for the Einstein field equations, either formulated as variational equations or as Hamiltonian (operator norm) minimization problem (which is the "action minimization" variational problem), ([VeW] 2.1.2, 2.1.3). We claim that this is the final confirmation to the dispute of H. Weyl and A. Einstein in support of H. Weyl.

Gravitational singularities

A gravitational singularity or space-time singularity is a location where the quantities that are used to measure the gravitational field become infinite in a way that does not depend on the coordinate system. These quantities are the scalar invariant curvature of space-time, which includes a measure of the density of matter. The Einstein-Cartan-Sciama-Kibble theory of gravity naturally averts the gravitational singularity at the Big Bang. This theory extends general relativity to matter with intrinsic angular momentum (spin) by removing a constraint of the symmetry of the affine connection and regarding its anti-symmetric part, the torsion tensor, as a variable in varying the action. The minimal coupling between torsion and Dirac spinors generates a spin-spin interaction in fermion matter, which becomes dominant at extremely high densities and prevents the scale factor of the Universe from reaching zero.

Philosophical aspects

We basically refer to papers from H. Weyl and M. Heidegger, especially to the great book of H. Weyl ([WeH]).

There are different opinions/views on „what’s matter are?“, as its most likely best expressed by the well know mathematical vs. physical views on it, formulated by the mathematician R. Penrose versus the physicist S. Hawking.

We recall some quotes from [BIS]: "*What fills up space? The curious nature of things and their properties*";

... Kant thought, that if we can only know objects because of their potential effects on others, their powers, then it seems that we are only responsive to what they do but not responsive, necessarily, to what they are. He thought that there have to be „other intrinsic properties, without which the relational properties would not exist because there would be no subject, in which they inhered“. But it’s not clear how we can know about this „subject“.... Are we therefore cut off from the world as that? Then we would be caught in a „false imaginary world“, (Bishop Berkeley).

Michael Faraday thought, that we could just do without Kant’s „other intrinsic properties“. Suppose we try to distinguish a particle x from the powers or forces m whereby it makes its influence known. Then, Faraday writes,

„to my mind ... the x or nucleus vanishes, and the substance consists of the power, or m, and indeed what notion can we form of the nucleus independent of its power: what thought remains on which to hang the imagination of an x independent of the acknowledged forces? Why then assume the existence of that of which we are ignorant, which we cannot conceive, and for which there is no philosophical necessity?"

The problem which this is whether we can be satisfied with the idea that „the substance consists of the powers“, or whether contrary to Faraday there is some kind of philosophical necessity to posit a substance as well, a nucleus or thing that actually possesses the powers.

But there is an argument that we need Kant’s further category of intrinsic properties. We might call it the not-just-washing argument, after Bertrand Russell, who talks in his book „The Analysis of Matter“ of how „there are many possible ways of turning things hitherto regarded as „real“ into mere laws concerning the other things,“ and remarks, „Obviously there must be a limit to this process, or else all things in the world will merely be each other’s washing.“ The conclusion is that even if we have trouble understanding things apart from their powers, nevertheless we seem to need them. We seem to need them because otherwise we have no conception at all the actual world.

From [WeH2] we recall

p. 18: *"I am convinced that the substance has lost its role in physics"*

p. 19: *"the concept of "momentum" appears to be primarily to the concept of "mass/matter"*

p. 20. *"the mass of a body is determined by its state"*

p 31: *"when using a test particle to test/model the action of a field one already disturbs the field"*

p. 44: *"a strictly intuitive rational of a mathematical theory of the continuum (as drafted by Brouwer and Weyl) were required to build the continuum as a medium, where single particles can be identified, but where the set of particles can be resolved"*

p. 49: *"the today's relationship between matter and field is dynamical: the matter builds the field, the field acts on the matter"*

p. *"For Leibniz the "reality" of movement is not built on movements (change of the position), but on the causing force; "La substance est un etre capable d'action - une force primitive"*

p. 58: *"...the Leibniz agens theory of matter can be executed by the GRT. Based on this a matter particle is even not a point in the field space, even not any kind of something related to "space" (extensives)"*

p. 59: *"what is matter? After the perception of the concept of substance has been quashed, the today's beam vacillates between a dynamic and a field theory of matter"*

The conceptual background of a truly infinitesimal geometry, whereby "physics is the science which has geometry as its foundation" is probably best formulated in [WeH4], [ScE5] and parts (especially pdf's 3 & 4) in [PIJ].

We claim, that the mathematical concept of a "point particle", which is required to test the presence of (continuously "acting") forces, is the root cause for current conceptual miss matches between quantum and gravitation theory ([WeH1], [BeH]). For the GRT, as well as for the quantum theory point wise convergence of functions is of no interest. A to-be-developed mathematical GUT model needs to overcome the corresponding inherited constraints, basically caused by the concept of "particles", which goes along with the requirement to formulate a ("continuous") contact transformation between "objects" w/o extensions. "Particles" are e.g. required to describe (directly) its movement (which requires the conception of "continuity", leading e.g. to the famous paradox of Zenon) or (indirectly, as test particle) to define "forces" as a consequence of a potential, which is only then "reality", if there is a test particle". This then ends up to the paradox of continuous forces in combination with felt "continuous" actions, but with "discrete" energy quantum.

If our proposed (truly inner (!) infinitesimal geometry based) quantum gravity model, based on the proposed "new ground state energy model", is resp. becomes a valid model, some of the following conclusions would be the following:

"God does not throw dice, God do not measure "displacements / distances / extensions" by counting (Peano axiom system) the number of normalized (finite) gauges and God does not need integers and rational numbers (ratios of integers) to measure "subsets / ratio of distances" of such normalized gauges. Both concepts are required to define the axiom of Archimedes / Eudoxus, which ensures a distance measurement between zero and any real number x not equal to zero".

"For example the physical concept of "force" (through which physics represents reality) is an observable (source) of physical measurable attributes of matter, only, enabling "continuous" action transmission between "truly substances / monads (Leibniz)". As the Legendre transformation is no longer valid in the infinitesimal small only the Hamiltonian formalism is defined; the Lagrange formalism is not applicable in the infinitesimal small."

The wave-particle dualism is one of several expressions of the long history of thought about mind/spirit/souls and body/matter relationships. The most popular representative is Democritus with his philosophy of "atomism".

To spot on the view from G. W. Leibniz we recall from [FiK] the key message about the relationship of **mind and matter**:

Aristoteles: "mind / souls = purpose of body /matter"

Plato: " mind / souls = form of body"

Pythagoras / Leibniz: "mind = harmony / mass of body".

Leibniz's monads are characterized by the two "forces" of "souls" and "body". The unification of "souls" and "body", the monads can never be considered as Independent "entities" ([FiK] chapter IV, 1.2, [HeM]).

QUOTES

Quotes from [WeH] III, 15:

"the transformation rules of most of the physical quantities are connected with its differentials"

Quotes from [WeH] III, 14:

"the truly infinitesimal geometry is per definition transcendental, as "physics never can be lead back to geometry".

Quotes from [WeH] § 13 *the problem of relativity (p. 74)*

... "The widest group of automorphisms one can possibly envisage for a continuum consists of all continuous transformations; the corresponding geometry is called topology. ...Having explained automorphism we now come to a second phase of the relativity problem. How is it possible to assign to the points of a point-field marks or labels which could serve for their identification or distinction?" ...

Quotes from [WeH] §15, *Riemann's point of view, topology (p. 84-86)*

"The notations of dimensionality and sense are not restricted to metric Euclidean or affine space.One sees that both dimensionality and sense derive from the fact that affine geometry holds in the infinitely small. While topology has succeeded fairly well in mastering continuity, we do not yet understand the inner meaning of the restriction to differentiable manifolds. Perhaps one day physics will be able to discard it. At present it seems indispensable since the laws of transformation of most physical quantities are intimately connected with that of the differentials $dx(i)$. ..

Inspired by Gauss's theory of curved surfaces, Riemann assumed that Euclidean geometry holds in the infinitely small. Then the square of length ds of the infinitesimal vector

As the true lawfulness of nature, according to Leibniz's continuity principle, finds its expression in laws of nearby action, connecting only the values of physical quantities at space-time points in the immediate vicinity of one another, so the basic relations of geometry should concern only infinitely closely adjacent points ("near-geometry" as opposed to "far-geometry"). Only in the infinitely small may we expect to encounter the elementary and uniform laws, hence the world must be comprehended through its behavior in the infinitely small..."

Quotes from [WeH3]: §5 "Tensors"

*"In a Cartesian co-ordinate system the co-variant components coincide with the contra-variant components. It must again be emphasized that the contra-variant components alone are at our disposal in affine space, and that, consequently, wherever in the following pages we speak of the components of a displacement without specifying them more closely, the contra-variant ones are implied. ... 2. in opposition, however, to this representation there is another which, we nowadays consider, does more justice to the physical nature of force, inasmuch as it is based on the conception of **work**."*

Quotes from [WeH3]: §18 "Metrical Space from the Point of View of the Theory of Groups"

*"Whereas the character of affine relationship presents no further difficulties - ... - we have not yet gained a view of metrical structure that takes us beyond experience. It was long accepted as a fact that a metrical character could be described by means of a quadratic differential form, but this fact was not clearly understood. Riemann many years ago pointed out that the metrical ground form might, with equal right essentially, be a homogeneous function of the fourth order in the differentials, or even a function built up in some other way, and that it need not even depend rationally on the differentials. But we dare not stop even at this point. The underlying general feature that determines the metrical structure at the point P is the **group of rotations**. The metrical constitution of the manifold at the point P is known if, among the linear transformations of the vector body (i.e. the totality of vectors), those are known that are **congruent** transformations of themselves. There are just as many different kinds*

of measure-determinations as there are essentially differential groups of linear transformations (whereby essentially differential groups are such as are distinguished not merely by the choice of coordinate system). In the case of **Pythagorean metrical space**, which we have alone investigated hitherto, the group of rotations consists of all linear transformations that convert the quadratic ground form into itself. But the group of rotations need not have an invariant at all in itself (that is, a function which is dependent on single arbitrary vector and which remains unaltered after any rotations).

Let us reflect upon the natural requirements that may be imposed on the conception of rotation. ..."

M-Theory

[KaM] p.4, 10 quote: “*The powerful techniques of renormalization theory developed in quantum field theory over the past decades have failed to eliminate the infinities of quantum gravity.the problem has been, however, that even the powerful gauge symmetries of Yang-Mills theory and the general covariance of Einstein’s equations are insufficient to yield a finite quantum theory of gravity. At present, the most promising hope for a truly unified and finite description of these two fundamental theories is superstring theory and its latest formulation, M-theory. Roughly speaking the way in which superstring theory solves the riddle of infinities can be visualized by Fig. 1.1, where we calculate the scattering of two point particles by summing over an infinite set of Feynman diagrams with loops. **Unfortunately, the geometry of the superstring and membranes are some of the last features of the model to be developed.** This means that general relativity cannot be a renormalizable theory.*”

Therefore, as no geometry of the superstrings exists M-Theory is no final solution yet. What’s more concerning is the fact, that superstring theory is conceptually building on the existing particle concept, as the substance of matter. The consequences are gauge theories to be used for the description of elementary particles and their interactions. **Local gauge invariance is a central issue** (basically caused by the still unsolved “continuum” problem), leading to the concept of renormalizable gauge theories.

What’s the mathematical conceptual difference between today’s particle based paradox and the (still only promised) string based “solution” to overcome the open questions about the continuum?

It replaces the arte-fact of a (massless) particle with no extension and cardinality “zero” by a string with again no extension, but other cardinality, which is the same as the cardinality of the real numbers and the unit square (bigger than the cardinality of the integers). The “interface” and interactions between strings and interactions via “forces” still require an explanation, how “bodies” without extension “touch” each other, interact by forces or “create” observed forces (e.g. the paradox of a space-body environment with massless vacuum with energy or a gravity theory, which requires matter to generate a field and fields to model actions on (“dead”) matter; “the stage-actor problem”). This is another view on the fact that general relativity (and therefore, as a necessary consequence, also a GUT) cannot be a renormalizable theory.

The above (especially as the most important piece, a corresponding geometry, is still missing) looks very much that following further the road to a unifying (Einstein and quantum field theory) M-Theory geometry is walking through a dead-end street, where the dead-end signal has already been passed..

Our proposed alternative is to go back the street and challenge the assumptions, which jeopardize the building of a proper geometry. This follows the advice, which has been given by M. Kaku:

[KaM]: “*Because general relativity and quantum mechanics can be derived from a small set of postulates, one or more of these postulates must be wrong. The key must be to drop one or more of these assumptions about Nature on which we have constructed general relativity and quantum mechanics. Over the years several proposals have been made to drop some of our commonsense notions about the universe: **continuity**, causality, unitarity, locality, **point particles**.*”

Then the new way to go is guided by H. Weyl’s vision of a truly infinitesimal geometry. With respect to this, we quote the following from his book:

[WeH] III,22,E. "ATOMISM", (p. 184); quote : “*From the standpoint of a consistent substantial theory of matter there is no reason to see why, among the infinite continuous manifold of substantial spheres with all possible radii, just those few discrete possibilities are realized which correspond to the chemical elements; the mass however should be determined by the radius. We have seen before that experience is completely at variance with this requirement. The ether theory, on the other hand, imposes no restriction upon charge e and mass m of a body; here there is no collision with experience. Yet again it remains unexplained why of all these possibilities but a few are realized for the elementary particles. Only pure field theory holds out some hope that it might be able to explain this basic fact. For it could happen that its (non-linear) field laws were such as to possess no more than a discrete number of regular static spherically symmetric solutions.*”

Rational and opportunities

The MATHEMATICAL concept of "point particles" is applied and necessarily required in BOTH theories (which might be not anticipated by the "particle-wave" dualisms fans): not only the quantum theory, but also the mathematical field theory necessarily requires "test particles" to "verify" from a mathematical modeling perspective the "existence" and action of "field" forces. In mathematics, particles are identified by real numbers and the conception of continuous functions are "defined" in the context of identifying two functions as the "same", based on a "domain" of real numbers (see also GCa) below).

We refer to [LaD], [RiB], [CaG].

At the beginning of "Analysis" (Cauchy, Dedekind, Riemann, Weierstrass, Cantor ...) the conception of "continuous" function was developed in the context of trigonometric functions. B. Riemann asked the still unanswered question about a characterization of those functions, which can be represented as trigonometric series and are its Fourier series representation, at the same time; it ended up with today's "Standard Analysis" [LaD] p. 181 ff).

From [LiS] we quote

Quote (p.2): "Für den dreifach ausgedehnten Raum können die betreffenden Eigenschaften folgendermassen zusammengefasst werden:

die Bewegungen des dreifach ausgedehnten Raumes bilden eine Gruppe von reellen Transformationen, welche die folgende Eigenschaft besitzt: Wird ein reeller Punkt und ein reelles hindurchgehendes Linienelement festgehalten, so ist immer noch kontinuierliche Bewegung möglich; wird jedoch ausserdem ein durch das Linienelement gehendes reelles Flächenstück festgehalten, so bleiben alle Punkte des Raumes in Ruhe.

Diese Eigenschaft kommt der Gruppe der Euclidischen und der Gruppe der Nichteuklidischen Bewegungen, aber keiner anderen Gruppe zu.

In einem Raum mit mehr als drei Dimensionen lassen sich die beiden betreffenden Gruppen in ganz entsprechender Weise charakterisieren. Dagegen stellt sich die Sache wesentlich anders in einem zweifach ausgedehnten Raume; in der Ebene giebt es noch weitere Gruppen, welche die genannten Eigenschaften besitzen."

The papers of Riemann and Cantor lead to the conception of point wise convergence of series of "functions" in combination with the conception of "continuity" of functions. On the other side, point wise convergence is of no interest for mathematical physics. Alternatively the conception of convergence by quadratic means became quite successful in this area (GRT and quantum mechanics, as well), leading to the conception of Hilbert spaces ([LaD] p. 212).

"**Continuity**" comes along with the physical concept of movement, basically and at the beginning related to the 3-dimensional space, but later on also for more than 3 dimensional spaces, which don't Need to be an Euclidian space. The famous two papers of S. Lie provided a characterization of "movements" in such space frameworks (the space dimension $n=2$ showed different result!):

S. Lie (1. Abhandlung): *"Die Bewegungen des dreifach ausgedehnten Raumes bilden eine Gruppe von reellen Transformationen, welche die folgende Eigenschaft besitzt: Wird ein reeller Punkt und ein reelles durchgehendes Linienelement festgehalten, so ist immer noch kontinuierliche Bewegung möglich; wird jedoch ausserdem ein durch das Linienelement gehendes Flächenelement festgehalten, so bleiben alle Punkte des Raumes in Ruhe."*

"Diese Eigenschaft kommt der Gruppe der Euclidischen und der Gruppe der Nichteuklidischen Bewegungen, aber keiner anderen Gruppe zu. Der hiermit aufgestellte Satz hat ein bedeutendes Interesse, da er auf die Grundlagen der Geometrie Licht wirft."

S. Lie (2. Abhandlung): "*Nach dem Vorgange von Riemann und Herrn v. Helmholtz habe ich versucht die Axiome, welche der Geometrie zu Grunde liegen, auf ein Minimum zurückzuführen. ... In der nachstehenden Arbeit gebe ich nunmehr eine vollständige Darstellung meiner übrigen Untersuchungen auf dem Gebiete. Ich glaube durch diesselben die älteren Untersuchungen von Herrn v. Helmholtz und von Riemann wesentlich vervollständigt und verbessert zu haben.*"

Note: "wird **ein durch das Linienelement gehendes Flächenelement festgehalten, so bleiben alle Punkte des Raumes in Ruhe**". This means that in a corresponding physical "world" model no movements need to exist and cannot exist. S. Lie formulated this statement for space dimension $n=3$, but his results are also valid for $n>3$.

S. Lie (2. Abhandlung): "*Ich bin im Uebrigen, ich wiederhole das, der Ansicht, dass **die freie Beweglichkeit im Infinitesimalen** die Gruppe der Bewegungen in der einfachsten Weise charakterisiert,*"

The later ("die freie Beweglichkeit im Infinitesimalen") one sounds very much as an alternative definition of a monad/ ideal point.

We refer to [LuW]. It gives e.g. the kind of effects of the theory of infinitely small and infinitely large existing numbers to the theory of limits, to the conception of continuity and differentiability, to Euler's product for the sine function and the existence of a not measurable function in the sense of Lebesgue. The Nonstandard numbers have same cardinal number than the real numbers. Therefore there is no difference with respect to its ability to "support" a mathematical modeling of a "continuum". The only difference between both fields is about the no longer valid "Archimedean axiom". The "axiom of choice" is required to build the Non-standard number field. Its equivalent formulation as "Lemma of Zorn" is applied to prove the existence of basis of a Hilbert space. It enables the mathematics to handle with "objects", add/create "objects" to a mathematical framework, which are related to that framework "just" by its properties. So Robinson used it ([LuW] to "create" "objects", which are representing the "infinitely small delta" between dy/dx and $2x$ for $y = x^2$. The Axiom of choice also allows the Proposition, that each Hilbert space has an Basis, i.e. each element of a Hilbert space can be represented as linear combination of that Basis. This is basically the generalized Fourier series representation. The later one is related to the trigonometric series representation of a "continuous" function (see below).

We note that the field of real numbers has the same cardinality as the set of all subsets of the positive integer numbers \mathbb{N} . Therefore its cardinality is greater than then the cardinality of \mathbb{N} . If there exists a "number field" between both of each (the continuum hypothesis), cannot be decided (Paul Cohen).

We also note that the cardinality (e.g. the "number" of particles) in the unit square is the same as the "number of particles" in the interval $(0,1)$. The same is true for the fields of real (standard) numbers and its extension field of ideal (non-standard) numbers. It's "just" the Archimedean axiom, which makes the difference.

Radiation, Huygens' principle and Maxwell theory

In the context of radiation, Huygens' principle and Maxwell theory with respect to a characterization of a continuum with dimension $n=2$ and $n=4$ we note the distortion-free, traveling, spherical waves (Hadamard) conjecture ([CH II] VI, §10, 3 „radiation and Huygens' principle“):

„Distortion-free families of progressive/traveling, spherical waves of higher orders exist only if the Huygens' principle is valid; families of spherical progressive waves as such exist only for $n=2$ and $n=4$.“

The concept of "particles"

To weaken the necessary of an "existence" of a "particle" (in fact, to get rid of it) without losing the capability to model e.g. a single layer potential we propose to apply the Stieltjes/Lebesgue integral instead of the (generalized) Riemann integral conception, following the concept of J. Plemelj, who proposed an alternative definition of the normal derivative.

A. Robinson, (1966): Results and techniques from Non-Standard analysis ... *"appears to us today that the infinitely small and infinitely large numbers of a non-standard model of Analysis are neither more nor less real than, for example, the standard irrational numbers..."*

P. M. Dirac: *"I learnt to distrust all physical concepts as the basis for a theory. Instead one should put one's trust in a mathematical scheme even if the scheme does not appear at the first sight to be connected with physics. One should concentrate on getting an interesting mathematics"*.

Lemma of Zorn and Riemann's hypothesis, which lie at the bases of geometry

There are two fundamental, mathematical axioms/hypothesis, which build the foundation of today's mathematical-physical models:

- Zermelo's axiom of choice (resp. the lemma of Zorn) --> enabling e.g. the existence of a basis of Hilbert space, which builds the mathematical foundation of quantum theory

- Riemann's hypothesis which lie at the bases of geometry --> enabling e.g. PDE theory, manifolds and affine connection, which build the foundation gravity theory.

One essential crotch (going back *"the road of reality"* of R. Penrose), which finally ended up with Einstein's gravity theory, is going back to the [RiB], where he rejected the following conceptions:

- **the existence of the Delta functions** ([LaD] p. 205): originated by Cauchy and Dirac et. al., (defined in a $L(2)$ Hilbert space Framework)

- **the existence of an infinite large integer number i** (numerus infinitus i): originated and successfully applied by Euler-Leibniz, ([LaD], p. 293).

At the same time B. Riemann "calculated" on the conceptual level of today's (Cartan's) **"differential forms"**, as well as with **"limits"**.

Definition $SU(n)$, $SO(3)$

The special unitary group of degree n , denoted $SU(n)$, is the group of $n \times n$ unitary matrices with determinant 1. (In general, complex unitary matrices have complex determinants with modulus 1, but arbitrary phase.) The group operation is that of matrix multiplication. The special unitary group is a subgroup of the unitary group $U(n)$, consisting of all $n \times n$ unitary matrices, which is itself a subgroup of the general linear group $GL(n, \mathbf{C})$.

The Lie group $SO(3)$ models electromagnetic interactions. The group $SO(3)$ is the special orthogonal group for space dimension $n=3$, which is a 3-D rotation group. It is a diffeomorphism to the projective space $RP(3)$. The group $SO(3)$ is connected, but not simply connected. The simplest case $SU(1)$ is the trivial group, having only a single element.

The standard model of elementary particles (SMEP): $SU(3) \times SU(2) \times U(1)$ and quantum field theories of electromagnetism $U(1)$, the weak force $SU(2)$ and the strong force $SU(3)$.

Modern physical theories describe reality in terms of fields, e.g., the electromagnetic field, the gravitational field, and fields for the electron and all other elementary particles. A gauge theory is a type of field theory in which the Lagrangian is invariant under a continuous group of local transformations. Gauge theories are important as the successful field theories explaining the dynamics of elementary particles.

The importance of gauge theories in physics is exemplified in the tremendous success of the mathematical formalism in providing a unified framework to describe the quantum field theories of electromagnetism, the weak force and the strong force. This theory, known as the Standard Model, accurately describes experimental predictions regarding three of the four fundamental forces of nature, and is a non-abelian gauge theory with the gauge group $SU(3) \times SU(2) \times U(1)$. It has a total of twelve gauge bosons: the photon, three weak bosons and eight gluons.

We note $U(3) = U(1) \times SU(3)$.

The electrodynamics field: $U(1)$

The earliest field theory having gauge symmetry was Maxwell's formulation of electrodynamics. The symmetry group $U(1)$ is equivalent to the group of rotations in the plane. It has one gauge field, the electrodynamics four-potential, with the photon being the gauge boson. The gauge potential is essentially the 4-vector potential of electromagnetism.

The Quantum Electrodynamics field: $SU(2) \times U(1)$ (isospin \times isotropy symmetry)

Generalizing the gauge invariance of electromagnetism, a theory was constructed based on the action of the (non-abelian) $SU(2)$ symmetry group on the isospin doublet of photons and neutrons. This is similar to the action of the $U(1)$ group on the spinor fields of quantum electrodynamics (QED). The group $SU(2)$ is isomorphic to the group of quaternions of norm 1, and is thus diffeomorphic to the 3-sphere. Since unit quaternions can be used to represent rotations in 3-dimensional space (up to sign), there is a surjective homomorphism from $SU(2)$ to the rotation group $SO(3)$ whose kernel is $\{+I, -I\}$. It is the field model for the electroweak interaction.

The Quantum Chromo Dynamics field: Quark Flavor $SU(3)$

The special unitary group $SU(3)$ is the field model for Quantum Chromo Dynamics (QCD) a group on the color triplet of quarks. $SU(3)$ flavor symmetry. $SU(3)$ corresponds to special unitary transformation (3×3 matrices) on complex 3D vectors. The quarks (u, d, s) are all light (compared to hadron masses) and their interactions are dominated by the flavor-independent color force. The group $SU(3)$ provides a description of the exchange bosons (gluons) of QCD and allows the interactions between colored quarks to be calculated.

The Einstein (gravitation) field

The Einstein field equations are "just" an **axiom**. It can be described as Hilbert-Einstein functional, which corresponds to the Hamiltonian formalism, building on an appropriately defined energy resp. operator norm. According to the GRT an observed gravity force vanishes in a properly defined coordinate system, i.e. in the GRT an observed gravity force is just a phenomenon (which depends from the position of the observer), while the gravity itself is equivalent to the space-time curvature.

The measure of the space-time curvature at a point in the space-time continuum has the two components, the Weyl and the Ricci tensor. The **Weyl tensor** measures the local space-time curvature itself; The Weyl tensor is volume preservative. The **Ricci tensor** measures how the quantity of matter in a small ball (i.e. the density of the matter) at that point in space-time determines the inwards directed gravity acceleration of the matter particles. The size of the volume reduction of the matter particles is a measure for the Ricci curvature. The challenge is, that matter and curvature are not a consequence of the geometry (i.e. they are not intrinsic components of the geometry), but necessarily required physical elements: „*matter determines the curvature and curvature determines matter*“, and are not consequences of the geometry.

And here the loop is not closed, but close to, as the origin of the "Eichtheorien" goes back to H. Weyl (enabling the gauge theories), who is also the father of the idea of a truly infinitesimal small geometry (dispute with A. Einstein). Einstein's "challenge" (no congruence of infinitesimal small bodies") is closed now by the proposed (rotation invariant) gravity model, where we "just" exchanged the "displacement/vector" group by the rotation group. As this is, at the same time, a truly infinitesimal Euclidean (inner) geometry, the "beauty" requirement to an unified quantum gravity model is also fulfilled.

Differential geometry, minimal surfaces & varifold geometry

Differential geometry provides the mathematical framework for Einstein's field theory. The concept of least areas problems leads to the theory of minimal surfaces (integral varifold) as part of differential theory and geometric measure theory.

A standard minimal surface model is the „Plateau problem“. One existence proof of the „Plateau problem“ was given by R. Courant („Plateau's Problem and Dirichlet's Principle“, Ann. of Math., 38 (1937), 679-725): instead of minimizing the area $A(X)$ the energy integral (Dirichlet) $E(X)$ is minimized, applying the Dirichlet principle, which is basically an equivalent formulation of the Plateau problem. It holds: $E(X) \geq A(X)$, and $E(X)=A(X)$ if and only if X is weak conform.

Branching points are inevitable for space dimension greater than 3. Putting this in the context of the equivalence of energy and minimal surface might enable additional insight into the more or less unknown area of „singularities“ in the gravitation theory. (In case of $n < 4$ no branch point exists (for $n=2$ this is the proposition of the Riemann mapping theorem, in case of $n=3$ it is the result of the work of Osserman, Gulliver, Alt). The combination of the theory of minimal surfaces with differential forms results into the theory of „**Varifold Geometry**“ [AIF].

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