The three Millennium problem solutions, RH, NSE, YME, and a Hilbert scale based quantum geometrodynamics

Overview

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The homepage www.fuchs-braun.com provides solutions to the following Millennium problems

- the Riemann Hypothesis
- well-posed 3D-nonlinear, non-stationary Navier-Stokes equations
- the mass gap problem of the Yang-Mills equations.

A common underlying distributional Hilbert space framework provides an answer to Derbyshine's question ((DeJ) p. 295)(*): „ *What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles*?". It enables quantum gravity theory based on an only (energy related) Hamiltonian formalism, as the corresponding (force related) Lagrange formalism is no longer defined (due to the reduced regularity assumptions to the domains of the concerned operators).

The Bagchi Hilbert space reformulation of the Nyman, Beurling and Baez-Duarte RH criterion provides the link between the two solution areas above (BaB). The Zeta function on the critical line is an element of the distributional Hilbert space H_{-1}. Therefore, in order to verify the Hilbert-Polya conjecture any (weak) eigenfunction solution of a self-adjoint operator equation to verify the Hilbert-Polya conjecture needs to be an element of a $H_{-1/2}$. The imaginary part values ω_n of the zeros of the considered Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ (alternatively to $e^{2\pi inx}$) with its corresponding Mellin transform

$$M\left[{}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\right](s)=\int_0^\infty x^s\,{}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\frac{dx}{x}=\frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)}\quad,\quad 0<Re(s)<1$$

enjoy appropriate properties (SeA), e.g. $2n-1<2\omega_n<2n<\omega_n+\omega_{n+1}<2n+1$ satisfying the "Hadamard" gap" condition.

The corresponding analysis of the 2D-NSE for the 3D-NSE fails due to not appropriate Sobolev norm estimates. This is called the Serrin gap.

The classical Yang-Mills theory is the generalization of the Maxwell theory of electromagnetism where chromo-electromagnetic field itself carries charges. As a classical field theory it has solutions which travel at the speed of light so that its quantum version should describe massless particles (gluons). However, the postulated phenomenon of color confinement permits only bound states of gluons, forming massive particles. This is the Yang-Mills mass gap.

The current quantum state Hilbert space $H_0=L_2$ is extended to $H_{-1/2}$ to enable a Hilbert space based quantum gravity theory.

PART I:
Braun K., *A Kummer function based alternative Zeta function theory to solve the Riemann Hypothesis and the binary Goldbach conjecture*

PART II:
Braun K., *3D-NSE and YME mass gap solutions in a distributional Hilbert scale frame enabling a quantum gravity theory*

(*): ... "*The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles*?"

PART I

A Kummer/cot-function based alternative Zeta function theory to solve the Riemann Hypothesis

The Riemann Hypothesis states that the non-trivial zeros of the Zeta function all have real part one-half. The Hilbert-Polya conjecture states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self-adjoint operator. It is related to the Berry-Keating conjecture that the imaginary parts of the zeros of the Zeta function are eigenvalues of an „appropriate" Hermitian operator $H = \frac{1}{2}(xp + px)$ where x and p are the position and conjugate momentum operators, respectively, and multiplicity is noncommunative. The operator H is symmetric, but might have nontrivial deficiency indices (W. Bulla, F. Gesztesy, J. Math. Phys. 26 (1), October 1985), i.e. in a mathematical sense H is not Hermitian.

The key ingredients of the Zeta function theory are the Mellin transforms of the Gaussian function and the fractional part function. To the author´s humble opinion, the main handicap to prove the RH is the not-vanishing constant Fourier term of both functions. The Hilbert transform of any function has a vanishing constant Fourier term.

Let H and M denote the Hilbert and the Mellin transform operators. Replacing the Gaussian function $f(x) := e^{-\pi x^2}$ and the fractional part function by its Hilbert transforms enables an alternative Zeta function theory.

The Mellin transform of the Gaussian function is given by

$$M[f](s) = \frac{1}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) \ , \ \ M[-xf´(x)](s) = \frac{s}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) = \frac{1}{2}\pi^{-s/2}\Pi(\frac{s}{2}) \ .$$

The related Theta function properties (based on the Poisson summation formula) of

$$G(x) := \theta(x^2) := \sum_{-\infty}^{\infty} e^{-\pi n^2 x^2} = 1 + 2\sum_{1}^{\infty} e^{-\pi n^2 x^2} =: 1 + 2\psi(x^2) = \frac{1}{x}\sum_{-\infty}^{\infty} e^{-\pi\frac{n^2}{x^2}} = \frac{1}{x} G\frac{1}{x})$$

leads to the Riemann duality equation in the form (EdH) 1.8)

$$\xi(s) := \frac{s}{2}\Gamma\left(\frac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s) = (1-s)\cdot\zeta(s)M[-xf'(x)](s) = \zeta(s)\cdot M[-x(xf'(x))'](s) = \xi(1-s).$$

The Mellin transform for Riemann's auxiliary function

$$H(x) := -\frac{d}{dx}(x^2\frac{d}{dx})G(x)$$

is well defined and it holds

$$\int_0^{\infty} x^{1-s} H(x)\frac{dx}{x} = \int_0^{\infty} x^s H(x)\frac{dx}{x} \ .$$

(*) The Hilbert transform of the Gaussian function is given by the Dawson function

$$F(x) := e^{-x^2}\int_0^x e^{t^2}dt = \int_0^{\infty} e^{-t^2}\sin(2xt)\,dt = x\ {}_1F_1\left(1, \frac{3}{2}; -x^2\right) = xe^{-x^2}\ {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right) .$$

The appropriate related Mellin transform formulas are given by ((GrI) 7.612)

$$\int_0^{\infty} x^s\ {}_1F_1(\alpha, \beta; -x)\frac{dx}{x} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\Gamma(s)\frac{\Gamma(\alpha-s)}{\Gamma(\beta-s)} \ , \ 0 < Re(s) < Re(\alpha) \ ,$$

$$\int_0^{\infty} x^{\mu-1}e^{-\beta x^2}\sin(\gamma x)dx = \frac{\gamma e^{-\frac{\gamma^2}{4\beta}}}{2\beta^{\frac{\mu+1}{2}}}\ {}_1\Gamma(\frac{1+\mu}{2})F_1(1-\frac{\mu}{2}; \frac{3}{2}; \frac{\gamma^2}{4\beta}) \ , \ Re(\beta) > 0 \ , \ Re(\mu) > -1$$

leading to e.g., $\frac{1}{2}\int_0^{\infty} x^{s/2}\ {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right)\frac{dx}{x} = \frac{\frac{1}{2}\Gamma\left(\frac{s}{2}\right)}{1-s} = \frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)} = \frac{\Pi\left(\frac{s}{2}\right)}{s(1-s)}$, $0 < Re(s) < 1$. It indicates a replacement of the Gauss „Gamma" function definition ((EdH) p.8)

$$\Pi\left(\frac{s}{2}\right) := \Gamma\left(1+\frac{s}{2}\right) = \frac{s}{2}\Gamma\left(\frac{s}{2}\right) \qquad \rightarrow \qquad \Gamma^*\left(1+\frac{s}{2}\right) := \Gamma\left(\frac{s}{2}\right)tan\left(\frac{\pi}{2}s\right) = \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{\Gamma\left(1+\frac{s-1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma\left(1+\frac{s}{2}\right)}{(k-\frac{1}{2})^2-(\frac{s}{2})^2}.$$

We note the formula ((GRI) 3.511, 8.332) $\frac{2}{\pi}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2 dt = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\pi}{cosh\,(\pi t)}dt = 1$, i.e. $\Gamma(\frac{1}{2}+it) \in L_2(-\infty, \infty)$.

Formally it also holds

$$\int_0^\infty x^{-s}\left[(-\frac{d}{dx}x^2\frac{d}{dx})G(x)\right]dx = s(1-s)\int_0^\infty x^{-s}G(x)dx.$$

It implies that the invariant operator $x^{-s} \to \int_0^\infty x^{-s}G(x)dx$ is formally self-adjoint with the transform $2\xi(s)/(s(s-1))$. But this operator has no transform at all as the integrals do not converge, due to the not vanishing constant Fourier term of the Poisson summation formula ((EdH) 10.3). Replacing $f(x) \to f_H(x) := M[f](x)$ leads to an alternative entire Zeta function $\xi^*(s)$ in the form

$$\xi^*(s) := \frac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\,tan(\frac{\pi}{2}s)\cdot\zeta(s) = \zeta(s)\cdot M\left[\frac{d}{dx}[-x\cdot f_H(x)]\right](s)$$

with same zeros as $\xi(s)$, as it holds $s(1-s)\xi^*(s)\xi^*(1-s) = \pi\xi(s)\xi(1-s)$.

A similar situation is valid, if the duality equation is built on the fractional part function ([TiE] 2.1).

The Mellin transforms in the critical stripe for the distributional Fourier series representation of the cot –function in a distributional H_{-1} –sense are given by (*)

$$M[Cot^*](s) = \zeta(1-s)\cdot tan\left(\frac{\pi}{2}s\right) = \zeta(1-s)\cdot cot\left(\frac{\pi}{2}(1-s)\right)$$
$$M\left[\frac{1}{x}Cot^*(\frac{1}{x})\right](s) = M[Cot^*](1-s) = \zeta(s)\cdot cot\left(\frac{\pi}{2}s\right).$$

(*) The Bagchi Hilbert space based RH criterion is dealing with the fractional part function. Its Hilbert transform is given by

$$g(x) := \ln\left(2\sin\left(\frac{x}{2}\right)\right) = -\sum_{n=1}^\infty \frac{\cos(nx)}{n},$$

which is an element of H_0. Therefore, its related Clausen integral ((AbM) 27.8) is an element of H_1, and its first derivative, $\frac{1}{2}\cot\left(\frac{x}{2}\right)$ resp. $\cot(\pi x)$, joins the Zeta function on the critical line as an element of H_{-1}. The H_{-1} Hilbert space corresponds to the weighted l_2^{-1} –space as considered in (BhB). As $g(x) \in H_0 = H_0^*$, it holds

$$(g,v)_0 \cong (g',v)_{-\frac{1}{2}} = (S^1[g],v)_{-\frac{1}{2}} = (Cot,v)_{-1/2} < \infty,\quad \forall v \in H_0$$

i.e. the formally derived Fourier series representation of

$$Cot(x) = \sum_{n=1}^\infty \sin(nx) \quad \text{resp.} \quad Cot^*(x) = 2\sum_{n=1}^\infty \sin(2\pi nx)$$

is defined in a distributional H_{-1} –sense (see also (BeB) (17.12) (17.13)). For $a > 0$ and $0 < |Re(s)| < 1$ it holds ((GrI) 3.761)

$$\int_0^\infty x^s \sin(a)\frac{dx}{x} = \frac{\Gamma(s)}{a^s}\sin(\frac{\pi}{2}s) \quad , \quad \int_0^\infty x^s\cos(a)\frac{dx}{x} = \frac{\Gamma(s)}{a^s}\cos(\frac{\pi}{2}s).$$

Therefore the Mellin transforms of the H_{-1} – distributional Fourier series representation of the $Cot^{(*)}$- resp. $G_H(x)$ – functions are given by

$$M[Cot](s) = \Gamma(s)\sin(\frac{\pi}{2}s)\zeta(s) \text{ resp. } M[Cot^*](s) = 2(2\pi)^{-s}\Gamma(s)\sin(\frac{\pi}{2}s)\zeta(s)$$
$$M[G_H(x)](s) = 2\sqrt{\pi}\sum_1^\infty\int_0^\infty\int_0^\infty x^s e^{-\pi t^2}\sin(2\pi nxt)\,dt\frac{dx}{x} = 2M[\sum_1^\infty f_H(nx)](s)$$
$$= \sqrt{\pi}\int_0^\infty e^{-\pi t^2}\int_0^\infty x^s Cot^*(tx)\frac{dx}{x}dt = \sqrt{\pi}\left[\int_0^\infty t^{1-s}e^{-\pi t^2}\frac{dt}{t}\right]\cdot\left[\int_0^\infty x^s Cot^*(x)\frac{dx}{x}\right] = \pi^{\frac{s}{2}}\Gamma(\frac{1-s}{2})\cdot M[Cot^*](s)$$

In combination with the functional equation of the entire Zeta function in the form $\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\zeta(1-s)$ ((TiE) (2.1.1)) this leads to

$$M[Cot^*](s) = \zeta(1-s)\cdot tan\left(\frac{\pi}{2}s\right) \quad , M\left[\frac{1}{x}Cot^*(\frac{1}{x})\right](s) = M[Cot^*](1-s) = \zeta(s)\cdot cot\left(\frac{\pi}{2}s\right).$$

On the critical line $s = \frac{1}{2} + it$ holds $M[Cot^*](s)\cdot M[Cot^*](1-s) = \zeta(s)\cdot\zeta(1-s)\cdot[1 + tanh^2(\pi t)]$ because of

$$\sin\left(\frac{\pi}{2}s\right) = \frac{1}{\sqrt{2}}\left[\cosh\left(\frac{\pi}{2}t\right) + i\cdot\sinh(\frac{\pi}{2}t)\right] \text{ and } |\Gamma(s)|^2 = \frac{\pi}{\cosh(\pi t)}$$
$$\cot\left(\frac{\pi}{2}s\right) = \tan\left(\frac{\pi}{2}(1-s)\right) = 1 - i\cdot\tanh(\pi t) = 1 - 2i\cdot\sum_{k=1}^\infty(-1)^k e^{-2kt} \;(\; t > 0\;)$$
$$\cot\left(\frac{\pi}{2}(1-s)\right) = \tan\left(\frac{\pi}{2}s\right) = 1 + i\cdot\tanh(\pi t) = 1 + 2i\cdot\sum_{k=1}^\infty(-1)^k e^{-2kt} \;(\; t > 0\;)$$

From (TiE) 4.14)), (ObF) p. 182, and (EsR) p. 139, we recall the formulas

$$\zeta(s) - \sum_{n<x} n^{-s} = \sum_{n>x} n^{-s} = -\frac{1}{2i}\int_{x-i\infty}^{x+i\infty} z^{1-s}\cot(\pi z)\frac{dz}{z}, \quad Re(s) > 1;$$
$$M\left[\frac{1}{\pi}\frac{x^n}{1-x}\right](s) = cot(\pi s) \text{ (principle value) } \; -n < Re(s) < 1-n\;,\; n = 0, \pm 1, \pm 2, \ldots$$
$$F.p.(P.v.\int_0^\infty \frac{x^\alpha}{1-x}dx = \begin{cases} 0 & ,\alpha \in Z \\ \pi\cot(\pi\alpha) & ,else \end{cases}.$$

They are related to the operator $x^{-s} \to \int_0^\infty x^{-s} G_H(x)dx$ (in a distributional H_{-1} –sense) by

$$M[G_H(x)](s) = \pi^{\frac{s}{2}} \Gamma(\frac{1-s}{2}) M[Cot^*](s)$$

whereby it holds

$$M[-xG'_H(x)](s) = s[G_H(x)](s) \ , \ M[(xG_H)'(x)](s) = (1-s)[G_H(x)](s).$$

The Polya criterion is about the approximation of the Mellin transform integral over the half-line $(0,\infty)$ by integrals over *finite intervals* to obtain a theorem about zeros of the Mellin transforms ((EdH) 12.5), (PoG). The Mellin transform $M[G_H(x)](s)$ is a Müntz type representation, i.e. in a classical framework the Polya criterion cannot be applied.

We note the similar structure between the Polya RH criterion the automodel criterion ((EsR) p.57). The functions $k(x) \coloneqq cot(x)$ resp. $h(x) \coloneqq \frac{1}{x} cot\left(\frac{1}{x}\right)$ are slow varying functions (automodels) of order zero ((EsR) p.57) (*). Other slow varying functions are $-logx$ at $x = 0^+$ or $-\log(1-x)$ at $x = 1$ (SeE).

The functional analysis approach to prove the Prime Number Theorem (PNT) is based on Tauberian theorems, which are derived from the celebrated Wiener Tauberian theorem, that „*the closed linear hull of translates of a function f is the whole space L_1 if and only if its Fourier transform never vanishes*" (**).

In (PiS) Tauberian theorems for integral transforms are provided, which are of Mellin convolution type and whose kernels belong to suitable test function spaces. The result is based on the Wiener-Tauberian theorems for distributions as proven in (PiS1). In (ViJ) a corresponding functional analysis scheme for Tauberian problems is provided to *(prove)* the prime number theorem based on the Dirac delta measure δ_a ($(\delta_a, \varphi) = \varphi(a)$). It is built on the Delta function representation of

$$\psi'(x) = \sum_{n \le x} \Lambda(n)\delta(x-n) \in H_{-\frac{1}{2}-\varepsilon} \qquad \left(\psi(x) = \sum_{n \le x} \Lambda(n) = \int_{a-i\infty}^{a+i\infty} \left[-\frac{\zeta'(s)}{\zeta(s)}\right] x^s \frac{ds}{s} \approx x\right)$$

whereby the generalized Mellin transform of $\sum_{n=1}^\infty \delta(x-n)$ ($Re(s) < 0$) is given by $\zeta(1-s)$ ((ZeA) 4.3). It is proposed to replace the formal delta series $(f_x, \varphi) = \sum_{n=0}^\infty c_n \, \delta_{\frac{n}{x}}$ to the numerical series $\sum_{n=0}^\infty c_n$ by $(f_x, \varphi)_{-1/2} < \infty$,$\forall \varphi \in H_{-1/2}$. Conceptually this goes along with a replacement of the „dual" relationship $L_1 \leftrightarrow L_\infty$ by $H_{-1/2} \leftrightarrow H_{1/2}$(***). The latter Hilbert spaces are the appropriate framework for central functions in current Zeta function theory (****). For a corresponding generalized Mellin (integral) transformation in the form $F(s) = (f(x), x^s)_{-1/2}$ we refer to (ZeA). For $\vartheta(x) = \sum_{n \le x} \Lambda(n) \, log(\frac{x}{n})$ we note the related asymptotics (KoJ) (ViJ)

$$\lim_{\lambda \to \infty} \vartheta'(\lambda x) = \frac{d}{dx}\left[\sum_{n=1}^\infty \Lambda(n) \, log(\frac{\lambda x}{n})\right] = \lim_{\lambda \to \infty} \frac{\psi(\lambda x)}{\lambda x} = 1 \ .$$

(*) For $k(x) \coloneqq cot(x)$ resp. $h(x) \coloneqq \frac{1}{x} cot\left(\frac{1}{x}\right)$ it holds $\frac{xk'(x)}{k(x)} = -\frac{2x}{\sin(2x)}$ resp. $\frac{xh'(x)}{h(x)} = -1 + \frac{2/x}{\sin(2/x)}$;

(**) It is about the behavior of the function f, where the limit for the convolution integral $K[f](x)$ when $x \to \infty$ corresponds to $\hat{k}(0)$ ($\hat{k}$ denotes the Fourier transform of the kernel function $k(x)$);

(***) There is a similar differentiator between a proof of the PNT (****) (from which the convergence of the series $\sum_{n=1}^\infty \frac{\mu(n)}{n}$ can be derived) and a proof of the convergence of the series $\sum_{n=1}^\infty \frac{\mu(n)}{n} log(\frac{1}{n}) = 1$. Ikehara showed a Tauberian theorem for Dirichlet series in a L_1 – framework, which is equivalent to the statement that $d\psi(x) \sim dx$ as a Cesaro average.

„The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived" ((LaE) §160).

(****)
$$\rho(x) = x - [x] = \frac{1}{2} + \sum_{n=1}^\infty \frac{sin(2\pi n x)}{\pi n}, \ \rho_H(x) = \sum_{n=1}^\infty \frac{cos(2\pi n x)}{\pi n} = -\frac{1}{\pi} log(2\, sin(\pi x)), log\left(tan\left(\pi \frac{x}{2}\right)\right) \in L_2^\#(0,1),$$
$$\rho'_H(x) = -cot(\pi x) = -2\sum_{n=1}^\infty sin(2\pi n x), log'\left(tan\left(\pi \frac{x}{2}\right)\right) = \frac{\pi}{sin(\pi s)} \in H_{-1}^\#(0,1),$$
$$\|\Xi\|_{-1}^2 = \sum_{n=1}^\infty \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} = \int_0^1 \frac{log\, x}{x-1} dx = \left[\sum_{n=1}^\infty \frac{\mu(n)}{n^2}\right]^{-1}, \text{ i.e. } \Xi \in l_2^{-1}$$

(****) (EdH) 12.7: „The PNT is about the asymptotics equivalence of $\psi(x) = \sum_{n<x} \Lambda(n) \sim x$, which is equivalent to the statement that $d\psi(x) \sim dx$ as a Cesaro average in the context of Tauberian theorems. Hardy-Littlewood were able to prove the PNT by showing $d\psi(x) \sim dx$ as an Abel average, where a significant amount of work is done by a Tauberian theorem."

Alternatively to the usage of the Hardamard distribution function $\psi'(x)$ with the Dirac function domain $H_{-\frac{1}{2}-\varepsilon}$ we shall use distribution functions with a $log(\frac{x}{n})$ structure in combination with point measures enabling integer subsets with Snirelmann density ½.

The considered Hilbert space in (BaB) is about of all sequences $a=\{a_n | n \in N\}$ of complex numbers such that $\sum_{n=1}^{\infty} \theta_n |a_n|^2 < \infty$ with $\frac{c_1}{n^2} \le \theta_n \le \frac{c_2}{n^2}$, which is isomorph to the Hilbert space $H_{-1} \cong l_2^{-1}$. The real part values of the zeros of the considered Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ (alternatively to $e^{2\pi inx}$) enjoy appropriate behaviors (*). The linkage to convergent Dirichlet series

$$f(s) := \sum_1^{\infty} a_n e^{-s\log n} \quad g(s) := \sum_1^{\infty} b_n e^{-s\log n} \text{ , for } s > 0$$

to the (distributional) Hilbert spaces $H_{-1/2} \cong l_2^{-1/2}$ resp. $H_{-1} \cong l_2^{-1}$ is given by the inner products (**)

$$\left((f,g)\right)_{-1/2} := \lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(1/2+it) g(1/2-it) dt = \sum_1^{\infty} \frac{1}{n} a_n b_n$$
$$\left((f,g)\right)_{-1} := \lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(1+it) g(1-it) dt = \sum_1^{\infty} \frac{1}{n^2} a_n b_n .$$

For the Zeta function on the critical line $\Xi(t) \coloneqq \zeta\left(s=\frac{1}{2}+it\right) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ it holds

$$\Xi \in H_{-\frac{1}{2}-\varepsilon} \quad \text{resp.} \quad \|\Xi\|_{-1/2}^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\Xi(t)|^2 dt = \sum_{n=1}^{\infty} \frac{1}{n} = \zeta(1) = \infty.$$

Putting $w(t) \coloneqq \sum_{n=1}^{\infty} \log\left(\frac{1}{n}\right) \frac{\mu(n)}{n^s}$ it holds $((\Xi,w))_{-1/2} = 1$ from which it follows that $w \in H_{-\frac{1}{2}+\varepsilon}$.

The Hilbert space $H_{-\frac{1}{2}}^{\#} \cong l_2^{-1/2}$ enables a distributional form of the Snirelmann density $\lim_{n\to\infty} \frac{A(n)}{n}$ given by $\sum_{n=1}^{\infty} \frac{1}{n} a_n^2 = \|A\|_{-1/2}^2$ with $A = (a_n)_{n\in N} \in l_2^{-1/2}$. It puts another light on the dispersion method in binary additive number theory problems, where the binary Goldbach problem is inaccessible in the given form (LiJ).

What can derived from the PNT is the convergence of $\sum_{n=1}^{\infty} \frac{\mu(n)}{n}$. What cannot derived from the PNT is the convergence of the series $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} log(\frac{1}{n}) = 1$ (**). *„This theorem goes deeper than the PNT"* ((LaE2) §159). For the corresponding arithmetical function $\sigma(x) := \sum_{n\le x} \frac{\mu(n)}{n} \log\left(\frac{x}{n}\right)$ and $A(x) \coloneqq \sum_{n\le x} \frac{\mu(n)}{n}$ it holds $A(x) = o(1)$ ((ApT) p. 71) and for $x \ge 1$

$$\sigma(xy) + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} log(\frac{1}{n}) = \sigma(xy) + 1 = \sigma(x) + \sigma(y) \ , \quad \sigma'(x) = \frac{1}{x} \sum_{n\le x} \frac{\mu(n)}{n} \sim \frac{1}{x} \quad .$$

Its inverse mapping is given by

$$\sigma^{-1}(x) = \sum_{n\le x} \frac{1}{n} log\left(\frac{x}{n}\right).$$

(*) For the real part values ω_n of the zeros of ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ it holds (SeA) $2n-1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n+1 < 2\omega_{n+1} < 2(n+1)$ and the sequences $2\omega_n$ and $\omega_n + \omega_{n+1}$ fulfill the Hadamard gap condition

$$\frac{\omega_{n+1}}{\omega_n} > \frac{n+\frac{1}{2}}{n} = 1 + \frac{1}{2n} > q > 1 \quad \text{resp.} \quad \frac{\omega_{n+1}+\omega_{n+2}}{\omega_n+\omega_{n+1}} > \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > q > 1.$$

We mention the theorem of Kakeya (HuA) from which is follows that all zeros of $\sum_{k=1}^{n} s_k x^k = 0$ lie in the circular disk $\frac{1}{2} < |x| < 1$. We further mention the relationship to the uniform distribution of numbers mod 1(WeH).

(**) The average orders of $d(n), \varphi(n), \mu(n), \Lambda(n)$ orders are $D(x) \coloneqq \frac{1}{x} \sum_{n\le x} d(n) = \log x + (2\gamma - 1) + \frac{1}{x}\delta(x) \sim log(x)$ (with $\delta(x) = O(\sqrt{x})$), $\sum_{n\le x} \varphi(n) = \frac{3}{\pi^2} + x^2 + O(x\log x)$, zero and one. The latter two average order results are both equivalent to the PNT. Some weighted average orders of those two artithmetic functions are given by $\sum_{n\le x} \mu(n) \left[\frac{x}{n}\right] = 1$, $\sum_{n\le x} \Lambda(n) \left[\frac{x}{n}\right] = \sum_{p\le x} \left[\frac{x}{p}\right] logp = x\log x - x + O(\log x)$, (ApT) pp. 57, 66, 68. Dirichlet's asymptotics $\delta(x) = O(\sqrt{x})$ has been improved, but the exact order is still undetermined. The problem is closely related to that of the Riemann Zeta function: for $c > 0$ and x not an integer it holds $D(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta^2(z) x^z \frac{dz}{z}$ and therefore (for $0 < c' > 1$) $\delta(x) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \zeta^2(z) x^z \frac{dz}{z}$ (TiE) 12.1. It further holds $\sum_{n=1}^{\infty} d(n) n^{-s} = \zeta^2(s)$ resp. $(1-s) \int_1^{\infty} D(x) x^s \frac{dx}{x} = \sum_{n=1}^{\infty} \frac{d(n)}{n} n^s = \zeta^2(1-s)$ (NaC), and $d(n) = O(x^{\varepsilon})$, $\lim_{T\to\infty} \frac{1}{T} \int_1^T |\zeta(\sigma+it)|^2 dt = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^{2\sigma}}$ for $\sigma > 0$ (TiE) p. 171, 148.
The RH is true iff $\frac{1}{x} \sum_{n\le x} \mu(n) = O(x^{-\frac{1}{2}+\varepsilon})$ iff $\psi(x) - x = O(x^{-\frac{1}{2}+\varepsilon}$, (TiE) 14.25.
(**) (ApT): „mean value formulas for Dirichlet series", p. 240, (BiN1): „orthogonal polynomials on the unit circle; $l_2^{1/2}$ as subspace of l_2, and Szegö's theorem and its probabilistic descendants, new definition of long range dependence", (NaS): „$h(x) \coloneqq \frac{1}{x} \sum_{n\le x} d(n) - [\log x + (2\gamma - 1)] \in L_2(0,\infty)$, $H(s) \coloneqq M[h](s) = \frac{\zeta^2(1-s)}{1-s} \in L_2(-\infty,\infty)$ on the critical line, and a formula involving sums of the form $\sum d(n) f(n)$".

In the context of additive number theory problems the asymptotics of some related arithmetical functions for $x \geq 2$ (!) are given by ((LaE1) (*), (ApT) (**), (ScW) p. 216),

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} \sim \log x = \int_1^x d(\log t) \sim \Phi(x) = \sum_{n \leq x} \frac{1}{\varphi(n)} \sim \sum_{p \leq x} (1 - \frac{1}{p})^{-1}$$

$$\sum_{n \leq x} \frac{\mu(n)}{n \log n} \sim \log(\log x) = \int_1^{\log x} d(\log t) \sim \sum_{p \leq x} \frac{1}{p} \quad .$$

Putting $0 < \omega_0 := 1 - \omega_1 < \frac{1}{2}$ we consider the sequences $s_n^{(1)} := \frac{2\omega_n}{2n}$, $s_n^{(2)} := \frac{\omega_{n-1}+\omega_n}{2n-1}$ fulfilling

$$\min\left\{\frac{2n-1}{2n}, \frac{2n}{2n+1}\right\} = 1 - \frac{1}{2n} < s_n^{(1)}, s_n^{(2)} < 1 \ .$$

The related integer subsets

$$F_{1;2n} := \{\{[\omega_{n-1} + \omega_n]\} | n \in N\} = \{\{1, \{2n\}\} | n \in N\}, \ F_{2n-1} := \{\{[2\omega_n]\} | n \in N\} = \{\{2n-1\} | n \in N\}$$

do have the Snirelmann density $\sigma(F_{2n-1}) = \sigma(F_{2n}) = \frac{1}{2}$. They enable the definition of (binary additive) distributional functions for $x \geq 1$ (!) in the form

$$\sum_{\substack{n \leq x \\ n \in F_{1;2n}}} a_n \log(s_n^{(1)} \frac{x}{n}) + \sum_{\substack{n \leq x \\ n \in F_{2n-1}}} a_n \log(s_n^{(2)} \frac{x}{n}) \quad \text{resp.} \quad \sum_{\substack{n \leq x \\ n \in F_{1;2n}}} a_n \log(\frac{x}{2\omega_n}) + \sum_{\substack{n \leq x \\ n \in F_{2n-1}}} a_n \log(\frac{x}{\omega_{n-1}+\omega_n}) \ .$$

We note that the Snirelmann density is sensitive to the first values of a set. This is why the subset of even integers has a Snirelmann density zero, while the subset of odd integers has Snirelmann density $\frac{1}{2}$. Putting

$$\sigma^*(x) := \sum_{\substack{n \in F_{1;2n} \\ n \leq x}} \frac{\mu(n)}{n} \log\left(s_n^{(2)} \frac{x}{n}\right) + \sum_{\substack{n \in F_{2n-1} \\ n \leq x}} \frac{\mu(n)}{n} \log\left(s_n^{(1)} \frac{x}{n}\right)$$

$$= \log x + \sum_{\substack{n \text{ odd} \\ n \leq x}} \frac{\mu(n)}{n} \log\left(s_n^{(2)} \frac{x}{n}\right) + \sum_{\substack{n \text{ even} \\ n \leq x}} \frac{\mu(n)}{n} \log\left(s_n^{(1)} \frac{x}{n}\right) \ .$$

it follows ($x \geq 1$)

$$\sigma^*(xy) = \sigma^*(x) + \sigma^*(y) \quad , \quad \sigma^{*\prime}(x) = \frac{1}{x} + \sigma'(x) = \frac{1}{x} + \frac{1}{x} \sum_{n \leq x} \frac{\mu(n)}{n} \sim \frac{1}{x}$$

whereby $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$, $\left|\sum_{n \leq x} \frac{\mu(n)}{n}\right| \leq 1$ with equality holding only if $x < 2$ ((ApT) p.66, p. 97).

Let G_n denote the number of decompositions of an even integer n into the sum of two primes (whereby $p + q$ and $p + q$ are counted seperately); let $\tilde{G}_n := \frac{1}{2C} \frac{1}{\varphi(n)} \frac{n^2}{\log^2(n)}$ denotes the Stäckel approximation formula with $C := \frac{105 \cdot \zeta(3)}{2\pi^4} \sim 0{,}648 \ldots \sim \frac{1}{2}$. Then it holds (LaE1)

$$\frac{1}{2} \frac{x^2}{\log^2(x)} \sim \sum_{n=1}^{[x/2]} G_{2n} \sim \sum_{n=1}^{[x/2]} \tilde{G}_{2n} = \frac{1}{2C} \sum_{n=1}^{[x/2]} \frac{1}{\varphi(n)} \frac{n^2}{\log^2(n)}$$

whereby (*)

$$\frac{1}{2C} \sum_{n=1}^{[x/2]} \frac{\sigma_1(n)}{\log^2(n)} \leq \frac{1}{2C} \sum_{n=1}^{[x/2]} \frac{1}{\varphi(n)} \frac{n^2}{\log^2(n)} \leq \frac{\zeta(2)}{2C} \sum_{n=1}^{[x/2]} \frac{\sigma_1(n)}{\log^2(n)}.$$

Based on the above concept we propose the following alternative arithmetical function (**)

$$\sum_{n=1}^{[\omega_n+\omega_{n+1}]} \frac{\sigma_1(n)}{\log(\omega_n)} \frac{\sigma_1(n+1)}{\log(\omega_{n+1})} \ .$$

(*) With $C := \frac{105 \cdot \zeta(3)}{2\pi^4} \sim 0{,}648 \ldots \sim \frac{1}{2}$ and the Euler constant γ the asymptotics of

$$\Phi(x) := \sum_{n \leq x} \frac{1}{\varphi(n)} = \Phi_1(x) + \Phi_2(x) := \sum_{\substack{n \leq x \\ n \text{ odd}}} \frac{1}{\varphi(n)} + \sum_{\substack{n \leq x \\ n \text{ even}}} \frac{1}{\varphi(n)} = C\left[\log x + \gamma - \sum_p \frac{\log p}{p^2-p+1}\right] + \delta(x)$$

is given by $\lim_{x \to \infty} \delta(x) = o$ and (LaE1) $\Phi_1(x) = C \cdot \log x + c_1 + O(\frac{\log x}{x})$, $\Phi_2(x) = 2C \cdot \log x + c_2 + O(\frac{\log x}{x})$.

The related estimate to the sum of the divisors of n function $\sigma(n) = \sigma_1(n)$ is given by ((ApM) pp. 38, 57, 71), $\frac{\sigma(n)}{n^2} \leq \frac{1}{\varphi(n)} \leq \frac{\pi^2}{6} \frac{\sigma(n)}{n^2} = \zeta(2) \frac{\sigma(n)}{n^2}$, $n \geq 2$. The proof of the inequality is based on the formula $\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$ and the relation $1 + x + x^2 + \cdots = \frac{1}{1-x} = \frac{1+x}{1-x^2}$ with $x = \frac{1}{p}$. The average order of the divisor function $d(n)$ is given by $\log n$. The partial sums of the divisor function is given by $D(x) := \frac{1}{x} \sum_{n \leq x} d(n) = \log(x) + (2\gamma - 1) + O(1) \sim \log(x)$. It further holds $\sum_{n \leq x} \frac{d(n)}{n} \sim \frac{1}{2} \log^2(x)$, $\sum_{2 \leq n \leq x} \frac{1}{n \log n} \sim \log(\log((x))$.

(**) We note the two inequalities $\frac{1}{ab} > \frac{2}{a^2+b^2}$ & $\log 2 > \log \omega_1$ and $d(n) = O(x^{\varepsilon})$ (TiE) p. 171. *„The odd integers can be disregarded, as every odd integer n can be represented as sum of two primes, if $n - 2$ is a prime number only, otherwise not*" (LaE5).

The current tool trying to prove the tertiary and binary Goldbach conjecture is about the Hardy-Littlewood circle method. It is about a dissection of the circle $x = e^{2\pi i\alpha}$, or rather a smaller concentric circle, into „Farey arcs". The major arcs, or basic intervals, provide the main term in the asymptotic formula for the number of representations. Their treatment does not give rise to any very serious difficulties compared to the problems presented by the *„minor arcs"*, or *„supplementary intervals"*. The latter ones are analyzed by estimates of the Weyl (trigonometrical) sums

$$S(x) := \sum_n e^{2\pi inx}$$

without taking any (Goldbach) problem relevant information into account. We note that an asymptotic behavior in the form $O(N^{\frac{1}{2}+\varepsilon})$ of the Farey series is equivalent to to the Riemann Hypothesis (LaE5).

The Cesàro summable Fourier series representation (ZyA) VI-3, VII-1)

$$cot(\pi x) = 2\sum_{n=1}^{\infty} sin(2\pi nx)) \in H_{-1}^{\#}(0,1)$$

is related to the eigenfunctions $e^{2\pi inx} = e^{i\pi(2n)x}$. The proposed alternative Abel summable functions

$$cot^{(*)}(x) := \sum_{n=1}^{\infty} sin(\pi(2\omega_n)x) + sin\,(\pi(\omega_n + \omega_{n+1})x \in H_0^{\#}(0,1)$$

is related to the eigenfunctions pair $e^{i\pi(2\omega_n)x}$ and $e^{i\pi(\omega_n+\omega_{n+1})x}$ with corresponding alternative Weyl sums in the form

$$S_1^*(x) := \sum_n e^{i\pi(2\omega_n)x},\ \ S_1^*(x) := \sum_n e^{i\pi(\omega_n+\omega_{n+1})x}.$$

For the „weighted" $cot^{(*)}$ –function with the „alternative" harmonic numbers

$$2h_n := \sum_{k=1}^{n} \frac{2}{2k-1} = 2H_{2n} - H_n$$

the series

$$\sum \frac{2h_n}{n}(sin(2\pi\omega_n x) + sin\,(\pi(\omega_n + \omega_{n+1})x)$$

converges almost everywhere (*) (**) (***) (****).

The H_n are always fractions (except for $H_1 = 1$, $H_2 = 1.5$, $H_6 = 2.45$), the series is divergent, but the number n that the sum H_n past 100 is in the size of 10^{43}, i.e. a computer which takes 10^{-9} seconds to add each new term to the sum will have been completed in not less than 10^{17} (American) billion years (HaJ).

The extremly slow nondecreasing property on the interval $[1, n]$ might motivate the definition of an appropriate function to enable the corresponding Polya criterion (EdH) 12.5, (PoG)).

(*) For $T(x) := -\frac{\pi}{2} log\left(tan\left(\pi\frac{x}{2}\right)\right)$ the following series representation holds true (ElL) $T(x) = \sum \frac{2h_n}{n} sin\,(\pi(2n)x) = \sum c_n\, sin\,(2\pi nx)$, whereby $\sum_{n=1}^{\infty} c_n^2 < \infty$ i.e. $T(x) \in L_2^{\#}(0,1)$ resp. the formal Fourier series representation of its first derivative $log'\left(tan\left(\pi\frac{x}{2}\right)\right) = \frac{\pi}{sin\,(\pi x)} \in H_{-1}^{\#}(0,1)$. The convergent series $\sum_{n=1}^{\infty} c_n^2 = \frac{\pi^4}{32} < \infty$ in combination with the

Lemma (KaM1): Let $\{n_k\}$ be a sequence of integers satisfying the "Hadamard gap" condition, i.e. $\frac{n_{k+1}}{n_k} > q > 1$. Then the trigonometric gap series $\sum_{k=1}^{\infty} c_k \sin\,(2\pi n_k x)$ converges almost everywhere, if and only if, $\sum_{k=1}^{\infty} c_k^2 < \infty$

then proves that the series $\sum \frac{2h_n}{n}(sin(2\pi\omega_n x) + sin\,(\pi(\omega_n + \omega_{n+1})x)$ converges almost everywhere.

(**) We note the related potency series in the form (ChH) $\frac{1}{2} log^2\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty} \frac{2h_n}{n} x^{2n}$.

(***) Alternatively to $\frac{\sin\,(\pi x)}{\pi x}$ the Fourier theory of cardinal functions enables a correspondingly absolute convergent cardinal series in the form $C(x) := \frac{\sin\,(\pi x)}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{H_{2n} - \frac{1}{2}H_n}{x-n}$ (WhJ2).

(****) (RiB) p. 11: *„„.. denn so gross auch unsere Unwissenheit darüber ist, wie sich die Kräfte und Zustände der Materie nach Ort und Zeit um Unendlichkleinen ändern, so können wir doch sicher annehmen, dass die Functionen, auf welche sich die Dirichlet'sche Untersuchung nicht erstreckt, in der Natur nicht vorkommen"*

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PART II

3D-NSE and YME mass gap solutions in a distributional Hilbert scale frame enabling a quantum gravity theory

A common mathematical model of unified quantum and gravity theories requires a truly infinitesimal geometric framework. The Hilbert space based framework in quantum theory is certainly the more suitable geometric framework compared to Weyl's manifold based ones. At the same point in time both theories need to leave something out as they are not compatible. In quantum theory already in the simple quantum harmonic oscillator model the eigenvalues converge equidistant to infinity, i.e. the total energy is infinite as well (*). A similar situation is given by the concept of „wave packages" with other (less regular) domain as the H_1 domain for standard Fourier waves. A related concept is a about wavelets leading to the extended Hilbert space $H_{1/2}$ (**). The standard quantum theory Hilbert space is $H_0 = L_2$ in order to enable to full statistical analysis, which is basically statistical thermodynamics.

In a general Hilbert scale framework H_α $(\alpha \in R)$ a convergent momentum operator in a variational (quantum mechanical) representation would be given by $(u', v)_\alpha = (u, v)_{\alpha+1/2} < \infty$, $\forall\, v \in H_\alpha$. One of the current handicaps of quantum mechanics is the purely mathematically conditioned Dirac „point mass density" Hilbert space $H_{-n/2-\varepsilon}$, with $\varepsilon > 0$, where $n =$ denotes the space dimension (i.e. a point mass density in a one dimensional world (like the harmonic quantum oscillator) is different from a similar situation in a three dimensional (or even four-dimensional) model. The choice $\alpha := -1/2$ is proposed as new quantum state Hilbert space with its corresponding energy space $H_{1/2}$.

The GRT is built on Riemann´s mathematical concept of „manifolds"; we note that the mathematical model of the GRT even requires *„differentiable*" manifolds, whereby only *continuous* manifolds are required by physical GRT modelling aspects, w/o taking into account any appropriate quantum theoretical modelling requirements. Therefore, challenging the „continuity" concept, taking into account also its relationship to the quantum theory Hilbert space framework H_a and the related Sobolev embedding theorem, supports to the proposed replacement of the Dirac function concept by an alternative $H_{-1/2}$ −quantum state Hilbert space also from a GRT perspective.

The Lagrange formalism is related to the concept of „force", while the Hamiltonian formalism is related to the concept of „energy". Both formalisms are equivalent *only* (!) in case the Legendre (contact) transform can be applied. Our proposed „alternative energy (Hilbert space) concept" goes along with reduced regularity assumptions of the concerned operators (similar to the regularity reduction when moving from standard potential function (*„mass density*") definition to Plemelj's *„mass element*" concept ($\sim C^1 \rightarrow C^0$)), (PlJ).

The *„mass generation process*" is modelled as a „selfadjoint property" break down by the orthogonal projection $H_{1/2} = H_1 \otimes H_1^{\neg} \rightarrow H_1$, i.e. the closed subspace $H_1^{\neg}$ is the model for the ground state (vacuum) energy, which is and can be neglected in all („less granular") Lagrange formalism based physical models.

(*) With respect to the Kummer functions from the part I we note that the eigenvalue problem of the Schrödinger equation with a Coulomb potential is solved by confluent hypergeometric series

(**) ((BiN): „Traditionally, the subject of time series seemed to consist of two non-intercommunicating parts, „time domain" and „frequency domain" (known to be equivalent to each other via the Kolmogorov Isomorphism Theorem. The subject seemed to suffer from schizophrenia …. This unfortune schism has been healed by the introduction of wavelet methods." Other relations to the Hilbert space $H_{1/2}$ are given by $H_{1/2} \subset VMO \subset BMO$, $H_{1/2}$ occurs in work on topological degree and winding number, conformal mapping, analytical continuability of the Szegö function beyond the unit disc and scattering theory; (NaS): the $H_{1/2}$ space as first cohomology is fundamental to explain the properties of period mapping on the universal Teichmüller space (NaS).

We further note that the exterior Neumann problem admits one and only one generalized solution in case the related Prandtl operator of order one $P: H_r \rightarrow H_{r-1}$ is defined for domains with $1/2 \leq r < 1$ (LiI).

In a nutshell from a mathematical modelling perspective

in the proposed model the (standard) „calculus in the small" meets the „calculus in the large" (MoM) in a Hamiltonian formalism for classes of non-linear equations, where the kinetic (matter, Lagrange formalism) energy part is based on a Krein space setting/decomposition (GaA) of (H_0, H_1) within a $(H_{-1/2}, H_{1/2} = H^*_{-1/2})$ Hilbert space framework (*).

The Hilbert space subspaces (H_0, H_1) are *compactly* embedded into the Hilbert spaces $(H_{-1/2}, H_{1/2})$. This is about the same cardinality relationships as for the embededness of the set of *rational* numbers into the fields of *real* or *hyper-real* numbers (**).

In a nutshell from a physical modelling perspective

the physical concepts of „*time*" and „*change*" are different sides of the same coin, i.e. there is no „time" w/o „change" and there is no „change" w/o „time". In other words, the concepts of „time" and „change" are and need to be in scope of the „matter/kinetic" energy model H_1, while its complementary ground state (vacuum) energy model $H_1^{\neg}$ is per definition independent from the *thermodynamical* concept of „*time*" (***).

As H_1 is compactly embedded into $H_{1/2}$, and given an initial universe w/o any thermodynamical „time" (i.e. $H_1 = \{\ \}$, with only existing ground state energy state for the whole mathematical model system) the probability for „symmetry break down" events to generate mass were and are zero; obviously those events happened and will go on to be happen. At the same point in time the generated and still being generated „matter world" H_1 is governed by e.g. the „least action principle" (KnA), and the principles of „statistical thermodynamic" (ScE), whereby the classical action variable of the system determines the „time" (HeW).

In a nutshell from a philosophical perspective we refer to (HaJ), (KaI) p. 67 (****), (ScE1), (WeH3) appendix B.

(*) We note that the set of integers or rational numbers is „countable", while it is not for the fields of the real and hyper-real numbers. The corresponding (Cantor) cardinalities are given by $card(N) = card(Q) = \aleph$, $card(R) = card({}^*R) = 2^{\aleph}$.

(**) The standard energy Hilbert space H_1 enables a differentiation of "elementary particles" with and w/o mass (modelled by the orthogonal decomposition of the Hilbert spaces $H_{-1/2} = H_0 \otimes H_0^{\perp}$ resp. $H_{1/2} = H_1 \otimes H_1^{\perp}$). The Hilbert space H_1 is proposed to be interpreted as „ferminons mass/energy" space; $H_1^{\perp}$ is proposed to be interpreted as the orthogonal „bosons energy" space. Both together build the newly proposed quantum energy space $H_{1/2} = H_1 \otimes H_1^{\perp}$. The sub-space $H_1^{\perp}$ may be interpreted as zero point energy space containing "wave package" resp. „eigen-differential" „elements".

The concept of an optical function is an essential tool in the strategy to overcome technical difficulties to overcome the problems of „coordinates", and the „strongly nonlinear hyperbolic features of the Einstein equations" for a global stability of the Minkowski space (*). It is basically about appropriately modified Killing and conformal Killing vectorfields in the definition of the basic norm.

(HoM) 1.2: „The idea of wavelet analysis is to look at the details are added if one goes from scale a to scale $a - da$ with $da > 0$ but infinitesimal small. … Therefore, the wavelet transform allows us to unfold a function over the one-dimensional space R into a function over the two-dimensional half-plane $\boldsymbol{H}$ of positions and details (where is which details generated?). … Therefore, the parameter space $\boldsymbol{H}$ of the wavelet analysis may also be called the position-scale half-plane since if g localized around zero with width Δ then $g_{b,a}$ is localized around the position b with width $a\Delta$. The wavelet transform itself may now be interpreted as a mathematical microscope where we identify $b \leftrightarrow$ position; $(a\Delta)^{-1} \leftrightarrow$ enlargement; $g \leftrightarrow$ optics.
The continuous wavelet transform with the complex Shannon wavelet can be considered via solutions of Cauchy problems for PDE in the context of the construction of wavelets for an analysis of non-stationary wave propagation in inhomogeneous media ((PoE).

(***) The macroscopic and microscopic state of quanta relate to corresponding frequencies of its vibrations. The corresponding action variables of the system ((HeW) II.1.c) define the related kinematical (physical) and thermodynamical concept of "time" (RoC), (SmL)), (RoC1), section 13)
(PeR): „one of the deepest mysteries of our universe is the puzzle of whence it came."
(RoC1), section 13: "Our interaction with the world is partial, which is why we see it in blurred way. To this blurring is added quantum indeterminacy. The ignorance that follows from this determines the existence of a particular variable - thermal time - and of an entropy that quantifies our uncertainty. Perhaps we belong to a particular subset of the world that interacts with the rest of it in such a way that this entropy is lower in one direction of our thermal time."

(****) (KaI) I. Convolut, V. Bogen, 4. Seite: *„Selbst der Gebrauch der Mathematik in Ansehung der Anschauungen a priori in Raum u. Zeit gehört zur Transc. Phil.. Es sollte nicht mit Newton heissen Philosophiae naturalis principia mathematica (den es gibt eben so wenig mathematische Principien der Phil. als philos. der Mathematik) sondern phil. transscend. princ. vel mathem. vel phil. als genus. Transc. Phil. ist das subjective Prinzip der vereinigt theoretisch//speculativen and moralisch//practischen Vernunft in einem System der Ideen von einem All der Wesen unter dem Prinzip synthetischer Sätze a priori worin es eben so wenig mathematische Principien der Philosophie als philosophische der Mathematik giebt. Transc: Phil. ist das Prinzip eines Systems der Ideen der synthetischen Erkenntnis a priori aus Begriffen wodurch das Subject sich selbst zum Objecte constituirt (Aenesidemus) und das Formale der Wahrnehmungen zum Behuf moglicher Erfahrung anticipirt"*

A common Hilbert space framework for all quantum gravity related physical mathematical models requires common conceptual building principles for problem specific mathematical-physical PDE system models.

The following changes to current building principles are proposed:

1. a classical PDE system is an „only" approximation model to its corresponding physical relevant variational representation, and not the other way around

2. only the Hamiltonian formalism is valid (*), but not the Lagrange formalism (both formalisms are equivalent, if the Legendre transform is valid), because of only physical (energy related) relevant, but no longer mathematically (force related) assumed regularity assumptions to the variational solution. In this context *we note that „continuity" is one of the commonsense notions, which should be dropped out of the assumptions list of ground principles of the Universe (KaM) p. 12*)); consequently, the physical concept of „*force*" stays to be a phenomenon of the considered PDE (problem specific physical model) system, but is no longer a conceptual element of the overall „physical world reality" (i.e. it is not a notion as part of the stage of theoretical physics).

3. The „Newton/Dirac" „point/particle *mass density*" concept (whereby the regularity of the Dirac „function" depends from the space dimension) ist being replaced by the „Leibniz/Plemelj" „ideal point/differential *mass element*" concept.

The proposed common Hilbert space framework enables variational methods for nonlinear operators (VaM) for the considered mathematical physics models. It overcomes the (claimed) common purely mathematical handicaps for problem adequate solutions in alignment with the *purpose* of physical models. From a *physical* modelling perspective it is about a replacement of Dirac's model of the „*density*" of an idealized point mass or point charge, which is called the Dirac or Delta „function". It is a distribution equal to zero everywhere except for zero, and whose integral over the entire line is equal to one. The Dirac model of the „*density*" of an idealized point mass is replaced by Plemelj's concept of a „*mass element*" (PlJ), with the essential consequence, that the regularity requirement for those distributions $\mathrm{d}\mu$ are independent from the space-dimension in opposite to the Dirac function:

the regularity of Dirac's model of the point mass density of an idealized point mass is $\delta \in H_{-n/2-\varepsilon}$ ($\varepsilon > 0$, n = space dimension), while for Plemelj's mass element definition it holds $d\mu \in H_{-1/2}$.

From a mathematical point of view this means that a Lebesgue integral is replaced by a Stieltjes integral. The corresponding $\mathrm{H}_{-1/2}$ quantum state model (alternatively to the standard $\mathrm{L}_2 = \mathrm{H}_0$ model) goes along with a corresponding quantum energy Hilbert space model $\mathrm{H}_{1/2}$. Its definition follows the same building principles as for the standard Laplace operator in a $\mathrm{L}_2 = \mathrm{H}_0$ framework with its corresponding Dirichlet (energy inner product) integral $\mathrm{D}(\mathrm{u}, \mathrm{v}) = (\nabla u, \nabla v)_0 = (u, v)_1$.

(*) Non-linear minimization problems can be analyzed as saddle point problems on convex manifolds in the following form (VeW): $J(u): a(u,u) - F(u) \to min$, $u - u_0 \in U$. Let $a(\cdot,\cdot): V \times V \to R$ a symmetric bilinear form with energy norm $\|u\|^2 := a(u,u)$. Let further $u_0 \in V$ *and* $F(\cdot): V \to R$ a functional with the following properties:

i) $F(\cdot): V \to R$ is convex on the linear manifold $u_0 + U$, i.e. for every $u, v \in u_0 + U$ it holds $F((1-t)u + tv) \leq (1-t)F(u) + tF(v)$ for every $t \in [0,1]$

ii) $F(u) \geq \alpha$ for every $u \in u_0 + U$

iii) $F(\cdot): V \to R$ is Gateaux differentiable, i.e. it exits a functional $F_u(\cdot): V \to R$ with

$$\lim_{t \to 0} \frac{F(u+tv) - F(v)}{t} = F_u(v).$$

Then the minimum problem is equivalent to the variational equation $a(u,\phi) + F_u(\phi) = 0$ for every $\phi \in U$ and admits only an unique solution. In case the sub-space U and therefore also the manifold $u_0 + U$ is closed with respect to the energy norm and the functional $F(\cdot): V \to R$ is continuous with respect to convergence in the energy norm, then there exists a solution. We note that the energy functional is even strongly convex in whole V.

The decompositions

$$\mathrm{H}_{-1/2} = H_0 \otimes H_0^{\perp} = H_{1/2}^{*},\ H_{1/2} = H_1 \otimes H_1^{\neg} = H_{-1/2}^{*}$$

distinguish between elementary particle states & energy with or w/o „observed/measured mass". The „*symmetry break down*" model to „generate/explain" physical „mass" is replaced by a „projection of a *self-adjoint operator onto the observation/measure space* H_0" (**). In other words, the matter particles (fermions) are the manifestations of the vacuum energy (bosons).

The corresponding mass/energy Hilbert space is given by the decomposition $H_{1/2} = H_1 \times H_1^{\neg}$ into the „fermions" space and the orthogonal „bosons" space, including a Hilbert space based model of the Higgs boson, as well as a Cauchy problem representation of the Einstein-Vacuum field equation with an initial „inflation-field" with regularity $g_{Inflation} \in H_1^{\neg}$ without singularities for $t \to 0$, avoiding current early universe state model singularities.

The proposed common distributional Hilbert space framework $H_{-1/2}$ resp. its corresponding energy dual space

$$\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp} = H_1^{repulsive} \otimes H_1^{attrac} \otimes H_1^{\perp}$$

enables a common (Zeta function and quantum gravity) spectral theory providing an answer to Derbyshine's question ((DeJ) p. 295) (*).

The („physical") Hilbert space pairs (H_0, H_1) resp. the („meta-physical") closed subspaces $H_0^{\perp}, H_1^{\perp})$ of (H_0, H_1) are being governed by Fourier waves resp. Calderón´s wavelets (**). The current "*symmetry break down*" model to generate matter is replaced by a "*self-adjointness break down*" effect defined by the orthogonal projection from $H_{1/2}$ onto H_1. Consequently, the (kinetic) energy driven „inverse" is a kind of entropy operator with a „discrete/compactly embedded" Hilbert space domain to its complementary closed subspace of $H_{1/2}$ (***).

The current quantum state Hilbert space $L_2 = H_0$ is extended to the Hilbert space $H_{-1/2}$, including „fluid", plasma", „fermion", „photon" and „boson" states. The dual space $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp}$ of $H_{-1/2}$ provides the corresponding quantum energy space, whereby the „mass-less EPs" (hot plasma and photons/bosons) are (meta-physical, ground state (dark) energy) „elements" of the closed subspace $H_1^{\perp}$ of $\mathrm{H}_{1/2}$. The standard (variational) energy space H_1 is defined by the selfadjoint Friedrichs extension of the Laplacian operator in the standard $H_0 = L_2 -$ variational (statistics) framework. It keeps being valid for the quantum energy of the EPs *with* mass, including cold plasma.

(*) *(CoR) p. 765: „Huygens' principle stipulates that the solution at a point does not depend on the totality of initial data within the conoid of dependence but only on data on the characteristic rays through that point … It is proven, that for the wave equation in 3,5,7,… space dimensions, and for equivalent equations, the Huygens' principle is valid. For differential equations of second order with variable coefficients Hadamard's conjecture states that the same theorem holds even if the coefficients are not constant. Examples to the contrary show that this conjecture cannot be completely true in this form although it is highly plausible that somehow it is essentially correct. … Altogehter, the question of Huygens's principle for second order equations should be considered in the light of the much more comprehensive problem of the exact domain of dependence and influence for any hyperbolic problem, a problem which is still completely open." Altogether, the question of Huygens' principle for second order equations should be considered in the light of the much more comprehensive problem of the exact domain of dependence and influence for any hyperbolic problem (see §5), a problem which is still completely open."*

(**) *"The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?"*

(***) (HoM) 1.2: „The idea of wavelet analysis is to look at the details are added if one goes from scale a to scale $a - da$ with $da > 0$ but infinitesimal small. … Therefore, the wavelet transform allows us to unfold a function over the one-dimensional space R into a function over the two-dimensional half-plane $\boldsymbol{H}$ of positions and details (where is which details generated?). … Therefore, the parameter space $\boldsymbol{H}$ of the wavelet analysis may also be called the position-scale half-plane since if g localized around zero with width Δ then $g_{b,a}$ is localized around the position b with width $a\Delta$. The wavelet transform itself may now be interpreted as a mathematical microscope where we identify $b \leftrightarrow$ position; $(a\Delta)^{-1} \leftrightarrow$ enlargement; $g \leftrightarrow$ optics.

(***) *The continuous wavelet transform with the complex Shannon wavelet can be considered via solutions of Cauchy problems for PDE in the context of the construction of wavelets for an analysis of non-stationary wave propagation in inhomogeneous media ((PoE).*

The Friedrichs extension of the Laplacian operator is a selfadjoint, bounded operator B with domain H_1. Thus, the operator B induces a decomposition of H into the direct sum of two subspaces, enabling the definition of a potential and a corresponding „grad" potential operator. Then a potential criterion defines a manifold, which represents a hyperboloid in the Hilbert space H_1 with corresponding hyperbolic and conical regions ((VaM) 11.2). The direct sum of the corresponding two subspaces of $H = H_1$ are proposed as a model to define a decomposition of the „fermions" space H_1 into

$$H_1 = H_1^{repulsive} \otimes H_1^{attrac} =: H_1^{(-)} \otimes H_1^{(+)}$$

whereby the potential criterion defines repulsive resp. attractive elementary mass particles. Then the corresponding proposed quantum energy Hilbert space (including attractive gravitons $\in H_1^{(+)}$) is given by (*)

$$H_{1/2} = H_1^{(-)} \otimes H_1^{(+)} \otimes H_1^{\perp}.$$

The $H_{1/2}$ space as first cohomology is fundamental to explain the properties of period mapping on the universal Teichmüller space (**).

We note that a vector space and any linear subspace are convex cones, i.e. the tool „*convex analysis and general vector spaces*" can be applied. Morse's calculus of variations in the large enables a calculus of variations in the large e.g. on varifolds ((MoM), (SeH), (AlF)).

(*) The theory of Hilbert spaces with an indefinite metric is provided in e.g. ((DrM), (AzT), (DrM), (VaM)). Following the investigations of Pontrjagin and Iohvidov on linear operators in a Hilbert space with an indefinite inner product, M. G. Krein proved the Pontrjagin-Iohvidov-Krein theorem (FaK).

In case of a Hilbert space H, this is about a decomposition of H into an orthonal sum of two spaces H^1 and H^2 with corresponding projection operators P^1 and P^2 (see also the problem of S. L. Sobolev concerning Hermitean operators in spaces with indefinite metric, (VaM) IV). We note, that for a vector space H, the empty set, the space H, and any linear subspace of H are convex cones.

For x being an element of H this is about a defined "potential" ((VaM) (11.1))

$$\varphi(x) := \left((x)\right)^2 = \|P^1 x\|^2 - \|P^2 x\|^2$$

and a corresponding "grad" potential operator $\boldsymbol{W}(x)$, given by (VaM) (11.4)

$$\boldsymbol{W}(x) = \frac{1}{2}\mathrm{grad}\varphi(x) := P^1(x) - P^2(x).$$

The potential criterion $\varphi(x) = c > 0$ defines a manifold, which represents a hyperboloid in the Hilbert space H with corresponding hyperbolic and conical regions. It provides a model for „symmetry break down" phenomena by choosing $P^1 := P$, $P^2 := I - P$ for the orthogonal projections $P: H_{-1/2} \to H_0$, $P: H_{1/2} \to H_1$, leading to the decompositions $\mathrm{H}_{-1/2} = H_0 \otimes H_0^{\perp}$, $H_{1/2} = H_1 \otimes H_0^{\perp}$.
The tool set for an appropropriate generalization of the above "grad" definition in case of non-linear problems is about the (homogeneous, not always linear in h) Gateaux differential (or weak differential) $\boldsymbol{VF}(x,h)$ of a functional $\boldsymbol{F}$ at a point x in the direction h ((VaM) §3)).

If there exists an operator A with $D(A) = H_1$, $R(A) = H_0$ and $\|x\|_1 = \|Ax\|_0$, whereby the operator A is positive definite, self-adjoint and A^{-1} is compact, the corresponding eigenvalue problem $A\varphi_i = \sigma_i \varphi_i$ has infinite solutions $\{\sigma_{i,}\varphi_i\}$ with $\sigma_i \to \infty$ and $(\varphi_i, \varphi_k) = \delta_{i,k}$. For each element $x \in H_1 = A^{-1}H_0$ it holds the representation $x = \sum_{i=1}^{\infty}(x,\varphi_i)\,\varphi_i$. Inner products with corresponding norms of a distributional Hilbert scale can be defined based on the eigen-pairs of an appropriately defined operator in the form

$$(x,y)_\alpha := \sum_i^\infty \lambda_i^\alpha (x,\phi_i)\,(y,\phi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i.$$

Additionally, for $t > 0$ there can be an inner product resp. norm defined for an additional governing Hilbert space with an "exponential decay" behavior in the form $e^{-\sqrt{\lambda_i}t}$ given by

$$(x,y)_{(t)}^2 := \sum_{i=1} e^{-\sqrt{\lambda_i}t}(x,\phi_i)(y,\phi_i) \quad , \quad \|x\|_{(t)}^2 := (x,x)_{(t)}^2$$

The approximation "quality" of the proposed $\mathrm{H}_{-1/2}$ –quantum state Hilbert space with respect to the „observable space" norm of H_0 is governed by the inequality

$$\|x\|_{-1/2}^2 \leq \delta\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2 = \delta\|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i}\delta} x_i^2.$$

The estimate is valid for all $\alpha > 0$ in the form $\|x\|_{-\alpha}^2 \leq \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2$, which follows from the inequality $\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{t(\delta^{-1}-\sqrt{\lambda})}$, being valid for any $t, \delta, \alpha > 0$ and $\lambda \geq 1$. For a related approximation theory we refer to (BrK8), (NiJ), NiJ1). Applying the mathematical wavelet (microscopic view) tool is then about an analysis of a quantum state $x = x_0 + x_0^{\perp} \in H_0 \otimes H_0^{\perp}$. Putting $\sigma := \|x_0^{\perp}\|_{-1/2}^2$ the approximation "quality" of a quantum state with respect to the „observable space" norm of H_0 is governed by the inequality $\|x\|_{-1/2}^2 \leq \sigma\|x\|_0^2 + e\|x\|_{(\sigma)}^2 = \sigma\|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i}\sigma} x_i^2$.

(**) (NaS): „The Hilbert space $H_{1/2}$ can be interpreted as the first cohomology space with real coefficients of the „universal Riemann surface" – namely the unit disk – in a Hodge-theoretic sense."

The central part to prove the well-posedness of the 2D non-linear, non-stationary Navier-Stokes equations is a proper energy norm inequality estimate. It does not lead to blow-up effects for $t = T$ and do not show a Serrin gap with respect to the corresponding Sobolev norm estimates. We note that the energy norm of the non-linear terms of the 2D-NSE vanishes, which is appreciated from a mathematical point of view, but seems to be questionable from a physical point of view. The corresponding analysis for the 3D-NSE fails due to not appropriate Sobolev norm estimates. The analog analysis in a $H_{-1/2}$ variational framework (including a not-vanishing non-linear energy term) works out well, due to the appropriate Sobolewski estimates. In case of $\alpha = -1/2$ one gets from Sobolevskii-estimates ((GiY) lemma 3.2) the corresponding generalized "energy" inequality, given by

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq |(Bu,u)_{-1/2}| \leq \|u\|_{-1/2}\|Bu\|_{-1/2} \cong \|u\|_{-1/2}\left\|A^{-1/4}Bu\right\|_0.$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets $y'(t) \leq c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$, resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \leq \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \leq c\{\|u_0\|_{-1/2} + \|u_0\|_0^2\},$$

which ensures global boundedness by the a priori energy estimate provided that $u_0 \in H_0$.

The classical Yang-Mills theory is the generalization of the Maxwell theory of electromagnetism where chromo-electromagnetic field itself carries charges. As a classical field theory it has solutions which travel at the speed of light so that its quantum version should describe massless particles (gluons). However, the postulated phenomenon of color confinement permits only bound states of gluons, forming massive particles. This is the mass gap. Another aspect of confinement is asymptotic freedom which makes it conceivable that quantum Yang-Mills theory exists without restriction to low energy scales. A variational Maxwell equations representation in a $H_{-1/2}$ Hilbert space framework includes also "gluon" bosons and corresponding "*self-adjointness break downs*", i.e. there is no mass gap anymore.

The quantum gravity model also addresses the dilemma, as pointed out by E. Schrödinger: "*Since in the Bose case we seem to be faced, mathematically, with simple oscillator of Planck type, we may ask whether we ought not to adopt for half-odd integers quantum numbers rather than integers. Once must, I think, call that an open dilemma. From the point of view of analogy one would very much prefer to do so. For, the „zero-point energy" of a Planck oscillator is not only borne out by direct observation in the case of crystal lattices, it is also so intimately linked up with the Heisenberg uncertainty relation that one hates to dispense with it*".

The formalism of 2-"*spinors*" as an alternative to the standard vector-tensor calculus (Penrose R., Rindler W.) is proposed to be physically re-interpreted and mathematically applied in the context of a H_1-space decomposition into repulsive and attractive fermions subspaces, whereby it holds $spin(4) = SU(2)xSU(2)$.

„The two-component „spinor" calculus is a very specific calculus for studying the structure of space-time manifolds…. Space-time point themselves cannot be regarded as derived objects from spinor algebra, but a certain extension of it, namely the twistor algebra, can indeed be taken as more primitive than space-time itself. ... The programme of twistor theory, in fact, is to reformulate the whole of basic physics in twistor terms" (Penrose R., Rindler W. Volume II).

The point of departure for the twistor theory is the (classical) twistor equation (with a similar form as the continuity equation). Its corresponding weak variational represention with respect to the proposed $H_{-1/2}$ quantum state inner product leads to the Friedrichs extension of the classical Dirac spinor operator with domain $H_{-1/2}$, which is about the square root operator of order one of the Laplacian operator. The corresponding singular integral operator representation is about the Calderón-Zygmund integrodifferential operator ((EsG) example 3.5).

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With respect to the NSE, the ground state (or dark) energy and quantum gravity topics we also refer to

http://www.navier-stokes-equations.com/

https://quantum-gravitation.de/