The Calderón-Zygmund integrodifferential operator with symbol $|\nu|$ ([GEs] (3.17), (3.35)) is defined by

$$(\Lambda u)(x)=(\sum_{k=1}^{n}R_k D_k u)(x)=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\sum_{k=1}^{n}p.v.\int_{-\infty}^{\infty}\sum_{k=1}^{n}\frac{x_k-y_k}{|x-y|^{n+1}}\frac{\partial u(y)}{\partial y_k}dy=-\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}p.v.\int_{-\infty}^{\infty}\frac{\Delta_y u(y)}{|x-y|^{n-1}}dy=-(\Delta\Lambda^{-1})u(x)$$

whereby R_k denotes the Riesz operators ([HAb] p. 19, 106, [BPe] example 9.9)

$$R_k u=-i\frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}p.v.\int_{-\infty}^{\infty}\frac{x_k-y_k}{|x-y|^{n+1}}u(y)dy\ .$$

It holds ([GEs] (3.15))

$$\Lambda^{-1}u=\frac{\Gamma(\frac{n-1}{2})}{2\pi^{(n+1)/2}}\int_{-\infty}^{\infty}\frac{u(y)dy}{|x-y|^{n-1}}\quad,\quad n\geq 2\ .$$

The Riesz operators fulfill certain properties with respect to commutation with translations and homothesis. From [BPe] we note:

If $j\neq j$ then $R_j R_k$ is a singular convolution operator. On the other hand it holds

$$R_j^2=-(1/n)I+A_j$$

where A_j is a convolution operator. It further holds

$$\|R_j\|=1\ ,\ R_j^*=-R_j\quad,\ \sum R_j^2=-I\quad,\ \sum\|R_j u\|^2=\|u\|^2\ ,u\in L_2\ .$$

Another remarkable property is related to rotations ([BPe] example 9.9, 9.10, [ESt]):

let $$m:=m(x):=(m_1(x),...m_n(x))$$

be the vector of the Mikhlin multipliers of the Riesz operators and $\rho=\rho_{ik}\in SO(n)$, then

$$m(\rho(x))=\rho(m(x))$$

whereby $$R_k u=-ic_n p.v.\int_{-\infty}^{\infty}\frac{x_k-y_k}{|x-y|^{n+1}}u(y)dy\quad \text{with}\quad c_n:=\frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$$

$$m_j(\rho(x))=\sum\rho_{jk}m_k(x)$$

and $$m(\rho(x))=c_n\int_{S^{n-1}}(\frac{\pi i}{2}sign(x\rho^{-1}(y))+\log\left|\frac{1}{x\rho^{-1}(y)}\right|)\frac{y}{|y|}d\sigma(y)$$

$$=c_n\int_{S^{n-1}}(\frac{\pi i}{2}sign(xy)+\log\left|\frac{1}{xy}\right|)\frac{y}{|y|}d\sigma(y)\ .$$

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