The three Millennium problem solutions (RH, NSE, YME) and a Hilbert scale based quantum geometrodynamics

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This paper addresses the problem & solution areas

A. The Riemann Hypothesis (RH)
B. The 3D-Navier-Stokes equations (NSE)   navier-stokes-equations.com
C. The Yang-Mills equations (YME)
D. Plasma dynamics
E. Geometrodynamics

building on a common Hilbert scale framework.
The proposed framework provides an answer to Derbyshine's question, ("Prime Obsession")

... "*The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?*"

The story line of this page is structured as follows:

1. A Kummer function based alternative Zeta function theory to solve the Riemann Hypothesis and the binary Goldbach conjecture

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   c. An alternative Zeta function $\zeta^*(s)$ based on the $\cot(\pi s)=\tan\left(\pi(\frac{1}{2}-s)\right)$ function
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2. The $\mathrm{H}_{-1/2}$ Hilbert space and corresponding arithmetic functions

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1. A Kummer function based alternative Zeta function theory to solve the Riemann Hypothesis and the binary Goldbach conjecture

1.a Introduction and Overview

In order to prove the Riemann Hypothesis (RH) the Polya criterion can not be applied in combination with the Müntz formula ((TiE) 2.11). The Müntz formula is divergent in the critical stripe due to the asymptotics behavior of the exponential function. The conceptual challenge is about the not vanishing constant Fourier term of the Gaussian function and its related impact on the Poisson summation formula resp. on the corresponding Riemann duality equation ((EdH) 1.7). The proposed alternative "baseline" function is the Hilbert transform of the Gaussian function, which is the Dawson function ((OlF) p. 44)

$$F(x) := e^{-x^2} \int_0^x e^{-t^2} dt \ .$$

Let H and M denote the Hilbert and the Mellin transform operators. For the Gaussian function $f(x)$ it holds

$$M[f](s) = \frac{1}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) \ , \quad M[-xf'(x)](s) = \frac{s}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) = \frac{1}{2}\pi^{-s/2}\Pi(\frac{s}{2}) \ .$$

The corresponding entire Zeta function is given by ([EdH] 1.8)

$$\xi(s) := \frac{s}{2}\Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s) = (1-s)\cdot\zeta(s)M[-xf'(x)](s) = \xi(1-s).$$

Replacing the Gaussian function by the Dawson function leads to an alternative entire Zeta function $\xi^*(s)$ in the form

$$\xi^*(s) := \frac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\, tan(\frac{\pi}{2}s)\cdot\zeta(s) = \zeta(s)\cdot M\left[\frac{d}{dx}[-x\cdot f_H(x)]\right](s)$$

with same zeros as $\xi(s)$, as it holds $s(1-s)\xi^*(s)\xi^*(1-s) = \pi\xi(s)\xi(1-s)$.

The Dawson function approach enables an *„analysis of zeros of certain trigonometric integrals and entire higher genus-1 functions*" (PoG2), based on the identities ((GrI) 3.896)

$$F(x) := e^{-x^2}\int_0^x e^{-t^2}dt = \int_0^\infty e^{-t^2}\sin(2xt)dt = x\ {}_1K_1\left(1,\frac{3}{2};-x^2\right) = e^{-x^2}H(x)$$

with $H(x) := x\ {}_1K_1\left(\frac{1}{2},\frac{3}{2};x^2\right)$.

As the Hilbert transform defines a convolution operator, The Dawson function approach also enables *„zeros of entire functions*" analysis techniques in the context of the Hilbert-Polya conjecture (e.g. (CaD)).

Replacing the Mellin transform of the Gaussian function by the Mellin transform of the Dawson function goes along with a replacement of the „Riemann error function" related $\xi(s)$ –term $\Gamma\left(1+\frac{s}{2}\right)$ ((EdH) 1.16)

$$\int_x^\infty \frac{dt}{t(t^2-1)logt} = \int_x^\infty \frac{1}{tlogt}(\sum t^{-2n})dt = \sum_{n=1}^\infty \int_x^\infty t^{-2n}\frac{dt}{tlogt} = \frac{1}{2\pi i}\frac{1}{logx}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log\Gamma\left(1+\frac{s}{2}\right)}{s}\right]x^s ds$$

by

$$(*)\quad \Gamma\left(1+\frac{s}{2}\right) \quad\rightarrow\quad \Gamma^*\left(\frac{s}{2}\right) := \Gamma\left(\frac{s}{2}\right)tan\left(\frac{\pi}{2}s\right) = \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{\Gamma\left(1+\frac{s-1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{2}{\pi}\sum_{k=1}^\infty \frac{\Gamma\left(1+\frac{s}{2}\right)}{(k-\frac{1}{2})^2-(\frac{s}{2})^2}.$$

$$log'\Gamma^*(s) = log'\Gamma(s) + log'\tan\left(\frac{\pi}{2}s\right) = -\gamma + \frac{\pi}{\sin(\pi s)}$$

It results into a correspondingly modified Riemann approximation error function with now newly three summation terms governed by product fomula

$$\Gamma(1\pm s) = \prod(1+\frac{1}{n})^{\pm s}(1\pm\frac{s}{n})^{-1}$$

with improved ($|li(x)-\pi(x)|$ error) convergence behavior.

In the context of the $Li(x)$ function RH approximation criterion we also note that the Dawson function $F(\sqrt{x})$ enjoys an only polynomial asymptotics in the form $O(x^{-1/2})$. This, as well as the asymptotics $2dF(x) \sim \frac{dx}{x} = dlogx$, is a consequence of the identity $F'(x) + 2xF(x) = 1$ ((LeN) (9.13.3)).

The fractional part function related Zeta function theory is provided in ((TiE) II). The Hilbert transform of the fractional part function is given by the $\log(sinx)$ –function. A $log(sinx)$ –function based Zeta function theory deals with a „*Zeta*" function $\zeta^*(s)$ the form

$$\zeta^*(s) = \frac{\cot(\frac{\pi}{2}s)}{1-s} \cdot \zeta(s) = \frac{\tan(\frac{\pi}{(2}(1-s))}{1-s} \cdot \zeta(s) \quad .$$

The correspondingly defined distributional ("periodical") Hilbert space framework deals with the Hilbert spaces $H_0^\#(0,1)$ $H_{-1/2}^\#(0,1)$, $H_{-1}^\#(0,1)$, with its relationship to the Bagchi reformulation of the Nyman-Beurling RH criterion. This criterion basically becomes a „*dense embedding*" argument of $H_{-1/2}^\#(0,1)$ into $H_{-1}^\#(0,1)$, as the cot and the Zeta function on the critical line are $\in H_{-1}^\#(0,1)$.

The RH is equivalent to the Li criterion governing a sequence of real constants, that are certain logarithmic derivatives of $\xi(s)$ evaluated at unity (LiX). This equivalence results from a necessary and sufficient condition that the logarithmic of the function $\xi(\frac{1}{1-s})$ be analytic in the unit disk resp. that the sequence of real constants

$$\sigma_n := \frac{1}{\Gamma(n)} \frac{d^n}{ds^n} [s^{n-1} log\xi(s)]_{s=1}$$

are not-negative (CoM). The proof of the Li criterion is built on the two representations of the Zeta function, its (product) representation over all its nontrivial zeros ((HdE) 1.10) and Riemann's integral representation derived from the Riemann duality equation, based on the Jacobi theta function ((EdH) 1.8). Based on Riemann's integral representation involving Jacobi's theta function and its derivatives in (BiP) some particular probability laws governing sums of independent exponential variables are considered. In (CoF), (KeJ) corresponding Li/Keiper constants are considered. The proposed alternative entire Zeta function $\xi^*(\frac{1}{1-s})$ is suggested to verify the corresponding Li criteron.

The sequences ω_n (the imaginary parts of the Kummer function zeros) and $s_n := \frac{\omega_n}{n}$ enjoy the following properties

i) $2n-1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n+1$, $\quad -\frac{1}{2(n+1)} < s_{n+1} - s_n < \frac{1}{2n}$

ii) $f_n := \frac{1}{\omega_n + \omega_{n+1}} \in \left(\frac{1}{2n+1}, \frac{1}{2n}\right)$, $g_n := \frac{1}{2\omega_n} \in \left(\frac{1}{2n}, \frac{1}{2n-1}\right)$

iii) $2\omega_n$ and $\omega_n + \omega_{n+1}$ fulfill the Hadamard gap condition, i.e.

$$\frac{\omega_{n+1}}{\omega_n} > \frac{n+\frac{1}{2}}{n} = 1 + \frac{1}{2n} > q > 1 \quad \text{resp.} \quad \frac{\omega_{n+1}+\omega_{n+2}}{\omega_n+\omega_{n+1}} > \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > q > 1 \; .$$

For the harmonic numbers $2h_n = \sum_{k=1}^n \frac{2}{2k-1} = 2H_{2n} - H_n$, $H_n = \sum_{k=1}^n \frac{1}{k}$ resp. $c_n := \frac{2h_n}{n}$ it holds the Fourier series representation (ElL)

$$\frac{\pi}{2} \log\left(tan\left(\frac{\pi}{2}x\right)\right) = -\sum \frac{2h_n}{n} \sin(2\pi nx) = -\sum c_n \sin(2\pi nx) \in L_2^\#(0,1)$$

i.e. $\sum_{n=1}^\infty c_n^2 < \infty$.

The sequence c_n therefore enables the following replacements

$$2n \rightarrow 2\omega_n \quad , \quad e^{i\pi(2n)x} \rightarrow e^{i\pi(2\omega_n)x} \quad , \quad S(x) := \sum_n e^{2\pi inx} \rightarrow S^{(1)}(x) := \sum_n e^{i\pi(2\omega_n)x}$$
$$\rightarrow \omega_n + \omega_{n+1} \qquad \rightarrow e^{i\pi(\omega_n+\omega_{n+1})x} \qquad \rightarrow S^{(2)}(x) := \sum_n e^{i\pi(\omega_n+\omega_{n+1})x}$$

and

$$cot(\pi x) = 2\sum_{n=1}^{\infty} sin(2\pi nx)) \in H_{-1}^{\#}(0,1)$$

$$\rightarrow \quad cot^{(1)}(\pi x) := \sum sin(\pi(2\omega_n)x) \in H_0^{\#}(0,1)$$

$$\rightarrow \quad cot^{(2)}(\pi x) := \sum sin(\pi(\omega_n + \omega_{n+1})x \in H_0^{\#}(0,1) \ .$$

and

$$-\frac{\pi}{2}\log\left(tan\left(\frac{\pi}{2}x\right)\right) = \sum \frac{2h_n}{n}\sin(2\pi nx)) \in L_2^{\#}(0,1) \qquad \rightarrow$$

$$\rightarrow \qquad -\frac{\pi}{2}\log\left(tan^{(1)}\left(\frac{\pi}{2}x\right)\right) := \sum \frac{h_n}{n}\sin(\pi(2\omega_n)) \in H_1^{\#}(0,1)$$

$$\rightarrow \qquad -\frac{\pi}{2}\log\left(tan^{(2)}\left(\frac{\pi}{2}x\right)\right) := \sum \frac{h_n}{n}\sin(\pi(\omega_n + (\omega_{n+1})) \in H_1^{\#}(0,1)$$ and

and

$$\rightarrow \qquad \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} z^{1-s}(-\pi\, cot^{*}(\pi z)\frac{dz}{z} \quad \text{resp.} \quad \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} z^{1-s}(-\pi\, cot^{**}(\pi z)\frac{dz}{z}.$$

With respect to the binary Goldbach conjecture the above enables a new circle method on the unit circle, going along with corresponding *additive* number theory problem adequate arithmetic functions in the form (see also e.g. (BrK4) Notes O27/37)

$$\sigma^{*}(x) := \frac{1}{2}\left[\sigma_{(odd)}^{(1)}(x) + \sigma_{(even)}^{(2)}(x)\right]$$

with

$$\sigma_{(odd)}^{(1)}(x) := \frac{1}{x}h^{(1)}\left(\frac{1}{x}\right)\sum_{\substack{n\le x\\ n\ odd}} h^{(1)}\left(\frac{x}{2\omega_n}\right) \ , \ \sigma_{(even)}^{2}(x) := \frac{1}{x}h^{(2)}\left(\frac{1}{x}\right)\sum_{\substack{n\le x\\ n\ even}} h^{(2)}\left(\frac{x}{\omega_n+\omega_{n+1}}\right)$$

and

$$h^{(1)}(x) := -(\pi x)cot^{(1)}(\pi x) \quad , \quad h^{(2)}(x) := -(\pi x)cot^{(2)}(\pi x).$$

For a subset $\boldsymbol{A}$ of the set of integers $\boldsymbol{N}$ integers, if

i) the integer "1" is not an element of $\boldsymbol{A}$, the Snirelmann density of $\boldsymbol{A}$ is $= 0$

ii) if the integer "2" is not an element of $\boldsymbol{A}$, the Snirelmann density of $\boldsymbol{A}$ is $\le \frac{1}{2}$

iii) if $\boldsymbol{A} = \boldsymbol{N}$, the Snirelmann density of $\boldsymbol{A}$ is $= 1$.

Therefore, the set $\{2n - 1 = [2\omega_n]\}$ of odd integers has Snirelmann density $\le \frac{1}{2}$, while the set $\{2n = [\omega_n + \omega_{n+1}]\}$ of even integers has Snirelmann density $= 0$.

For the smallest prime $p > n$, it holds $p < 2n$, i.e. $\pi(2n) - \pi(n) \ge 1$.

1.b An alternative entire Zeta function $\xi^*(s)$ based on Kummer / Dawson and $\tan\left(\pi\frac{s}{2}\right)=\cot\left(\pi\frac{1-s}{2}\right)$ functions

Let H and M denote the Hilbert and the Mellin transform operators. For the Gaussian function $f(x)$ it holds

$$M[f](s)=\frac{1}{2}\pi^{-s/2}\Gamma(\frac{s}{2})\ ,\ \ M[-xf'(x)](s)=\frac{s}{2}\pi^{-s/2}\Gamma(\frac{s}{2})=\frac{1}{2}\pi^{-s/2}\Pi(\frac{s}{2})\ .$$

The corresponding entire Zeta function is given by ([EdH] 1.8)

$$\xi(s):=\frac{s}{2}\Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s)=(1-s)\cdot\zeta(s)M[-xf'(x)](s)=\xi(1-s).$$

Putting

$$G(u):=\sum_{-\infty}^{\infty}e^{-\pi n^2u^2}=\sum_{-\infty}^{\infty}f(nu)$$

Riemann's functional equation implies, that the invariant operator $g(x)\to\int_0^\infty g(ux)G(u)du$ is formally self-adjoint. A valid invariant operator would prove the Hilbert Polya conjecture. But the operator has no transform at all (due to the not vanishing constant Fourier term of the Gaussian function); that is the integral $\int_0^\infty u^{-s}G(u)du$, which is formally represented in the form ((EdH) 10.3)

$$\int_0^\infty u^{-s}G(u)du=\frac{2\xi(s)}{s(s-1)}$$

does not converge for any s. The central idea is to replace $M[-xf'(x)](s)\to M[-f_H(x)](s)$ with $f_H(x):=H[f](x)$ ($\hat{f}_H(0)=0$) and

$$M[f_H(x)](s)=M\left[2\pi x\ {}_1F_1\left(1,\frac{3}{2},-\pi x^2\right)\right](s)=M[2\sqrt{\pi}F(\sqrt{\pi}x)](s)=\frac{1}{2}\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\,tan(\frac{\pi}{2}s)\ ,$$

leading to the alternative *entire* Zeta function $\xi^*(s)$ in the form

$$\xi^*(s):=\frac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\,tan(\frac{\pi}{2}s)\cdot\zeta(s)=\zeta(s)\cdot M\left[\frac{d}{dx}[-x\cdot f_H(x)]\right](s)\ .$$

The link to the Zeta functions $\xi(s)$ resp. $\zeta(s)$ is given by the equation

$$\xi^*(s)\xi^*(1-s)=\pi\frac{\xi(s)\xi(1-s)}{s(1-s)}\ ,$$

i.e. $\xi(s)$ and $\xi^*(s)$ do have the same set of zeros in the critical stripe.

The corresponding invariant operator $f(x)\to\int_0^\infty f(ux)G_H(u)du$ is built on

$$G_H(u):=\sum_{-\infty}^{\infty}e^{-\pi n^2u^2}=\sum_{-\infty}^{\infty}f_H(nu)$$

Which replaces Riemann's auxiliary function ((EdH) 10.3) $H(u):=\frac{d}{du}\left[u^2\frac{d}{du}G(u)\right]$.

The link to the normal distribution $\bar{F}(x)=\int_{-\infty}^{\infty}e^{-u^2}du$ resp. to the topic of „Convolution Operators and the Zeros of Entire Functions" (CaD), is given by by the formula

$$h_n(z)=\int_{-\infty}^{\infty}(z-is)^n d\bar{F}(s)=\int_{-\infty}^{\infty}(z-is)^n e^{-s^2}ds=2^{-n}H_n(z).$$

The link of the Hermite polynomials to the corresponding Hilbert transform of the Gaussian (the Dawson function) is given by (AbM), 7.1.15,

$$\frac{1}{\sqrt{\pi}}H\left[e^{-x^2}\right]=F(x)=\frac{1}{\sqrt{\pi}}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{H_k^{(n)}}{x-x_k^{(n)}}\approx\frac{1}{x}.$$

where $x_k^{(n)}$ and $H_k^{(n)}$ are the zeros and weight factors of the Hermite polynomials.

In ((BuH) p.184) the product representation of the Kummer functions

$$ {}_1F_1(a,c,z) = \frac{1}{\Gamma(c)} e^{\frac{a}{c}z} \prod (1-\frac{z}{\alpha_n}) e^{z/\alpha_n} $$

are provided, where α_n denotes the infinite set of zeros of ${}_1F_1(a,c,z)$, i.e. the Kummer functions are elements of the Laguerre-Polya class LP (CaD), which is related to the topic *„zeros of certain trigonometric integrals in the context of entire transcendental higher genus 1 functions*" (PoG2). The Kummer function related Mellin transforms can be derived from the corresponding formulas provided in the appendix.

Riemann inverted the formula ((EdH), 1.13)

$$ J(x) = \sum_{n \le x} \Lambda(n) logn = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} log\zeta(s) x^s \frac{ds}{s} = Li(x) - \sum_\rho Li(x^\rho) - log2 + \int_x^\infty \frac{1}{t(t^2-1)} \frac{dt}{logt} $$

by means of the Möbius inversion formula, getting ((EdH), 1.17)

$$ \pi(x) = Li(x) - \sum_{n=2}^\infty Li\left(x^{1/n}\right) \ , \qquad Li(x) := \int_2^x \frac{dt}{logt} - Li(2),\ Li(2) \sim 1.04 \ldots \ . $$

He suggested a more natural and better approximation in the form

$$ \pi(x) \sim R(x) := Li(x) - \sum_{n=2}^\infty \frac{\mu(n)}{n} Li\left(x^{1/n}\right) \ . $$

However, he was aware of the defects of this approximation and his analysis of it, which is basically due to fact, that he has no estimate at all of the size of the „periodic" terms $\sum_{n=1}^\infty \sum_\rho Li(x^{\rho/n})$ ((EdH), 1.17). The $Li(x)$ function RH approximation criterion is given by

$$ |Li(x) - \pi(x)| = O\left(\sqrt{x} logx\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right); \varepsilon > 0. $$

The challenge to prove the $Li(x)$ function approximation criterion is about the (exponential) asymptotics of the Gaussian function (which we propose to be replaced by the Dawson function). Based on the product representations of the Gamma function

$$ \Gamma(1+s) = \prod \frac{(1+\frac{1}{n})^s}{(1+\frac{s}{n})} \quad , \quad \Gamma(1-s) = \prod \frac{(1+\frac{1}{n})^{-s}}{(1-\frac{s}{n})} \quad , \ \Gamma\left(1+\frac{s}{2}\right) = \prod \frac{(1+\frac{1}{n})^{s/2}}{(1+\frac{s}{2n})} $$

the Riemann approximation error function ((EdH) 1.16)

$$ \int_x^\infty \frac{dt}{t(t^2-1)logt} = \int_x^\infty \frac{1}{tlogt} (\sum t^{-2n}) dt = \sum_{n=1}^\infty \int_x^\infty t^{-2n} \frac{dt}{tlogt} = \frac{1}{2\pi i} \frac{1}{logx} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{log\Gamma\left(1+\frac{s}{2}\right)}{s} \right] x^s ds $$

is calculated from the terms

$$ \frac{d}{ds} \left[\frac{log\Gamma\left(1+\frac{s}{2}\right)}{s} \right] \ \text{and} \ \ H(\beta) := \frac{1}{2\pi i} \frac{1}{logx} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{log(1-\frac{s}{\beta})}{s} \right] x^s ds \ , \ (H(1) = Li(x) - i\pi) \ . $$

As the alternative entire $\xi^*(s)$ function is going along with the replacement (*) above, this results into a correspondingly modified Riemann approximation error function with now newly three summation terms with improved ($|li(x) - \pi(x)|$ error) convergence behavior.

We mention the Kummer function based representation of the $li(x)$-function in the form $li(x) = x\ {}_1F_1(1,1;-logx)$ ((LeN) (9.13.7)) with the asymptotics

$$ li(x) = -x\ {}_1F_1(1;1,-log\,x) = Ei(logx) \approx log^{-1}(x). $$

1.c An alternative Zeta function $\zeta^{*}(s)$ based on the $\cot(\pi s) = \tan\left(\frac{\pi}{2}(1-2s)\right)$ function

The fractional part function related Zeta function theory is provided in ((TiE) II). The Hilbert transform of the fractional part function is given by the $\log(sinx)$ –function. The correspondingly defined distributional ("periodical") Hilbert space framework enables the Bagchi reformulation of the Nyman-Beurling RH criterion, which then becomes basically a „*being dense embedded*“ argument of $H^{\#}_{-1/2}(0,1)$ into $H^{\#}_{-1}(0,1)$.

One proof of the Riemann functional equation is based on the fractional part function $\rho(x)$, whereby the zeta function $\zeta(s)$ in the critical stripe is given by the Mellin transform

$$\zeta(1-s) = M\left[-x \cdot \rho'(x)\right](s-1) \text{ , (TiE) (2.1.5).}$$

The functional equation is given by ((TiE) (2.1.12)

$$\zeta(s) = \chi(s)\zeta(1-s) = \pi^{s-1/2}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s) \text{ .}$$

The Hilbert transform of the fractional part function $\rho(x) = x - [x] = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{\sin(2\pi n)}{\pi n} \in L^{\#}_{2}(0,1)$ is given by

$$\rho_H(x) = \sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{\pi n} = -\frac{1}{\pi}\log(2\sin(\pi x)) \in L^{\#}_{2}(0,1)$$

with

$$\rho'_H(x) = -\cot(\pi x) = -2\sum_{n=1}^{\infty}\sin(2\pi nx) \in H^{\#}_{-1}(0,1)$$

whereby the $\cot(\pi x)$ series representation is Cesàro summable (mean of order 1) (BrK4) Note O25, (ZyA) VI-3, VII-1).

The corresponding alternative $\zeta^{*}(s)$ function is given by ((BrK4) lemma 1.4, lemma 3.1 (GrI) 1.441, 3.761, 8.334, 8.335)

$$\zeta^{*}(1-s) = \frac{\tan(\frac{\pi}{2}s)}{s}\cdot\zeta(1-s) \quad \text{resp.} \quad \zeta^{*}(s) = \frac{\cot(\frac{\pi}{2}s)}{1-s}\cdot\zeta(s) = \frac{\tan(\frac{\pi}{2}(1-s))}{1-s}\cdot\zeta(s)$$

resp.

$$log\zeta^{*}(s) = \log(\tan\frac{\pi}{2}(1-s)) + log\left(\frac{1}{1-s}\right) + log\zeta(s).$$

We note the series representation (TiE) 4.14,

$$\sum_{n>x}\frac{1}{n^s} = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} z^{1-s}(-\pi\, cot(\pi z)\frac{dz}{z} \quad .$$

The functions $\log(2\sin(\pi x))$, $cot(\pi x)$, $sin^{-2}(\pi x)$ are also related to the convolution kernels of the model adequate Pseudo-Differential operators in the context of the proposed new ground state energy model of the harmonic quantum oscillator model problem.
The function $\pi\, cot(\pi x)$ is holomorphic except the pole $z = 1$.

From the representation (ElL)

$$T(x) := -\frac{\pi}{2}\log\left(tan\left(\pi\frac{x}{2}\right)\right) = -\frac{\pi}{2}\log\left(cot\left(\pi\frac{1-x}{2}\right)\right) = \sum\frac{2h_n}{n}\sin(\pi(2n)x)$$

with the harmonic numbers $2h_n := \sum_{k=1}^{n}\frac{2}{2k-1} = 2H_{2n} - H_n$ and $H_n = \sum_{k=1}^{n}\frac{1}{k}$ it follows

$$T(x) \in L^{\#}_{2}(0,1) \quad , \quad T'(x) = \frac{\pi}{\sin(\pi s)} \in H^{\#}_{-1}(0,1)$$

i.e., $T(x)$ is of same regularity as the fractional part function, while $T'(x) = \frac{\pi}{\sin(\pi s)}$ is of same regularity as the cot – and the Zeta function on the critical line.

1.d Alternative arithmetical functions based on the Kummer function seros

The function $\frac{\pi}{2}\log(tan(\frac{\pi}{2}x))$ enjoys the following Fourier series representation (ElL)

$$\frac{\pi}{2}\log\left(tan\left(\frac{\pi}{2}x\right)\right) = -\sum\frac{2h_n}{n}\sin(2\pi nx) = -\sum c_n\sin(2\pi nx) \in L_2^{\#}(0,1)$$

with $2h_n = \sum_{k=1}^{n}\frac{2}{2k-1} = 2H_{2n} - H_n$, $H_n = \sum_{k=1}^{n}\frac{1}{k}$ (harmonic numbers), $c_n := \frac{2h_n}{n}$ and

$$\sum_{n=1}^{\infty} c_n^2 < \infty \text{ and } \int_0^1 \log(tan(\frac{\pi}{2}x))\ cos(k\pi x)\,dx = \begin{cases} -1/k & k\ odd \\ 0 & k\ even \end{cases}.$$

FRom (LaJ) we mention the

„*Elementary RH criterion*": The RH holds true iff for each $n \geq 1$ $\sum_{d|n} d \leq H_n + e^{H_n}\log(H_n)$.

The sequences ω_n (the imaginary parts of the Kummer function zeros) and $s_n := \frac{\omega_n}{n}$ enjoy the following properties ((BrK4) Notes O5-O7, O13, O15-O17)

iv) $n - \frac{1}{2} < \omega_n < n$, $\frac{1}{2} < \omega_1 < a := s_n := \frac{\omega_n}{n} \to 1$ $n \in N$

v) $2n - 1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n + 1$, $-\frac{1}{2(n+1)} < s_{n+1} - s_n < \frac{1}{2n}$

vi) $f_n := \frac{1}{\omega_n + \omega_{n+1}} \in \left(\frac{1}{2n+1}, \frac{1}{2n}\right)$, $g_n := \frac{1}{2\omega_n} \in \left(\frac{1}{2n}, \frac{1}{2n-1}\right)$

vii) the sequences $2\omega_n$ and $\omega_n + \omega_{n+1}$ fulfill the Hadamard gap condition
((BrK4) Notes S43-49, O5-7, O24-25, O27/37, (KaM1), (KoA) (ZyA))

$\frac{\omega_{n+1}}{\omega_n} > \frac{n+\frac{1}{2}}{n} = 1 + \frac{1}{2n} > q > 1$ resp. $\frac{\omega_{n+1}+\omega_{n+2}}{\omega_n+\omega_{n+1}} > \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > q > 1$

viii) $\theta := \frac{1}{4} < \alpha_{n+1} - \alpha_n < 1 - \frac{1}{4} = 1 - \theta$ for $\alpha_n := \frac{\omega_n}{2}$

Lemma (KaM1): Let $\{n_k\}$ be a sequence of integers satisfying the "Hadamard" gap" condition, i.e. $\frac{n_{k+1}}{n_k} > q > 1$. Then the trigonometric gap series $\sum_{k=1}^{\infty} c_k\sin(2\pi n_k x)$ converges almost everywhere, if and only if, $\sum_{k=1}^{\infty} c_k^2 < \infty$.

Therefore, with property iv) it follows

Lemma: the series $\sum\frac{2h_k}{k}\sin(2\pi\omega_k x)$ and $\sum\frac{2h_k}{k}\sin(\pi(\omega_k + \omega_{k+1})x)$ converge almost everywhere.

Indicating the following replacements

$cot(\pi x) = 2\sum_{n=1}^{\infty} sin(2\pi nx)) \in H_{-1}^{\#}(0,1)$ Cesàro summable (ZyA) VI-3, VII-1)

$\to$ $cot^{(1)}(\pi x) := \sum sin(\pi(2\omega_n)x) \in H_0^{\#}(0,1)$ Abel summable

$\to$ $cot^{(2)}(\pi x) := \sum sin(\pi(\omega_n + \omega_{n+1})x \in H_0^{\#}(0,1)$ Abel summable

and

$-\frac{\pi}{2}\log\left(tan\left(\frac{\pi}{2}x\right)\right) = \sum\frac{2h_n}{n}\sin(2\pi nx)) \in L_2^{\#}(0,1)$ $\to$

$\to$ $-\frac{\pi}{2}\log\left(tan^{(1)}\left(\frac{\pi}{2}x\right)\right) := \sum\frac{h_n}{n}\sin(\pi(2\omega_n)) \in H_1^{\#}(0,1)$

$\to$ $-\frac{\pi}{2}log\left(tan^{(2)}\left(\frac{\pi}{2}x\right)\right) := \sum\frac{h_n}{n}sin(\pi(\omega_n + (\omega_{n+1})) \in H_1^{\#}(0,1)$

and

$\to$ $\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} z^{1-s}\left(-\pi\, cot^{(1)}(\pi z)\frac{dz}{z}\right.$ resp. $\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} z^{1-s}\left(-\pi\, cot^{(2)}(\pi z)\frac{dz}{z}\right.$.

Property ii) gives the Hardy-Littlewood condition $|s_{n+1}-s_n|<\frac{K}{n}$, i.e. the sequence s_n has a defined Abel average ((EdH) 12.7)

$$\lim_{r\uparrow 0}\frac{s_1 r+s_1 r^2+s_3 r^3+\cdots\ldots}{r+r^2+r^3+\cdots\ldots}=L \quad .$$

As a consequence, the sequence s_n enables a point measure, which is s_n at n and zero elsewhere, i.e.

$$d\sigma(x)=d\sigma_n(x)=\begin{cases} s_n & x=n \\ 0 & else \end{cases}$$

enjoying the identities

$$\lim_{r\uparrow 1}\frac{\int_0^\infty r^x d\sigma(x)}{\int_0^\infty r^x dx}=L \ , \ \lim_{A\to\infty}\frac{\int_0^A d\sigma(x)}{\int_0^A dx}=L \ .$$

which is a restated Abel average representation of the sequence s_n. With respect to the relationship of series with Hadamard gaps, launary sequences, Abel summability and Tauberian theorems for distributional point values to local behavior of distributions we also refer to ((ViJ1) pp. 98, 119, 123, 125).

(EdH) 12.7: „*The PNT is about the asymptotics equivalence of $\psi(x)\sim x$, which is equivalent to the statement that $d\psi(x)\sim dx$ as a Cesaro average in the context of Tauberian theorems. Hardy-Littlewood were able to prove the PNT by showing $d\psi(x)\sim dx$ as an Abel average, where a significant amount of work is done by a Tauberian theorem.*"

The Mangoldt resp. Landau density functions are given by

$$\psi(x)=\sum_{n<x}\Lambda(n) \quad \text{resp.} \quad \vartheta(x)=\sum_{n=1}^\infty \Lambda(n)\log(\frac{x}{n}) \ .$$

We propose an alternative density function (based on the sequence s_n) to enable a proof in which all of the work is done by a Tauberian theorem:

What cannot derived from the PNT is the convergence of the series

$$\sum_{n=1}^\infty \frac{\mu(n)}{n} log(\frac{1}{n})=1 \ .$$

„*The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived*" ((LaE) §160). In order to anticipate this finding we suggest to apply the point measures $d\sigma_n(x)$ and

$$\vartheta^*(x)=\sum_{n=1}^\infty \Lambda(n)\log(\frac{x}{\omega_n}) \quad , \quad \vartheta^*(x)=\sum_{n=1}^\infty \frac{\mu(n)}{n}\log(\frac{x}{\omega_n}) \qquad (\frac{x}{n}<\frac{x}{\omega_n}=\frac{x}{n}\frac{1}{s_n}<2\frac{x}{n})$$

(whereby $d\vartheta=d\vartheta^*=d\vartheta^{**}$), in combination with the Ikehara theorem ((EdH) 12.7).

The convergence of the series

$$\sum_{n=1}^\infty \frac{\mu(n)}{n} log(\frac{1}{n})=1 \ .$$

is related to the proposed Hilbert space $H_{-\frac{1}{2}}$ by the identity

$$1=\sum_{n=1}^\infty \frac{1}{n}\mu(n)\,log(\frac{1}{n})=\sum_1^\infty \frac{1}{n}a_n b_n=:((u,v))_{-\frac{1}{2}}:=\lim_{\omega\to\infty}\frac{1}{2\omega}\int_{-\omega}^{\omega}u\left(\frac{1}{2}+it\right)v\left(\frac{1}{2}-it\right)dt$$

where

$$\left(u\left(\frac{1}{2}+it\right):=\sum_{n=1}^\infty \frac{\mu(n)}{n^s}\in H_{-\frac{1}{2}} \quad , \quad v\left(\frac{1}{2}-it\right):=\sum_{n=1}^\infty \frac{log(1/n)}{n^s}\in H_{-\frac{1}{2}} \quad , \quad s=\frac{1}{2}+it \ .$$

Property v) in combination with the theorem of E. Landau (LaE3) applied for $\alpha_n, \varphi_n := e^{2\pi i\alpha_n}=e^{\pi i\omega_n}$ leads to the estimate

$$\forall\varepsilon>0 \quad cot\left(\frac{\pi}{2}\theta\right)-\varepsilon=cot\left(\frac{\pi}{8}\right)-\varepsilon<S_m:=\left|\sum_{n=1}^m \varphi_n\right|=\left|\sum_{n=1}^m e^{\pi i\omega_n}\right|<cot\left(\frac{\pi}{8}\right)=cot\left(\frac{\pi}{2}\theta\right) \ .$$

1.e New arithmetical functions for additive number theory

From the section above we propose to take advantage of property iii) above (which are kind of alternative major arcs, (BrK4) Note O22)

$$f_n := \frac{1}{\omega_n+\omega_{n+1}} \in \left(\frac{1}{2n+1},\frac{1}{2n}\right) \text{ , } g_n := \frac{1}{2\omega_n} \in \left(\frac{1}{2n},\frac{1}{2n-1}\right)$$

replacing

$$\beta(x) := \sum_{n<x} a_n log\left(\frac{x}{\omega_n}\right) \quad \rightarrow \quad \beta^*(x) := \frac{1}{2}\left[\sum_{\substack{n<x \\ n\ odd}} a_n \log\left(\frac{x}{2\omega_n}\right) + \sum_{\substack{n<x \\ n\ even}} a_n \log\left(\frac{x}{\omega_n+\omega_{n+1}}\right)\right]$$

with $d\beta = \frac{dx}{x} = d\beta^*$. Property ii) $2n-1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n+1$ in the form

$$4n-1 = \left(2n-\frac{1}{2}\right) + \left(2n-\frac{1}{2}\right) < \{2\omega_n\} + \{\omega_n + \omega_{n+1}\} = \frac{1}{g_n} + \frac{1}{f_n} < \left(2n+\frac{1}{2}\right) + \left(2n+\frac{1}{2}\right) = 4n+1 \text{ .}$$

provide an adequate „binary prime sum" counting sequences.

For the fractional part function $\rho(x) = x - [x] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin(2\pi n)}{\pi n} \in L_2^{\#}(0,1)$ we recall the following theorem of E. Landau (PoG1):

Theorem: Let q_n denote a divergent sequence of positive numbers $0 < q_1 \leq q_2 \leq q_3 \leq \cdots$ $\lim_{n\to\infty} q_n = \infty$, $\tau(x)$ the corresponding counting function of the numbers of q_n less than $\leq x$ and $w(x)$ a positive, non decreasing function with

$$\lim_{n\to\infty} \frac{w(2x)}{w(x)} = 1, \text{ and } \lim_{n\to\infty} \frac{\tau(x)w(x)}{x} = 1.$$

Then

$$\lim_{n\to\infty} \frac{1}{\tau(x)} \sum_{q\leq x} \rho\left(\frac{x}{q}\right) = 1-\gamma \text{ .}$$

We mention that a consequence of the condition $\lim_{n\to\infty} \frac{w(2x)}{w(x)} = 1$ is $\lim_{n\to\infty} \frac{w(bx)}{w(ax)} = 1$ for any positive $a, b > 0$ and the formulas $\frac{\sin(2x)}{sinx} = 2cosx$ resp. $\frac{\cot(2x)}{cotx} = \frac{1}{2}\frac{\cos(2x)}{sin^2(x)}$.

This Landau theorem above is a special case of the following generalized number theoretical function theorem (PoG1):

Lemma: Let $w(x)$ a positive, non decreasing function with $\lim_{n\to\infty} \frac{w(\beta x)}{w(\alpha x)} = 1$ with α, β positive numbers. Then

$$\lim_{n\to\infty} \frac{w(x)}{x} \sum_{n\leq x} f\left(\frac{x}{n}\right) = \int_0^1 f(t)dt \text{ .}$$

For $h(x) := (\pi x)cot(\pi x)$ it holds $h(x) \rightarrow_{x\to 0} -1$ and $\int_0^1 h\left(\frac{x}{2}\right)dx = log2$ ((GrI) 3.747).

Then for $w(x) := h\left(\frac{1}{x}\right)$ and $f(x) := h\left(\frac{x}{2}\right)$ one gets

$$\frac{w(x)}{x} f\left(\frac{x}{n}\right) = \frac{1}{x} h\left(\frac{1}{x}\right) h\left(\frac{x}{2n}\right) \quad \text{and} \quad \int_0^1 f(t)dt = log2 = -log\xi(0).$$

Putting $h^{(1)}(x) := -(\pi x)cot^{(1)}(\pi x)$, $h^{(2)}(x) := -(\pi x)cot^{(2)}(\pi x)$ this enables the following arithmetic function definitions

$$\sigma\ (x) := \frac{1}{x} h\left(\frac{1}{x}\right) \sum_{n\leq x} h\left(\frac{x}{2n}\right) \quad \sigma^{(1)}(x) := \frac{1}{x} h^{(1)}\left(\frac{1}{x}\right) \sum_{n\leq x} h^{(1)}\left(\frac{x}{2\omega_n}\right) \text{ , } \sigma^2(x) := \frac{1}{x} h^{(2)}\left(\frac{1}{x}\right) \sum_{n\leq x} h^{(2)}\left(\frac{x}{\omega_n+\omega_{n+1}}\right) \text{ .}$$

With respect to the additive number theory we propose to deal with

$$\sigma^*(x) := \frac{1}{2}\left[\sigma^{(1)}_{(odd)}(x) + \sigma^{(2)}_{(even)}(x)\right].$$

The regular varying function concept was introduced by Karamata (SeE).

Lemma ((VlV), I.3): if $v(x)$ is differentiable for $x \geq x_0$ and there is a limit $\lim_{x\to\infty} \frac{xu'((x)}{u(x)} \to \alpha$, then $v(x)$ is automodel (or regular varying) of order α, i.e. $\lim_{x\to\infty} \frac{u(bx)}{u(x)} = b^{\alpha}$.

Therefore, a further candidate for $w(x)$ is given by $k(x) := -cot\left(\pi \frac{x}{2}\right)$, which is a slow regular varying (automodel) of order zero ((EsR) 3.9.8). The above criterion is fulfilled, as $\frac{x\boldsymbol{k}'((x)}{k(x)} = \frac{\pi x}{\sin(\pi x)}$.

Further candidates for $f(x)$ are e.g.

i) $f_1(x) := -\log(sin\pi x)$ with $\int_0^1 f_1(t)dt = log2$ (GrI) 4.384

ii) $f_2(x) := -\mathrm{h}\left(\frac{x}{2}\right) = -h(x) := \frac{1}{x} - \left(\frac{\pi}{2}x\right) cot\left(\frac{\pi}{2}x\right)$ with $\int_0^1 f_2(t)dt = \log(\frac{\pi}{2})$ (GrI) 3.788

iii) $f_3(x) := \log\left(a \cdot tan\left(\pi \frac{x}{2}\right)\right)$ with $\int_0^1 f_3(t)dt = \log a$, $a > 0$ (GrI) 4.227.

Typical examples of slow varying functions are positive constants or functions converging to a positive constant, logarithms and iterated logarithms. Specifically the function $-logx$ is slowly varying at $x = 0^+$ ((SeE) p. 47), which will be applied in the following section.

Every regular varying function f of order α has a representation $f(x) = x^{\alpha}L(x)$, where L is some slow varying function (MiT).

A general representation of slow regular varying functions is given by the

Theorem 1.2 (SeE): A positive measurable function L on $[x_0, \infty[$ is a slow varying function if and only if it can be written in the form

$$L(x) = e^{v(x) + \int_{x_0}^{x} \epsilon(y)\frac{dy}{y}}$$

where $v(\cdot)$ is a measurable bounded function, such that $\lim_{x\to\infty} v(x) = c$ $(|c| < \infty)$ and $\varepsilon(x) \to 0$ as $x \to \infty$.

Comparison Tauberian theorems are about the asymptotics behavior of the ratio of some integral transforms of two functions (distributions), if the asymptotic behavior of the ratio of some other integral transform is given ((EsR), (VlV)). In ((VlV) the Abel and Cesaro series summation with respect to an automodel weight, as well as asymptotic properties of solutions of convolution equations are considered.

Karamata's Tauberian theorem involving regular variations are provided in ((SeE) 2.2)

Theorem 2.3: Let $U(x)$ be a monotone non-decreasing function on $[0, \infty[$ such that $w(x) = \int_{0^+}^{\infty} e^{-xu} dU(u)$ is finite for all $x > 0$. Then, if $\rho \geq 0$, and L is a slowly varing function,

i) if $w(x) = x^{-\rho}L(\frac{1}{x})$ as $x \to 0^+$ then $U(x) \sim x^{\rho} \frac{L(x)}{\Gamma(\rho+1)}$ all $x \to \infty$

ii) if $w(x) = x^{-\rho}L(x)$ as $x \to \infty$ then $U(x) \sim x^{\rho} \frac{L(\frac{1}{x})}{\Gamma(\rho+1)}$ all $x \to 0^+$

The corresponding „density" extension of this theorem is provided in

Theorem 2.4: Let $U(x)$, defined and positive on $[A, \infty[$ for some given A sufficiently large, be given by $U(x) = \int^{\infty} u(y)dy$, where $u(y)$ is ultimately monotone. Then for $\rho \geq 0$, if $U(x) = x^{\rho}L(x)$ then $\frac{xu(x)}{U(x)} \to \rho$ as $x \to \infty$.

Another such theorem, using both parts i) and ii) of theorem 2.3 above is provided in theorem 2.5, which is about the asymptotics $\int_{0^-}^{\infty} \frac{dA(t)}{(t+x)^{\rho}} \sim x^{\rho}L(x)$.

1.f Dirichlet series and the (distributional) Hilbert space $H^{\#}_{-1/2} \cong l_2^{-1/2}$

In this section we are concerned with Hilbert scales $H^{\#}_{\alpha} \cong l_2^{\alpha}$, $\alpha \in R$, which is built on the 2π −periodic Hilbert space $L_2^{\#}(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit circle sphere. Then for $u \in L_2^{\#}(\Gamma)$ and for real $\beta \in R$, $n \in Z$ the Fourier coefficients

$$u_n := \frac{1}{2\pi} \oint u(x) e^{inx} dx$$

enable the definition of the norms ((BrK), (BrK3,6,7))

$$\|u\|_{\beta}^2 := \sum_{n=-\infty}^{\infty} |n|^{2\beta} |u_n|^2 .$$

With the notation of [LaE] §227, Satz 40, the for $s > 0$ convergent Dirichlet series (in a classical L_∞ resp. C^0 sense, where H_k is a subset of C^0 for $k \geq \frac{1}{2} + \varepsilon, \varepsilon > 0$)

$$f(s) := \sum_1^{\infty} a_n e^{-s \log n} \qquad g(s) := \sum_1^{\infty} b_n e^{-s \log n}$$

are linked to the (distributional) Hilbert space $H^{\#}_{-1/2} \cong l_2^{-1/2}$ by ((EdH) 9.8, (NaS))

$$((f,g))_{-1/2} := \lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(1/2+it) g(1/2-it) dt = \sum_1^{\infty} \frac{1}{n} a_n b_n .$$

For $\Gamma := S^1(R^2)$, the operators defined by the the single layer potential, the normal derivative of the double layer potential and the Hilbert transform ((KrR) theorems 8.20, 8.21, (LiI), $\alpha = -\frac{1}{2}$) leads to the following representations ($\alpha \in R$):

$$S_{-1}: H^{\#}_{\alpha} \to H^{\#}_{\alpha+1} \qquad S_{-1}[u](x) := 2\int_0^1 \log \left| \frac{1}{2\sin(\pi(x-y))} \right| u(y) dy \ , \ -log 2 \sin(\pi x) = \sum_{n=1}^{\infty} \frac{\cos(2n\pi x)}{n}$$

$$S_1: H^{\#}_{\alpha+1} \to H^{\#}_{\alpha} \qquad S_1[u](x) := \frac{1}{2}\int_0^1 \frac{u(y)}{sin^2(\pi(x-y))} dy \ , \quad \left[\frac{\pi}{\sin(\pi x)}\right]^2 = \sum_{-\infty}^{\infty} \frac{1}{(x-n)^2}$$

$$S_0: H^{\#}_{\alpha+1/2} \to H^{\#}_{\alpha+1/2} \qquad S_0[u](x) := \int_0^1 cot(\pi(x-y)) u(y) dy, \quad \pi cot(\pi x) = \frac{1}{x} + \sum_{\substack{-\infty \\ n \neq 0}}^{\infty} \left[\frac{1}{x-n} + \frac{1}{n}\right].$$

For the two self-adjoint operators S_{-1}, S_1 it holds $(S_{-1}[u],v)_{\beta} \cong (u,v)_{\beta-1/2}$, $(S_1[u],v)_{\beta} \cong (u,v)_{\beta+1/2}$, for the Hilbert transform operator it holds $(S_0[u],v)_{\beta} = -(u,S_0[v])_{\beta}$, i.e. the three convolutions integrals are Pseudo-Differential operators of order -1, 1, 0, defining corresponding isomorphisms between the corresponding domains and ranges.

As the function $log\frac{1}{x}$ is slowly varying at $x = 0^+$ ((SeE) p. 47), the kernel function of $S_{-1}[u]$ is slowly varying at

$$x = \begin{cases} 2\mu - 1 & +\mu \in N \\ 2\mu & -\mu \in N \end{cases}$$

With respect to the kernel functions of $S_1[u]$, $S_0[u]$ we note the relationships

$$\frac{d^3}{dx^3}[-\log(sinx)] = \frac{d^2}{dx^2}[-\text{cotx}] = \frac{d}{dx}\left[\frac{1}{sin^2 x}\right] = (-cotx)\frac{1}{sin^2 x} = (-cotx)\frac{d}{dx}[-cotx] = \frac{1}{2}\frac{d}{dx}[cot^2 x]$$

with its relationship to the Claussen function ((AbM) 27.8, (BrK4) Notes O27,28).

For the (transformed) Zeta function on the critical line Ξ it holds $\Xi \in H^{\#}_{-1}$, i.e. there exists a *convolution* integral representation of the Zeta function by an unique $\omega \in H_0^{\#}$ with $S_1[\omega](x) := \Xi(x)$ (CaD). It leads to the corresponding variational representation in the form

$$(S_1[\omega],v)_{-1} = (\omega,v)_{-1/2} = (\Xi,v)_{-1} \ , \ \forall v \in H^{\#}_{-1}$$

The distributional $H_{-1/2}$ Hilbert space framework enables the Bagchi RH criterion, which is a reformulation of the Nyman-Beurling RH criterion (BaB). It is basically a standard density argument of the Hilbert sub-space $H^{\#}_{-1/2}$, which is densely embedded into $H^{\#}_{-1}$ with respect to the $H^{\#}_{-1}$ – norm ((BrK4) remark 3.5, notes S21 & 24, see also „Riesz theory" ((KrR) chapter 3). The corresponding approximating sequence ${\Xi_n}^*$ of the Zeta function Ξ is defined by

$$(\omega_n^*, v)_{-1/2} = ({\Xi_n}^*, v)_{-1} \ , \forall v \in H^{\#}_{-1/2} \ .$$

From (LiI) 1.2.34, we mention

$$S_1[a_n \cos(2\pi nx) + b_n \sin(2\pi nx)](x) = -2n[a_n \cos(2\pi nx) + b_n \sin(2\pi nx)]$$

With its relationship to the concept of logarithmic capacity of sets and convergence of Fourier series of functions fulfilling $\sum_1^\infty n[a_n^2 + b_n^2]$ and harmonic analysis ((ZyA) V-11, (BrK4) remarks 4.1, 4.2, Notes S37/38).

With respect to the Bagchi criterion and its related Hilbert space $H^{\#}_{-1}$ we further mention the relationship to the Brownian motion $B(t) = \int_0^t dB(\tau)$, which is obtained as the integral of the white noise signal $dB(t)$, which is distribution. Its spectral density $E_0 = \left|Fourier[B'](\omega)\right|^2 = constant$ is flat. Therefore the energy spectrum of the Brownian motion is given by $E(\omega) = |Fourier[B](\omega)|^2 = \frac{E_0}{\omega^2}$.

The Wiener-Ikehara theorem was devised to obtain a simple proof of the PNT. In this theorem the boundary behavior of a Laplace transform in the complex plane plays a crucial role. The distributional version of this theorem shows that local pseudofunction boundary behavior, which allows mild singularities, is necessary and sufficient for the desired asymptotic relation. It follows that the twin-prime conjecture is equivalent to a pseudofunction boundary behavior of a certain analytic function (KoJ).

In (ViJ), (ViJ1) a proof of the PNT is provided, based on the Dirac function $\delta \in H_{-1/2-\varepsilon}$ in combination with the concept of quasi-asymptotically bounded distributions defined by

$$\langle \frac{f(\vartheta \cdot)}{\rho(\vartheta)}, \varphi(\cdot) \rangle = O(1) \text{ for } \vartheta \to \infty \ .$$

The density of the Mangoldt function $\psi(x) = \sum_{n<x} \Lambda(n)$ is then given by ((BrK4) Note S17)

$$\psi'(x) = \sum_{n<x} \Lambda(n)\delta(x-n) \in H_{-\frac{1}{2}-\varepsilon}$$

We propose an alternative quasi-asymptotically bounded $H_{-1/2}$ distributions concept defined by

$$\lim_{\vartheta \to \infty} \left(\frac{f(\vartheta \cdot)}{\rho(\vartheta)}, \varphi(\cdot) \right)_{-1/2} = O(1) \ , \ \ \forall \varphi \in H_{-1/2} \to \infty$$

leading to a replacement in the form ((BrK4) Note S19)

$$\psi' \in H_{-\frac{1}{2}-\varepsilon} \quad \to \ \widetilde{S_1}[\psi] \in H_{-\frac{1}{2}}: \ (\widetilde{S_1}[\psi], v)_{-1/2} = (\psi, v)_0 \ \ \forall v \in H_{-1/2}.$$

The approximation by polynomials in a complex domain leads to several notions and theorems of convergence related to Newton-Gaussian and cardinal series. The latter one are closely connected with certain aspects of the theory of Fourier series and integrals. Under sufficiently strong conditions the cardinal function can be resolved by Fourier's integral. Those conditions can be considerably relaxed by introducing Stieltjes integrals resulting in (C,1) summable series ((WhJ1) theorems 16 & 17, (BrK4) Remarks 3.6/3.7). With respect to interpolations theory with points of sequences c_n (with respect to the Newton-Gauss and cardinal series), the corresponding cardinal series theory with certain aspects of the theory of Fourier series and integrals, especially Fourier-Stieltjes series and related convergent series in the form

$$\sum_{n=1}^{\infty}\frac{1}{n}(|a_n|+|a_{-n}|)<\infty$$

we refer to ((BrK4) Remarks 3.6 & 3.7, pp 105 ff, WhJ1).

In the context of representations of coefficient sums by integrals we recall from (LaE2), §86 the

Lemma: Let $D(s)=\sum_{n=1}^{\infty}\frac{b_n}{n^s}$ denote an absolute convergent Dirichlet series ($a:=Re(s)$) and

$$f(x):=\begin{cases}\sum_{n=1}^{x}b_n & for \quad x \quad not\ integer\\ \sum_{n=1}^{x}b_n-b_x/2 & for \quad x \quad integer\end{cases}$$

Then

$$f(x)=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{a-it}^{a+it}D(s)x^s\frac{ds}{s} \quad \text{resp.} \quad \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{D(s)}{s}x^s\frac{ds}{s}=\sum_{n=1}^{x}b_n\log(\frac{x}{n}) \ .$$

With respect to the physical aspects we refer to (NaS), where the $H_{-1/2}$ dual space of $H_{-1/2}$ on the circle (with its inner product defined by a Stieltjes integral) is considered in the context of Teichmüller theory and the universal period mapping via quantum calculus. For the corresponding Fourier series analysis we refer to ((ZyA) XIII, 11).

The common denominator of the alternatively proposed Hilbert space framework $H_{-1/2}$ goes along with the definition of a correspondingly defined "momentum" operator (of order 1) P: $H_{1/2} \rightarrow H_{-1/2}$ defined in a variational form.

In the one-dimensional case the Hilbert transform H (in the n>1 case the Riesz operators R) is linked to such an operator given by $(Pu,v)_{-1/2)} = (\mathrm{H}u,v)$. With respect to quantum theory this indicates an alternative Schrödinger momentum operator (where the gradient operator "grad" is basically replaced by the Hilbert transformed gradient, i.e. **P**:=-i***R**(grad) and a corresponding alternative commutator representation Q**P**-**P**Q in a weak $H_{-1/2}$ form.

We note that the Riesz operators **R** commute with translations and homothesis and enjoy nice properties relative to rotations.

The theory of spectral expansions of non-bounded self-adjoint operator is connected with the notions "Lebesgue-Stieltjes integral" and "functional Hilbert equation for resolvents ((LuL) (7.8). The corresponding Hilbert scale framework plays also a key role on the inverse problem for the double layer potential. The corresponding model problem (w/o any compact disturbance operator) with the Newton kernel enjoys a double layer potential integral operator with the eigenvalue 1/2 (EbP).

The incomplete Gamma function play a key role to compute the action of the Leray projection operator on the Gaussian functions (LeN1). Those action formulas can be applied to derive in the context of the well-posedness topic of the NSE and related (based on tempered distribution and a Carleson measure characterization of the BMO space) estimates ((LeN1), (KoH), theorems 1 and 2, see also (BrK4) pp. 26, 58, 64, 99, 121).

For the related equations with respect to the incomplete Gamma function we refer to (OlF1) 7.2.2, 8.4.15, (AbM) 6.5.12, 13.6.10).

The RH is connected to the quantum theory via the Hilbert-Polya conjecture resp. the Berry-Keating conjecture. It is about the hypothesis, that the imaginary parts t of the zeros $1/2 + it$ of the Zeta function $Z(t)$ corresponds to eigenvalues of an unbounded self-adjoint operator, which is an appropriate Hermitian operator basically defined by $QP + PQ$, whereby Q denotes the location, and P denotes the (Schrödinger) momentum operator. In (BrK3) the corresponding model (convolution integral) operator S_1 (of order 1 with "density" $dcot$ for the one-dimensional harmonic quantum oscillator model is provided.

In the context of the Berry-Keating conjecture the Gaussian function $f(x)$ can be characterized as "minimal function" for the Heissenberg uncertainty inequality. Applying the same solution concept as above then leads to an alternative Hilbert operator based representation in $H_{-1/2}$, resp. to a H_{-1} based definition of the commutator operator with extended domain.

1.g A $H^{\#}_{-1/2} \cong l_2^{-1/2}$ based alternative circle method on the unit circle and a Kummer function zero based approach to prove the binary Goldbach conjecture

The current tool trying to prove the tertiary and binary Goldbach conjecture is about the (Hardy/Littlewood) circle method. Vinogradov's „*basic intervals*" correspond in principle to Hardy and Littlewood's „*major arcs*" ((ViI) p.61). Hardy and Littlewood dissected the circle $x = e^{2\pi i\alpha}$, or rather a smaller concentric circle, into „Farey arcs". The major acrs, or basic intervals, provide the main term in the asymptotic formula for the number of representations. Their treatment does not give rise to any very serious difficulties compared to the problems presented by the „*minor arcs*", or „*supplementary intervals*". The latter ones are analyzed by the Weyl (trigonometrical) sums

$$S(x) := \sum_n e^{2\pi inx}.$$

On the one hand side, the probability that the binary Goldbach conjecture is true, is 100%, while on the other side the current (circle method) tool failed, because of not sufficient (already optimal) Weyl estimates. As those Weyl estimates are w/o any information (i.e. relevance) regarding the underlying (Goldbach) problem, the probability, that the circle method is not adequate is also 100%.

The natural Hilbert space framework (related to generalized Fourier analysis techniques and Dirichlet series, but also related to convergent Weyl sums series) is $\in H^{\#}_{-1/2}(0,1)$ (e.g. BrK9). The Cesàro summable $\cot(\pi x) = 2\sum_{n=1}^{\infty} \sin(2\pi nx)) \in H^{\#}_{-1}(0,1)$ (ZyA) VI-3, VII-1), is related to the eigenfunctions $e^{2\pi inx} = e^{i\pi(2n)x}$, while the proposed alternative Abel summable functions

→ $cot^{(1)}(\pi x) := \sum \sin(\pi(2\omega_n)x) \in H_0^{\#}(0,1)$

→ $cot^{(2)}(\pi x) := \sum \sin(\pi(\omega_n + \omega_{n+1})x \in H_0^{\#}(0,1)$

are related to the eigenfunctions pair $e^{i\pi(2\omega_n)x}$ and $e^{i\pi(\omega_n+\omega_{n+1})x}$ resp. to the alternative Weyl sums

$$S_1^*(x) := \sum_n e^{i\pi(2\omega_n)x} \ , \ S_1^*(x) := \sum_n e^{i\pi(\omega_n+\omega_{n+1})x}.$$

With the notation of (LaE1) the prime pair (p, q) counting function $H(x)$ with the condition $p + q \leq x$ is given by

$$H(x) = \sum_{n=1}^{x} G_n = \sum_{p \leq x} \pi(x-p) \sim \frac{x}{logx}\int_0^{x/2} \frac{dt}{logt} \sim \frac{1}{2}(\frac{x}{logx})^2$$

The (improved) Stäckel formula (based on the Euler $\varphi(n)$ –function) shows the asymptotics in the form

$$\tilde{G}_{2n} \sim \frac{1}{2}\frac{105 \cdot \zeta(3)}{\pi^4}\frac{n}{logn}\cdot\frac{n}{logn}\cdot\frac{1}{\varphi(n)} \sim 0{,}648 \ldots \cdot \frac{1}{\varphi(n)}\cdot\frac{n}{logn}\cdot\frac{n}{logn}.$$

Therefore, Landau predicted a proof of the binary Goldbach conjecture „with high probability" (LaE1).

Related to the Stäckel formula we recall from (ApM) p. 71, the following estimates

$$\frac{\sigma(n)}{n^2} \leq \frac{1}{\varphi(n)} \leq \frac{\pi^2}{6}\frac{\sigma(n)}{n^2} = \zeta(2)\frac{\sigma(n)}{n^2} \quad , \ n \geq 2$$

whereby $\sigma(n) = \sigma_1(n)$ denotes the sum of the divisors of n ((ApM) p. 38).

With respect to the factor $\zeta(2)$ we recall the related $cot\left(\frac{\pi}{2}x\right)$ estimate from the previous section

$$\left|\frac{\pi}{2}cot\left(\frac{\pi}{2}x\right)-\frac{1}{x}\right|\leq\frac{\pi^2}{6}=\zeta(2)\ ,\quad |x|\leq 1\ .$$

With respect to the Zeta function itself $\zeta(s)$ we recall the related $cot(\pi x)$ representation from the previous section (TiE)

$$\sum_{n>x}\frac{1}{n^s}=\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}z^{1-s}(-\pi\,cot(\pi z)\frac{dz}{z}\quad ,\ s=\frac{1}{2}+ix$$

and its link to the Zeta function is given by

$$(1-s)\cdot\zeta^*(s)=cot(\frac{\pi}{2}s)\cdot\zeta(s)=tan(\frac{\pi}{(2}(1-s))\cdot\zeta(s)$$

whereby it holds

$$\pi\,cot(\pi x)=\frac{\pi}{2}\left[cot\left(\frac{\pi}{2}x\right)-\frac{1}{2}cot(\frac{\pi}{2}(1-x))\right]=\frac{\pi}{2}\left[cot\left(\frac{\pi}{2}x\right)-\frac{1}{2}tan(\frac{\pi}{2}x)\right].$$

and

$$log'(tan\left(\frac{\pi}{2}s\right)=-log'(cot\left(\frac{\pi}{2}s\right)=\frac{\pi}{sin(\pi s)}\ .$$

In the following we shall deal with the special Kummer functions

$$K_a(z):=\ {}_1K_1(a,a+1;z)\text{ with }0<a\leq 1.$$

The asymptotics of ${}_1K_1(a,a+1;z)$ is given by ((OlF) 10.3, (AbM) 13.5.1.)

$${}_1K_1(a,a+1;z)\sim\frac{1}{\Gamma(a)}\frac{e^z}{z}\ ,\ z\to\infty,ph(z)=0\ ,\ {}_1K_1(a,a+1;z)\sim\frac{1}{(-z)^a}\ ,\ z\to-\infty,ph(-z)=0$$

where $ph(z)$ denotes the phase or the argument of z. For the real case ($x\in R$) we deal with the function $F_a(x):=c\cdot\ {}_1F_1(a,a+1;x)$ with a given constant c fulfilling the following properties

i) ${}_1F_1(a,c;x)\sim\frac{1}{\Gamma(a)}\frac{e^x}{x^{c-a}}$ resp. ${}_1F_1(a,a+1;x)\sim\frac{1}{\Gamma(a)}\frac{e^x}{x}\ ,\ x\to\infty$

ii) $\frac{d}{dx}F_a(x)=\frac{a}{a+1}F_{a+1}(x)$

iii) $F_a(x)\sim\frac{c}{\Gamma(a)}\frac{e^x}{x}\ ,\ x\to\infty$ ((OlF), 7 §10.1, (AbM) 13.5.1.)

iv) $\frac{1}{2}\frac{d}{dx}F_a^2(x)=\frac{a}{a+1}F_a(x)\cdot F_{a+1}(x)\sim\frac{1}{a+1}\frac{1}{\Gamma^2(a)}(\frac{e^x}{x})^2$.

From (AbM) 13.2.9, we mention the Barnes-type contour integral representation

$${}_1F_1(a,a+1;z)=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a}{a+s}\Gamma(1-s)(-z)^s\frac{ds}{s},\quad |\arg(-z)|<\frac{\pi}{2},\ a\neq 0,-1,-2,\ldots$$

where the contour must separate the poles of $\Gamma(-s)$ from those of $\Gamma(a+s)$.

The special Kummer functions $K_a(z):=\ {}_1K_1(a,a+1;z)$ go along with the Hurwitz generalization of the Zeta function ((TiE) 2.17)

$$\zeta(s,a)=\sum_{n=1}^{\infty}\frac{1}{(n+a)^s}\ ,\ Re(s)>1\ ,\ 0<a\leq 1$$

resp. the corresponding generalized Dirchlet series.

In the following we first shall deal with the choice

$$\frac{1}{2} \le a := a_n := \frac{2n-1}{2n} \to 1$$

and

$$a := \frac{2n-1}{2n} \text{ , } a_n := \frac{1}{a+1} = \frac{2n}{4n-1} \text{ , } b_n := \frac{1}{\Gamma^2(a)} = \frac{1}{\Gamma^2\left(1-\frac{1}{2n}\right)} \text{ , } c := \sqrt{2/3}.$$

The corresponding definition of $H^*(x)$ is motivated by the replacement

$$H(x) \sim \frac{x}{logx} \int_2^{\frac{x}{2}} \frac{dt}{logt} \frac{d}{dx} \;\; \to \;\; H^*(x) \sim \frac{d}{dx} F_{a_n}(logx) \cdot \int_{log2}^{\log(\frac{x}{2})} \frac{d}{dt} F_{a_n}(t) dt \text{ .}$$

fulfilling the

Lemma („Chebychev" inequality):

$$\frac{1}{3\pi} (\frac{x}{logx})^2 \le H_n^*(x) \le \frac{1}{2} (\frac{x}{logx})^2.$$

Proof: It holds $a_1 = \frac{3}{2}$, $a_\infty = \frac{1}{2}$, $b_1 = \frac{1}{\pi}$, $b_\infty = 1$ and therefore

$$\frac{1}{2\pi} \le a_n \cdot b_n \le \frac{3}{2} \text{ resp. } \frac{1}{3\pi} \le \frac{2}{3} a_n \cdot b_n \le 1.$$

Chosing $c := \sqrt{2/3}$ then proves the lemma.

We note that for $n \ge 4$ it holds $1 - \frac{1}{8} < 1 - \frac{1}{2n}$, $1 + \frac{1}{2n} < 1 - \frac{1}{8}$, with ist relationship to the Chebychev inequality

$$1 - \frac{1}{8} < \frac{\pi(x)}{\frac{x}{logx}} < 1 + \frac{1}{8} \text{ .}$$

The Kummer function zeros related inequality $2n - 1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n + 1$ indicates a replacement

$$\frac{1}{2} \le a_n \le 1 \qquad \to \qquad \frac{1}{2} \le \frac{2n-1}{2n} < d_n := \frac{\omega_n + \omega_n}{\omega_n + \omega_{n+1}} < \frac{2n}{2n+1} \le 1$$

leading to

$$H_n^{**}(x) := \frac{1}{2} \sum_{n<x} \frac{d}{dx} F_{d_n}^2(logx)$$

With respect to a Hilbert space framwork $H_{-1/2}$ (see also below) we note the identities $((dK^2, v))_{\alpha-1/2} = (K^2, v)_\alpha$ for $K^2, v \in H_\alpha$.

The circle method (based on the interior of the unit disk) is concerned with complex-valued functions $f(z) = f(r \cdot e(\varphi))$ $(e(\varphi) := e^{2\pi i\varphi}$, $\varphi \in (0,1))$ resp. $e_n(\varphi) := e(n\varphi)$ fulfilling

$$\int_0^1 e_n(\varphi)d\varphi = \begin{cases} 1 & n= \quad 0 \\ 0 & n= \quad otherwise \end{cases} \quad \text{and} \quad |1-e(\varphi)|^2 = 4sin^2(\pi\varphi).$$

It is about Fourier analysis of complex-valued power series functions

$$f(x) = \sum_{n=0}^{\infty} a_n z^n \ , \ \ |z| < 1$$

i.e. defined for the interior unit circle domain. The underlying mathematical tool set is based on the formula ((ViI1) chapter I, lemma 4, Notes)

$$r^n a_n = \int_0^1 f(re^{2\pi it})e^{-2\pi int}dt \ , \ 0 < r < 1.$$

The corresponding (convergence) requirements are ensured by the

Theorem ((OlF) 3.3): let $\sum_{n=0}^{\infty} a_n z^n$ converge when$|z| < r$. Then for fixed k

$$\sum_{n=k}^{\infty} a_n z^n = O(z^k) \quad \text{in any disk } |z| \le \rho \text{ such that } \rho < r.$$

In line with the concepts of the previous section we propose a correspondingly modified circle method, which is about a Fourier analysis of complex-valued (generalized) Fourier series representations on the unit circle, based on a corresponding mathematical tool set enabled by the formula

$$-nf_n(x) = \frac{1}{2}\int_0^1 g(x-t)f_n(t)dt$$

with $f_n(t) = a_n\cos(2\pi nt) + b_n\sin(2\pi nt)$ and $g(x) := sin^{-2}(\pi x)$. In the corresponding Hilbert scale framework $H_\alpha^\#(0,1)$ the mapping formula

$$v = S[u] \text{ with } v(x) := \frac{1}{2}\int_0^1 g(x-t)u(t)dt$$

defines an (Pseudo-Differential) integral convolution operator $S: H_\alpha^\#(0,1) \to H_{\alpha-1}^\#(0,1)$ of order 1, where the Fourier transform of the kernel function is given by $\hat{g}(\omega) = -\omega$. It leads to a variational representation in the form

$$(S[v], w)_{\alpha-1/2} = (v, w)_\alpha, \ \forall v, w \in H_\alpha^\#(0,1).$$

The corresponding convolution kernel of the inverse operator $S_{-1}: H_{\alpha-1}^\#(0,1) \to H_\alpha^\#(0,1)$ is given by $k(x) := -log|2sin(\pi x)|$ ((BrK3), (BrK4) remarks 3.6, 3.9, Notes S38, S48, O23-30).

Regarding orthogonal polynomials on the unit circle, built on a non-negative, integrable (in a Lebesgue-sense) function $f(\theta)$ of period 2π, we mention the case $f(\theta) = \{g(\theta\}^{-1}$ where $g(\theta)$ is a positive trigonometric polynomial of degree m ((SzG) 11.2, (BrK4) Note S49).

The newly proposed elements (e.g. (BrK4) Notes O27/37), is now about a circle method on the unit circle, enabled by the arithmetical function

$$\sigma^*(x) := \frac{1}{2}\left[\sigma_{(odd)}^{(1)}(x) + \sigma_{(even)}^{(2)}(x)\right]$$

with

$$\sigma_{(odd)}^{(1)}(x) := \frac{1}{x}h^{(1)}\left(\frac{1}{x}\right)\sum_{\substack{n\le x \\ n\ odd}} h^{(1)}\left(\frac{x}{2\omega_n}\right) \ , \ \sigma_{(even)}^{2}(x) := \frac{1}{x}h^{(2)}\left(\frac{1}{x}\right)\sum_{\substack{n\le x \\ n\ even}} h^{(2)}\left(\frac{x}{\omega_n+\omega_{n+1}}\right)$$

with

$$h^{(1)}(x) := -(\pi x)cot^{(1)}(\pi x) \ , \ h^{(2)}(x) := -(\pi x)cot^{(2)}(\pi x)$$

and

$$cot^{(1)}(\pi x) := \sum \sin(\pi(2\omega_n)x) \in H_0^\#(0,1) \quad , \quad cot^{(2)}(\pi x) := \sum \sin(\pi(\omega_n + \omega_{n+1})x \in H_0^\#(0,1) \ .$$

For each positive real number x the Snirelmann density is defined for a subset $\boldsymbol{A}$ of the set of positive integers $\boldsymbol{N}$ by

$$0 \leq \sigma(\boldsymbol{A}) := inf_n \frac{A(n)}{n} \leq 1 \quad \text{with} \quad A(x) := \sum_{\substack{a \in \boldsymbol{A} \\ a \leq x}} 1.$$

It holds $A(n) \geq kn$ for $\sigma(\boldsymbol{A}) = k$; $A(1) = 0$ (and therefore $\sigma(\boldsymbol{A}) = 0$), if 1 is not an element of $\boldsymbol{A}$; and $A(n) = n$ (and therefore $\sigma(\boldsymbol{A}) = 1$), if $\boldsymbol{A} = \boldsymbol{N}$.

The Snirelmann-Goldbach theorem states, that the set $\boldsymbol{A} := \{0,1\} \cup \{p + q; p, q \; prim\}$ has positive Snirelmann density. In case of a Snirelmann density ½ the binary Goldbach conjecture would be proven.

For a subset $\boldsymbol{A}$ of the set of integers $\boldsymbol{N}$ integers, if

i) the integer "1" is not an element of $\boldsymbol{A}$, the Snirelmann density of $\boldsymbol{A}$ is $= 0$

ii) if the integer "2" is not an element of $\boldsymbol{A}$, the Snirelmann density of $\boldsymbol{A}$ is $\leq \frac{1}{2}$

iii) if $\boldsymbol{A} = \boldsymbol{N}$, the Snirelmann density of $\boldsymbol{A}$ is $= 1$.

Therefore, the set $\{2n - 1 = [2\omega_n]\}$ of odd integers has Snirelmann density $\leq \frac{1}{2}$, while the set $\{2n = [\omega_n + \omega_{n+1}]\}$ of even integers has Snirelmann density $= 0$.

For the smallest prime $p > n$, it holds $p < 2n$, i.e. $\pi(2n) - \pi(n) \geq 1$.

From section 1 (property ii) we recall the (non-integer Kummer function zeros related) inequality

$$2n < r_n + s_n < 2n + 1 \quad \text{with} \quad r_n := \omega_n + \frac{1}{4} \quad \text{and} \quad s_n := \frac{\omega_n + \omega_{n+1}}{2} + \frac{1}{4}$$

motivating the following definition

$$\boldsymbol{A}^* := \{0,1\} \cup \{r_n + s_n\}$$

and a related kind of „Snirelmann-Stieltjes integral" density value $\sigma^*(\boldsymbol{A}^*) = \frac{1}{2}$.

The „Snirelmann-Stieltjes integral" density concept is related to another method to analyze binary additive problems, which is about the „dispersion method", which is about a „correlation theory of binary problems" (LiJ). Unfortunately, in its current form this method cannot be applied to the binary Goldbach problem ((LiJ), chapter X, §2).

We claim, that the proposed circle method on the unit circle with its underlying related distributional Hilbert space framework $H^{\#}_{-\frac{1}{2}}(0,1)$ (going along with the Stieltjes/Plemelj „differential" potential density concept) provides the appropriate framework for a correspondingly adapted „truly" „dispersion / variance method", to solve the binary Goldbach conjecture. This is about the independence of the (number theoretical) „events" related to the (positive integer resp. real number) sets

$$\boldsymbol{A} := \{0,1\} \cup \{p + q; p, q \; prim\} \quad , \quad \boldsymbol{A}^* := \{0,1\} \cup \{2\omega_n + (\omega_n + \omega_{n+1} - 1)\} \; .$$

1h. The $H_{-1/2}$ Hilbert space and corresponding arithmetic functions

In (ViJ) the Prime Number Theorem (PNT) is proven on a distributional way, applying the Dirac function to derive the first derivative of the Chebyshev function. In line with the proposal of section 4.d) below we propose to replace the Dirac distributions space by the $H_{-1/2}$ Hilbert space, resulting in corresponding representation of concerned arithmetic functions.

This section is about the building of a distributional density function $\theta(x)$ with

$$\theta'(x)=S[\theta](x)=\sum_{n<x}a(n)\in H_{-\frac{1}{2}} \text{ in a weak } H_0 \text{ −sense,}$$

alternatively to the Mangoldt resp. Landau distribution functions

$$\psi(x)=\sum_{n<x}\Lambda(n) \quad \text{resp.} \quad \vartheta(x)=\sum_{n=1}^{\infty}\Lambda(n)\log(\tfrac{x}{n})$$

with

$$\psi'(x)=\sum_{n<x}\Lambda(n)\delta(x-n)\in H_{-\frac{1}{2}-\varepsilon} \text{ resp. } \vartheta'(x)=\frac{1}{x}\sum_{n<x}\Lambda(n).$$

The Delta function representation of $\psi'(x$ is applied in (ViJ) for „a quick distributional way to *(prove)* the prime number theorem" (see also (BrK4) Note S19).

For

$$\theta(x):=\sum_{n<x}\Lambda(n)\cdot\log(\sin(\pi\frac{x}{2\omega_n}))\in H_{1/2}$$

one gets with the abbreviation $p^*(x):=-(\frac{\pi}{2}x)cot(\frac{\pi}{2}x)$

$$\theta'(x)=\frac{1}{x}\sum_{n<x}\Lambda(n)(-\pi\frac{x}{2\omega_n}\cot(\left(\pi\frac{x}{2\omega_n}\right))=\frac{1}{x}\sum_{n<x}\Lambda(n)\cdot(p^*\left(\frac{x}{\omega_n}\right))\in H_{-1/2} \text{ in a strong sense.}$$

For the convergence $\lim_{n\to\to\infty}\theta'(x)$ we refer to the previous sections.

The baseline formulas for the following are the representations

i) $p(x):=(\frac{\pi}{2}x)cot(\frac{\pi}{2}x)=1+\sum_{n=1}^{\infty}\left[\frac{x}{x+n}+\frac{x}{x-n}\right]$

ii) $w(x):=\frac{p(x)}{x}=(\frac{\pi}{2})cot(\frac{\pi}{2}x)=\frac{1}{x}+\sum_{n=1}^{\infty}\left[\frac{1}{x+n}+\frac{1}{x-n}\right]$

iii) $\psi(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[-\frac{\zeta'}{\zeta}(s)\right]x^s\frac{ds}{s}=\sum_{n<x}\Lambda(n)$ and $-\frac{\zeta'}{\zeta}(s)=\int_0^{\infty}x^{-s}d\psi=\sum_{n=2}^{\infty}\Lambda(n)n^{-s}$, $(Re(s)>1)$, (EdH) 3.2

iv) $\vartheta(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[-\frac{\zeta'}{s\cdot\zeta}(s)\right]x^s\frac{ds}{s}=\int_0^x\psi(t)dlog(t)=\sum_{n=1}^{\infty}\Lambda(n)\log(\frac{x}{n})$ and $-\frac{\zeta'}{s\cdot\zeta}(s)=\int_0^{\infty}x^{-s}d\vartheta$, $(Re(s)>0)$

The Euler conjecture, i.e. the convergence of the series

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}-\cdots=0$$

can be derived from the PNT ((LaE) §159). This series is linked to the zeta function by the identities

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \quad , \quad -\frac{\zeta'(s)}{s}=\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{n^s}$$

i.e.

$$-\frac{\zeta'(s)}{s\zeta(s)}=\left[\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right]\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{n^s}\right] \quad , \quad \frac{\zeta'(s)}{\zeta(s)}=\left[\sum_{n=1}^{\infty}\mu(n)\frac{1}{n^s}\right]\cdot\left[\sum_{n=1}^{\infty}\log\left(\frac{1}{n}\right)\frac{1}{n^s}\right].$$

What cannot derived from the PNT is the convergence of the series

$$(*) \quad \sum_{n=1}^{\infty}\frac{\mu(n)}{n}log(\frac{1}{n})=1 \quad .$$

„The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived" ((LaE) §160).

The Landau statement above corresponds to the proposed replacement of the Dirac distribution by appropriate distributions of the Hilbert space $H_{-\frac{1}{2}}$, going along with the following identities ((ApT), (BrK4), (LaE) §227), while at the same time enabling the Bagchi RH criterion (BaB),

$$1=\sum_{n=1}^{\infty}\frac{1}{n}\mu(n)\,log(\frac{1}{n})=\sum_{1}^{\infty}\frac{1}{n}a_n b_n=:\big((u,v)\big)_{-\frac{1}{2}}:=\lim_{\omega\to\infty}\frac{1}{2\omega}\int_{-\omega}^{\omega}u\left(\frac{1}{2}+it\right)v\left(\frac{1}{2}-it\right)dt$$

with ($s=\frac{1}{2}+it$)

$$u\left(\frac{1}{2}+it\right):=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\in H_{-\frac{1}{2}}\quad,\quad v\left(\frac{1}{2}-it\right):=\sum_{n=1}^{\infty}\frac{log(1/n)}{n^s}\in H_{-\frac{1}{2}}.$$

Let

$$z\left(\frac{1}{2}+it\right):=\sum_{n=1}^{\infty}\frac{1}{n^s}\in H_{-1}\quad,\quad w\left(\frac{1}{2}-it\right):=\sum_{n=1}^{\infty}\log\left(\frac{1}{n}\right)\frac{\mu(n)}{n^s}$$

$$r\left(\frac{1}{2}-it\right):=\sum_{n=1}^{\infty}\frac{\varphi(n)}{n}\frac{1}{n^s}\quad,\quad s\left(\frac{1}{2}-it\right):=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{nlogn}\frac{1}{n^s}\ .$$

Then it holds $((z,w))_{-\frac{1}{2}}=1$. Because of $\sum_{n=1}^{\infty}\frac{\varphi(n)}{n^2}<\infty$ and $z\in H_{-1}$, it follows that

$$\big((z,r)\big)_{-\frac{1}{2}}=\sum_{n=1}^{\infty}\frac{\varphi(n)}{n^2}<\infty\quad,\quad\big((w,s)\big)_{-\frac{1}{2}}\sim\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\frac{\mu(n)}{n}<\infty$$

and therefore $r\in H_0$, $w\in H_0$, $s\in H_{-1}$. For

$$\sigma(x):=\sum_{n\le x}\frac{\mu(n)}{n}\log(\frac{x}{n})$$

it holds

$$\sigma(xy)+1=\sigma(x)+\sigma(y)\quad,\quad\sigma'(x)=\frac{1}{x}\sum_{n\le x}\frac{\mu(n)}{n}$$

and the inverse mapping is given by

$$\sigma^{-1}(x)=\sum_{n\le x}\frac{1}{n}log(\frac{x}{n})\ ,\ x\ge 1\ .$$

The asymtotics of a related arithmetical function is given by ((ApT) 3.12)

$$\sum_{n\le x}\frac{1}{n}\frac{\varphi(n)}{n}+\sum_{n=1}^{\infty}\frac{1}{n}\frac{\mu(n)\,log\,n}{n}=\frac{6}{\pi^2}(log\,x+\gamma)++O(\frac{log\,x}{x}).$$

For the relationship to the alternative Zeta function theory below we note that the function $\pi\,cot(\pi x)$ is holomorphic except the pole $z=1$ and it holds

$$\frac{1}{2}\cot(\frac{x}{2})=\frac{d}{dx}\ln(2\sin(\frac{x}{2})=-\frac{d}{dx}\sum_{n=}^{\infty}\frac{\cos(nx)}{n}=\sum_{n=}^{\infty}\sin(nx)\in H_{-1}^{\#}(0,2\pi)$$

From the identity ($s=\frac{1}{2}+ix$)

$$\sum_{n\ge x}\frac{1}{n^s}=\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}z^{1-s}(-\pi\,cot(\pi z)\frac{dz}{z}=:g(s)$$

it follows $\big((z,g)\big)_{-1}<\infty$, i.e. $z,g\in H_{-1}$ resp. $\big((z,\omega)\big)_{-1},\big((g,\omega)\big)_{-1},\forall\omega\in H_{-1}$. The latter representation enables an orthogonal projections $z^{(*)},g^{(*)}\in H_{-1/2}$ in the form $\big((z^{(*)},\omega)\big)_{-1},\big((g^{(*)},\omega)\big)_{-1},\forall\omega\in H_{-1/2}$.

1i. Appendix: Formulas and properties

The proposed alternative "baseline" function is the Hilbert transform of the Gaussian function, which is the Dawson function ((OlF) p. 44)

$$F(x) := e^{-x^2}\int_0^x e^{-t^2}dt$$

Its relationship to the *„analysis of zeros of certain trigonometric integrals and entire higher genus-1 functions*" (PoG2), is given by the identities ((GrI) 3.896)

$$F(x) := e^{-x^2}\int_0^x e^{-t^2}dt = \int_0^\infty e^{-t^2}\sin(2xt)dt = x\ {}_1K_1\left(1,\tfrac{3}{2};-x^2\right) = e^{-x^2}H(x)$$

with $H(x) := x\ {}_1K_1\left(\frac{1}{2},\frac{3}{2};x^2\right)$. We note that the Hilbert transform of the $sin(ax)$ –function is given by $-cos(ax)$ –function and $\int_0^\infty e^{-t^2}\cos(2xt)dt = \frac{\sqrt{\pi}}{2}e^{-x^2}$ ((GrI) 3.896). From (GrI) 7.612) we mention the reciprocal formula

$$\int_0^\infty e^{-\frac{x^2}{2}}\ {}_1F_1\left(\tfrac{1}{2},1,\tfrac{x^2}{2}\right)sin(yx)dx = \frac{\sqrt{\pi}}{2}e^{-\frac{x^2}{2}}\ {}_1F_1\left(\tfrac{1}{2},1,\tfrac{x^2}{2}\right).$$

The Kummer function related Mellin transforms can be derived from the following formulas ((GrI) 7.612):

i) $\int_0^\infty x^s\ {}_1F_1(a,c,-x)\frac{dx}{x} = \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(c-s)}\frac{\Gamma(c)}{\Gamma(a)}$ $\qquad 0 < Re(s) < Re(a)$

ii) $\int_0^\infty x^s\ {}_1F_1(a,a+1,-x)\frac{dx}{x} = \frac{a}{(a-s)}\Gamma(s)$ $\qquad 0 < Re(s) < Re(a)$

iii) $\int_0^\infty x^{s-1/2}\ {}_1F_1\left(a,a+\frac{1}{2},-x\right)\frac{dx}{x} = \frac{\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)}\Gamma(s-\frac{1}{2})$ $\qquad \frac{1}{2} < Re(s) < Re(a+\frac{1}{2})$

iv) $\int_0^\infty x^{\frac{s}{2}}\ {}_1F_1(\frac{1}{2},\frac{3}{2},-x)\frac{dx}{x} = \frac{\Gamma(\frac{s}{2})}{1-s}$ $\qquad 0 < Re(s) < Re(1)$

v) $h(x) + 2xh'(x) = e^{-x}$ with $h(x) :=\ {}_1F_1(\frac{1}{2},\frac{3}{2},-x)$

vi) $M[-xh']\left(\frac{s}{2}\right) = \frac{s}{2}M[h]\left(\frac{s}{2}\right) = \frac{\Gamma(1+\frac{s}{2})}{1-s}$ $\qquad 0 < Re(s) < Re(1)$

vii) $M_{\frac{1}{4},\frac{1}{4}}(z) = z^{\frac{3}{4}}e^{-z/2}\ {}_1F_1(\frac{1}{2},\frac{3}{2},z)$, $e^{-\frac{z}{2}}\ {}_1F_1\left(\frac{1}{2},\frac{3}{2},z\right) = z^{-\frac{3}{4}}M_{\frac{1}{4},\frac{1}{4}}(z) = (-z)^{-\frac{3}{4}}M_{-\frac{1}{4},\frac{1}{4}}(-z)$

for the Whittaker functions $M_{\frac{1}{4},\frac{1}{4}}(z)$ ((GrI) 9.220, 9.231)

viii) For $-1/4 < Re(\vartheta) < 1/4$, $0 < 1/2 - 2\vartheta < 1$, $1/2 < 1 + 2\vartheta < 3/2$ it holds ((GrI) 7.612):

$$\int_0^\infty G_{2\vartheta}(x)sin(yx)dx = \frac{\sqrt{\pi}}{2}G_{2\vartheta}(x) \quad \text{with } G_{2\vartheta}(x) := x^{4\vartheta}e^{-\frac{x^2}{2}}\ {}_1F_1\left(\tfrac{1}{2}-2\vartheta,1+2\vartheta,\tfrac{x^2}{2}\right)$$

resp.

$$\int_0^\infty e^{-\frac{x^2}{2}}\ {}_1F_1\left(\tfrac{1}{2},1,\tfrac{x^2}{2}\right)sin(yx)dx = \frac{\sqrt{\pi}}{2}e^{-\frac{x^2}{2}}\ {}_1F_1\left(\tfrac{1}{2},1,\tfrac{x^2}{2}\right) \quad (\vartheta = 0)$$

ix) The asymptotics of the Kummer functions are given by ((OlF), 7 §10.1, (AbM) 13.5.1.)

$${}_1F_1(a,c;x) \sim \frac{1}{\Gamma(a)}\frac{e^x}{x^{c-a}} \qquad \text{resp.} \qquad {}_1F_1(a,a+1;x) \sim \frac{1}{\Gamma(a)}\frac{e^x}{x}\ ,\quad x \to \infty$$

x) The simple zeros of ${}_1F_1\left(1,\frac{3}{2};z\right)$ lie in the half-plane $Re(z) < -1/2$. The simple zeros of ${}_1F_1\left(\frac{3}{2},\frac{3}{2};z\right)$ lie in the half-plane $Re(z) > 1/2$. All zeros z_n of the Kummer function $K_a(z) :=\ {}_1K_1(a,a+1;z)$ $(0 < a \leq 1)$ are simple and satisfy the asymptotic formula (SeA)

$$z_n = 2\pi in + \left[(1-a) + \frac{(1-a)^2}{2\pi in}\right]\cdot\left[\log(2\pi|n|) + \frac{log\Gamma(a)}{1-a} \pm \left(-\frac{\pi}{2}\right)\right] - \frac{3a+1}{2\pi in} + O\left(\frac{log|n|}{n^2}\right),\quad n \to \pm\infty,$$

The zeros all lie in the horizontal stripe

$$(2n-1)\pi < |Im(z_n)| := |2\pi\omega_n| < 2n\pi \quad .$$

((SeA), see also ((BrK4) lemma A3, Notes O5-17, O22, O23, (BrK7) Note 11)).

For the Fourier inverse of the Zeta function on the critical line (in a distributional sense) we refer to (BrK4) Notes S21, S24. The related (classical) Zeta approximation series reprensentation is provided in (TiE) 4.14 resp. (BrK4) Notes S51, O9, O27. In (OlF1) 25.6.6, an integral value representation for $\zeta(2n+1)$ is provided with $\cot(\pi x)$ "density" function.

We note the corresponding Gamma function equivalent in the form

$$log'\Gamma(1-x) - log'\Gamma(x) = \pi\cot(\pi x) = \frac{1}{x}[1 - 2\sum_{k=0}^{\infty} s_{2k}\, x^{2k}] \quad \text{(NiN) §14 (5), §19 (16).}$$

The alternative „Gamma" function $\Gamma^*\left(\frac{s}{2}\right)$ fulfills the following properities

i) $\Gamma^*\left(\frac{s}{2}\right)\Gamma^*\left(\frac{1-s}{2}\right) = \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)$, $\Gamma'(1) = -\gamma$, $\Gamma^{*\prime}(1) = \pi$ ($\tan(\pi) = 0$ and $tan'(\pi) = \pi$)

ii) $\Gamma\left(1+\frac{s}{2}\right) \quad \rightarrow \quad \Gamma^*\left(\frac{s}{2}\right) := \Gamma\left(\frac{s}{2}\right) tan\left(\frac{\pi}{2}s\right) = \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{\Gamma\left(1+\frac{s-1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}$

iii) $log'\Gamma^*(x) = log'\Gamma(x) + log'\tan\left(\frac{\pi}{2}x\right) = -\gamma + \frac{\pi}{\sin(\pi x)}$

iv) $log'\Gamma^*\left(\frac{s}{2}\right) = log'\Gamma\left(\frac{s}{2}\right) + \frac{1}{2}\frac{1}{\sin(\pi s)} = log'\Gamma\left(\frac{s}{2}\right) + \frac{1}{2}\Gamma(s)\Gamma(1-s)$, $log'\Gamma\left(\frac{3}{4}\right) - log'\Gamma\left(\frac{1}{4}\right) = \pi$

v) $\frac{\pi}{2}\Gamma^*\left(\frac{s}{2}\right) = \sum_{k=1}^{\infty}\frac{\Gamma\left(1+\frac{s}{2}\right)}{(k-\frac{1}{2})^2-(\frac{s}{2})^2}$ (because of $tan\frac{\pi}{2}x = \frac{4x}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2-x^2}$).

The alternative „Gamma" function properties and some related lemmata in section 2 below in combination with the „value" property of Kummer resp. hypergeometric functions ((BeI), (WoJ)), might also enable a new tool to prove the irrationality or even the transcendence of e.g. the Euler constant γ. The approaches in ((BeI), (WoJ)) are both based on classical E-function theory, whereby the considered Kummer functions in (BeI) are explicitly excluding the considered Kummer function of this paper. The concept in (PoG4) (in combination with the conjecture in (PoG3) about power series with rational or integer coefficients in the context of a convergence radius one) might provide an alternative E-or G-function approach.

With respect to the alternative $\zeta^*(s)$ function

$$\zeta^*(1-s) = \frac{\tan(\frac{\pi}{2}s)}{s}\cdot\zeta(1-s) \quad \text{resp.} \quad \zeta^*(s) = \frac{\cot(\frac{\pi}{2}s)}{1-s}\cdot\zeta(s) = \frac{\tan(\frac{\pi}{(2}(1-s))}{1-s}\cdot\zeta(s)$$

resp.

$$log\zeta^*(s) = \log(\tan\frac{\pi}{2}(1-s)) + log\left(\frac{1}{1-s}\right) + log\zeta(s)$$

We onte the following properties

i) $\frac{1}{1-s} = \int_1^{\infty} x^{-s}dx = 1 - s\int_1^{\infty} x^{-s-1}xdx$

ii) $\frac{\pi}{2}\tan\left(\frac{\pi}{2}s\right) = \frac{1}{1-x} + O(|1-x|)$ in the neighborhood of $x = 1$

iii) $\log(\tan\left(\frac{\pi}{2}s\right)) = -4\sum h_n\frac{\sin(2\pi ns)}{\pi n}$

iv) $log'\Gamma\left((1-\theta) = \frac{3}{4}\right) - log'\Gamma\left(\theta = \frac{1}{4}\right) = \pi$

v) $log'(\tan\left(\frac{\pi}{2}s\right) = -log'(\cot\left(\frac{\pi}{2}s\right) = \frac{\pi}{\sin(\pi s)} = \frac{1}{x}[+2\sum_{k=0}^{\infty}\sigma_{2k}\,x^{2k}] = \beta(x) + \beta(1-x)$

with $\beta(x) := \sum_{k=0}^{\infty}(-1)^k\frac{1}{x+k} = \sum_{k=0}^{\infty}(-1^k)\sigma_{k+1}\,x^k$ (NiN) §14 (6), §19 (17).

Applying Riemann's building concept for the auxiliary function, defining a self-adjoint operator with Mellin tranform $\xi(s)$ ((EdH) 10.3), results into a replacement of

$$log(\sin x) = logx - \frac{1}{6}x^2 - \frac{1}{180}x^4 - \frac{1}{2835}x^6 - \cdots \quad \rightarrow \quad \log(tan\left(\frac{\pi}{2}x\right)) = logx + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 + \cdots \ldots$$

The density of prime numbers appears to be the Gaussian density $dg = \log\left(\frac{1}{t}\right)dt$ defining the corresponding prime number counting integral function ((EdH) 1.1 (3)). The Clausen density dw, based on the Clausen integral ((AbM) 27.8)

$$w(t) = \int_0^t \log\left(2sin\frac{t}{2}\right)dt \; , \quad 0 \le t \le \pi$$

is related to the Hilbert transform of the fractional part function ((BrK4) Note O28).

For $T(x) \coloneqq \log(tan(\frac{\pi}{2}x))$, we summaries a few properties

i) $\frac{\pi}{2}T(x) = -\sum \frac{2h_n}{n}\sin(2\pi nx) \in L_2^{\#}(0,1)$ (ElL)

with $2h_n = \sum_{k=1}^{n}\frac{2}{2k-1} = 2H_{2n} - H_n$ and $H_n = \sum_{k=1}^{n}\frac{1}{k}$ (harmonic numbers) and

$$\int_0^1 T(x)\, cos(k\pi x)\, dx = \begin{cases} -1/k & k\ odd \\ 0 & k\ even \end{cases}.$$

ii) the $log(tan\, x)$ –integral evaluated by series involving $\zeta(2n+1)$ is provided (ElL1)

iii) for the Hilbert transform evaluation of $T(x)$ we refer to (MaJ)

iv) from (GrI), 1.421,1.518, we recall the series representations

$$\mathrm{T(x)} = logx + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2^{2k-1}-1)2^{2k}B_{2k}x^{2k}}{k(2k)!} \quad , x^2 < (\frac{\pi}{2})^2$$

$$T(x) = logx + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 + \cdots \ldots.$$

$$log(\sin x) = logx - \frac{1}{6}x^2 - \frac{1}{180}x^4 - \frac{1}{2835}x^6 - \cdots ..$$

v) For the related Fourier expansion of the $log\Gamma(x)$ function we refer to (EsO) with coefficients $a_n = \frac{1}{2n}$, $b_n = \frac{A+logn}{2cn}$ and $a_0 = log\sqrt{2\pi}$.

vi) The counterpart of the asymptotics $log(tan\frac{\pi}{2}x) \sim \log(\sin(x) \sim logx$ with respect to the cot –function is given by the estimate

$$\left|\frac{\pi}{2}\cot\left(\frac{\pi}{2}x\right) - \frac{1}{x}\right| \le \frac{\pi^2}{6} = \zeta(2) \;\; (|x| \le 1),$$

which is a result of the following inequalities

$$\left|\frac{1}{2}cot\left(\frac{x}{2}\right) - \frac{1}{x}\right| = \left|\sum_{k=1}^{\infty}\frac{1}{x+2\pi k} + \frac{1}{x-2\pi k}\right| \le \left|\sum_{-\infty}^{\infty}\frac{x}{\pi(x-2\pi k)}\right| \le \frac{\pi}{2\pi^2}\sum_{-\infty}^{\infty}\left|\frac{1}{1-\frac{x}{2k\pi}}\right|\frac{1}{k^2} \le \frac{\pi}{6} \quad , \; |x| \le \pi$$

With respect to the distributional Fourier series representation of the $\cot(\pi x)$ function we note the product representation ((GrI), 1.392)

$$sin(nx) = \begin{cases} n \cdot sin(x) \cdot \prod_{k=1}^{(n-1)/2}(1 - \dfrac{sin^2(x)}{sin^2(\pi\frac{k}{n})} & n\ odd \\ \dfrac{n}{2} \cdot sin(2x) \cdot \prod_{k=1}^{(n-2)/2}(1 - \dfrac{sin^2(x)}{sin^2(\pi\frac{k}{n})} & n\, even \end{cases}$$

$$-log\left|2\cos\left(\frac{x}{2}\right)\right| = -\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{n} \qquad -log\left|2\sin\left(\frac{x}{2}\right)\right| = -\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}$$

$$-log\left|2\sin\left(\frac{x}{2}\right)\right| - log\left|2\cos\left(\frac{x}{2}\right)\right| = -log|2\sin(x)| = -\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}.$$

In (ChK) VI, §2, two expansions of $\cot(z)$ are compared to prove that all coefficients of one of this expansion $\frac{\zeta(2n)}{\pi}e^{-2n}$ are rational. Corresponding formulas for odd inters are unknown.

In (EsR), 3.8 (example 78), resp. (BrK4) Notes S51, a "finite part"-"principle value" integral representation of the $\frac{\pi}{2}\cot\left(\frac{\pi}{2}x\right)$ – is given (which is zero also for positive or negative integers)

$$F.p.(P.v.\int_0^\infty t^x \frac{t}{t^2-1}dt) = \begin{cases} 0 & for \quad x\in 2Z \\ \left(\frac{\pi}{2}\right)cot\left(\frac{\pi}{2}x\right) & otherwise \end{cases}.$$

It is used as enabler to obtain the asymptotic expansion of the p.v. integral, defined by the "restricted" Hilbert transform integral of a function $u(x)$ over the positive x-axis, only. In case $u(x)$ has a structure $u(x) = v(x)\sqrt{x}$ the representation enjoys a remarkable form, where the numbers $n + 1/2$ play a key role.

In the context of Landau's „generalized number theoretical function theorem" we note the following properties: for

$$g(x) := \frac{1}{sin^2(\pi x)} \quad , \quad p(x) := (\frac{\pi}{2}x)cot(\frac{\pi}{2}x) = 1 + \sum_{n=1}^\infty \left[\frac{x}{x+n} + \frac{x}{x-n}\right] \quad , \quad h(x) := x \cdot p(x) \ ,$$

it holds

i) $\cot(x) = \frac{1}{2}cot\left(\frac{x}{2}\right) - \frac{1}{2}cot(\frac{\pi-x}{2})$, $-\frac{1}{2}x\frac{g'(x)}{g(x)} = P(x) = -\pi\, cot(\pi x) \underset{x\to 0}{\to} 1$

ii) $\sin\left(\pi n + \frac{1}{n}\right) = O(\frac{1}{n})$, $\sin\left(O(\frac{1}{x})\right) = O(\frac{1}{x})$ $\quad \cos\left(O(\frac{1}{x})\right) = O(1)$ (OlF) 3.1

iii) $\int_0^1 h(x)dx = \frac{1}{2} - \sum_{k=1}^\infty \frac{\zeta(2k)}{2^{2k}(2k+1)}$, $\int_0^1 p(x)\,dx = \int_0^1 h(x)\frac{dx}{x} = \int_1^\infty h\left(\frac{1}{x}\right)\frac{dx}{x} = log2$ (GrI) 3.747

iv) $\int_0^1 h(x)\frac{dx}{x} + \int_1^\infty h\left(\frac{1}{x}\right)\frac{dx}{x} = 2log2$ and $h(x) \in L_1(0,1)$ (GrI) 3.748

resp. $\frac{1}{log2}\int_0^1 x(\frac{\pi}{2}x)cot(\frac{\pi}{2}x)\frac{dx}{x} = \frac{1}{log2}\int_1^\infty \frac{1}{x}(\frac{\pi}{2}\frac{1}{x})cot(\frac{\pi}{2}\frac{1}{x})\frac{dx}{x} = 1$, $\int_0^1 x\left[(\frac{\pi}{2}x)cot(\frac{\pi}{2}x)\right]dx = \frac{1}{2} - \sum_{k=1}^\infty \frac{\zeta(2k)}{2^{2k}(2k+1)}$.

From (ZyA) V.2, we recall for $0 < \beta < 1$ and $0 < x \le \pi$ the estimates

i) $\left|\sum_1^N \frac{\cos(nx)}{n^\beta}\right| \le c\frac{x^\beta}{x}$

ii) $\left|\sum_1^N \frac{\sin(nx)}{n^\beta}\right| \le c\frac{x^\beta}{x}$

iii) $\left|\sum_1^N \frac{\cos(nx)}{n}\right| \le \log\left(\frac{1}{x}\right) + C.$

The series $\sum_2^\infty \frac{\cos(nx)}{logn}$ is divergent, ist conjugate $\sum_2^\infty \frac{\sin(nx)}{logn}$ is not a Fourier series, and ((ZyA) V.1)

i) $\sum_2^\infty \frac{\cos(nx)}{logn} \sim \left[\frac{\pi}{2}log^{-1}(\frac{1}{x})\right]\frac{1}{x}log^{-1}(\frac{1}{x}) \qquad x \to +0$

ii) $\sum_2^\infty \frac{\sin(nx)}{logn} \sim \frac{1}{x}log^{-1}(\frac{1}{x}) \qquad x \to +0.$

Because of $\int_0^1 f(x)sin\,(\pi nx)dx \to 0$, $n \to \infty \quad \forall f \in L_2(0,1)$, the sequence $\{sin(\pi nx)\}$ converges weakly to zero, but not strongly, as $\|sin(\pi nx)\| = \frac{1}{\sqrt{2}}$. The same is true for

$$f(x) = \begin{cases} 1, & n < x < n+1 \\ 0, & otherwise \end{cases}$$

2. The $H_{-\frac{1}{2}} = H_0 \otimes H_0^{\perp}$ decomposition for a quantum space-time model

The proposed Hilbert space framework (with ist relationship to the Zeta function theory and the Hilbert-Polya/ Berry-Keating conjecture) enables a combined usage of spectral theory, variational methods for non-linear operators (VaM), Galerkin-Ritz approximation theory (VeW), and tools like Pseudo-Differential operators ((EsG), (LoA), (PeB)), degenerated hypergeometric functions (GrI), Hilbert (resp. Riesz) transform(s) and wavelets (HoM). The link between PDO and the Galerkin-Ritz approximation theory is given by the Garding inequality and the concept of hypoellipticity ((AzA), (GaL), (PeB)). The norms of the Hilbert scale H_α can be enriched with an additional norm enjoying an "exponential decay" behavior. Each Hilbert space norm with a<0 is governed by the norm of the Hilbert space H_0 and this "exp-decay" norm. This property is proposed to be applied in the context of the decomposition of the Hilbert space $H_{-1/2} = H_0 \otimes H_0^{\perp}$ ((BrK), (BrK1), (BrK3), (BrK7)).

We briefly sketch the key elements of the Hilbert space/scale theory with relationship to corresponding approximation theory ((NiJ), (NiJ1)), and wavelet theory, (FaM), (HoM).

If there exists an operator A with $D(A) = H_1$, $R(A) = H_0$ and $\|x\|_1 = \|Ax\|_0$, whereby the operator A is positive definite, self-adjoint and A^{-1} is compact.

Then the corresponding eigenvalue problem $A\varphi_i = \sigma_i \varphi_i$ has infinite solutions $\{\sigma_i, \varphi_i\}$ with $\sigma_i \to \infty$ and $(\varphi_i, \varphi_k) = \delta_{i,k}$, and for each element $x \in H_1 = A^{-1}H_0$ it holds the representation

$$x = \sum_{i=1}^{\infty} (x, \varphi_i)\, \varphi_i$$

Inner products with corresponding norms of a distributional Hilbert scale can be defined based on the eigen-pairs of an appropriately defined operator in the form

$$(x, y)_\alpha := \sum_i^\infty \lambda_i^\alpha (x, \varphi_i)(y, \varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

Additionally, for $t > 0$ there can be an inner product resp. norm defined for an additional governing Hilbert space with an "exponential decay" behavior in the form $e^{-\sqrt{\lambda_i}t}$ given by

$$(x, y)_{(t)}^2 := \sum_{i=1} e^{-\sqrt{\lambda_i}t} (x, \varphi_i)(y, \varphi_i) \quad , \quad \|x\|_{(t)}^2 := (x, x)_{(t)}^2$$

The distributional $H_{-\frac{1}{2}}$ – Hilbert space is proposed to model quantum states, alternatively to the Hilbert space H_0. A mathematical (wavelet microscopic) analysis of those states is then about an analysis of the "objects"

$$x = x_0 + x_0^{\perp} \in H_0 \otimes H_0^{\perp}$$

with

$$\|x_0\|_0 = 1 \qquad \sigma := \|x_0^{\perp}\|_{-1/2}^2$$

As it holds for any $t, \delta, \alpha > 0$ and $\lambda \geq 1$ the inequality

$$\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{t(\delta^{-1} - \sqrt{\lambda})}$$

the following inequality is valid for any $\in H_0$, governing the approximation "quality" of a quantum state with respect to the norm of H_0:

$$\|x\|_{-1/2}^2 \leq \delta \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2 = \sigma \|x\|_0^2 + e \|x\|_{(\sigma)}^2 = \sigma \|x\|_0^2 + \sum_{i=1}^{\infty} e^{1 - \sqrt{\lambda_i}\sigma} x_i^2$$

It balances the "continuous" view of the overall state with its "discrete" components.

For a related approximation theory we refer to (BrK8), (NiJ), NiJ1).

The decomposition of the quantum state space $H_{-1/2} = H_0 \otimes H_0^{\perp}$ resp. the quantum energy space $H_{1/2} = H_1 \otimes H_1^{\perp}$ goes along with the Fourier wave resp. the Calderón wavelet tool. The (wavelet) admissibility condition is strongly related tot he Hilbert space $H_{\frac{1}{2}}$. The admissibility condition ensures the validity of the inverse wavelet transform which then is valid for all Hilbert scale values.

A L_2 − based Fourier wave analysis is the baseline for statistical analysis, as well as for PDE and PDO theory. The Fourier transform of a wavelet transformed function f is given by:

$$\widehat{W_\vartheta[f]}(a,\omega) := (2\pi|a|)^{\frac{1}{2}} c_\vartheta^{-\frac{1}{2}} \hat{\vartheta}(-a\omega)\hat{f}(\omega) \quad .$$

For $\varphi, \vartheta \in L_2$®, $f_1, f_2 \in L_2(R)$,

$$0 < |c_{\vartheta\varphi}| := 2\pi \left| \int_R \frac{\hat{\vartheta}(\omega)\bar{\hat{\varphi}}(\omega)}{|\omega|} d\omega \right| < \infty$$

and $|c_{\vartheta\varphi}| \le c_\vartheta c_\varphi$ one gets the duality relationship $(W_\vartheta f_1, W_\varphi^* f_2)_{L_2\left(R^2, \frac{dadb}{a^2}\right)} = c_{\vartheta\varphi}(f_1, f_2)_{L_2}$ i.e.

$$W_\varphi^* W_\vartheta [f] = c_{\vartheta\varphi} f \quad \text{in a } L_2 \text{−sense.}$$

For $\varphi, \vartheta \in L_2$®, $f_1, f_2 \in L_2(R)$,

$$0 < |c_{\vartheta\varphi}| := 2\pi \left| \int_R \frac{\hat{\vartheta}(\omega)\bar{\hat{\varphi}}(\omega)}{|\omega|} d\omega \right| < \infty$$

and $|c_{\vartheta\varphi}| \le c_\vartheta c_\varphi$ one gets the duality relationship $(W_\vartheta f_1, W_\varphi^* f_2)_{L_2\left(R^2, \frac{dadb}{a^2}\right)} = c_{\vartheta\varphi}(f_1, f_2)_{L_2}$ i.e.

$$W_\varphi^* W_\vartheta [f] = c_{\vartheta\varphi} f \quad \text{in a } L_2 \text{−sense.}$$

This identity provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets in a (distributional) Hilbert scale framework where the "*microscope observations*" of <u>*two*</u> wavelet (optics) functions ϑ, φ can be compared with each other by the above "reproducing" ("duality") formula. The prize to be paid is about additional efforts, when re-building the reconstruction wavelet. We further note that for a convenient choice of the two wavelet functions the Gibbs phenomenon disappears.

We note the Gaussian function related "Mexican hat" (wavelet) function

$$g(x) := -\frac{d^2}{dx^2}(e^{-\frac{x^2}{2}}) = (1-x^2)e^{-\frac{x^2}{2}}.$$

has being successfully applied e.g. in wavelet theory. We further mention that the Hilbert transform of a wavelet is again a wavelet.

The decomposition of the quantum state space $H_0 \otimes H_0^{\perp}$ resp. the quantum energy space $H_1 \otimes H_1^{\perp}$ is very much related to the „hidden variables in quantum theory" concept of D. Bohm (BoD) with the notions of implicate and explicate order:

((BoD), A2): „*It is important to emphasize, however, that mathematics and physics are not being regarded here as separate but mutually related structures (so that, for example, one could be said to apply mathematics to physics as paint is applied to wood). Rather, it is being suggested that mathematics and physics are to be considered as aspects of a single undivided whole*".

The implicate or enfolded order is about „a new notion of order, that may be appropriate to a universe of unbroken wholeness. In the enfolded order, space and time are no longer the dominant factors determining the relationships of dependence or independence of different elements."

((BoD), A3): „*Implicate order is generally to be described not in terms of simple geometric transformations, such as translations, rotations, and dilations, but rather in terms of a different kind of operations*. … *What happens in the broader context of implicate order we shall call a metamorphosis. … An example of such a metamorphosis metamorphosis M is determed by the Green's function relating amplitudes at the illuminated structure to those at the photographic plate*".

In our case this related to the closed sub-spaces $H_0^{\perp}$ and $H_1^{\perp}$.

Rather, an entirely different sort of basic connection of elements is possible, from which our ordinary notions of space and time, along with those of separately existent material particles, are abstracted as forms derived from the deeper order. These ordinary notions in fact appear in what is called the explicate or unfolded order, which is a special and distinguished form contained within the general totality of all the implicate orders… Explicate order arises primarily as a certain aspect of sense perception and of experience with the content of such sense perception. It may be added that, in physics, explicate order generally reveals itself in the sensibly observable results of functioning of an instrument. … „*What is common to the functioning of instruments generally used in physical research is that the sensibly perceptible content is ultimately describable in terms of a Euclidean system of order and measure, i.e., one that can adequately be understood in terms of ordinary Euclidean geometry. … The general transformations are considered to be the essential determining features of a geometryin a Euclidean space of three dimensions; those are displacement operators, rotation operators and the dilatation operator* ((BoD) A.2).

In our case this is related to the Hilbert spaces H_0 and H_1. The Euclidean systems of order and measure are strongly related to the Archimedian principle.

„*Of course, in the quantum theory, the algebraic terms are interpreted as standing for 'physical observables' to which they correspond. However, in the approach that is being suggested here, such terms are not to be regarded as standing for anything in particular. … This means, of course, that we do not regard terms like 'particle', 'charge', 'mass', 'position', 'momentum', etc., as having primary relevance in the algebraic language. Rather, at best, they will have to come out as high-level abstractions.*" ((BoD) A4).

2a. The Berry-Keating conjecture and the $H_{-1/2}$ quantum state space

The Berry-Keating conjecture puts the zeros of the Zeta function (on the critical line, if the RH is true) in relationship to the (energy level) eigenvalues associated with the classical Hermitian operator $\mathrm{H}(\mathrm{x}, \mathrm{p} = \mathrm{x} \cdot \mathrm{p} \sim \mathrm{x} \cdot \frac{d}{dx}$, where x denotes the position coordinate and p the conjugate momentum. The Friedrichs extension of the variational representation of the Zeta function (on the critical) with L_2-test space indicated a $\mathrm{H}_{-1/2}$ quantum state space with related $\mathrm{H}_{1/2}$ energy space. The today's standard quantum state resp. energy spaces are $\mathrm{H}_0 = \mathrm{L}_2$ resp. H_1, i.e. those Hilbert spaces are compactly embedded subspaces of the proposed new ones.

Applying the physical quantum (fluid) Hilbert (state) space $\mathrm{H}_{-1/2}$ to the 3-D non-linear, non-stationary NSE enables a well posed variational representation of the NSE with appropriate valid energy inequality, closing the Serrin gap problem. The correspondingly variational representation of the Maxwell equations enables a quantum field model (modified YME), enabling a differentiation of "elementary particles" with and w/o mass (modelled by the orthogonal decomposition of the Hilbert spaces $\mathrm{H}_{-1/2} = H_0 \otimes H_0^{\perp}$ resp. $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp}$. It enables the concept of orthogonal projection, which can be interpreted as "mass generation process during observation". Purely energy interaction of the "EP" are "acting" in the orthogonal space (which might be interpreted as zero point energy pool, "wave package resp. eigen-differential space). The macroscopic and microscopic state of quanta relate to corresponding frequencies of its vibrations. The corresponding action variables of the system ((HeW) II.1.c) define the related kinematical (physical) and thermodynamical concept of "*time*", as described in ((RoC), (SmL)), (RoC1), section 13): "*Our interaction with the world is partial, which is why we see it in blurred way. To this blurring is added quantum indeterminacy. The ignorance that follows from this determines the existence of a particular variable - thermal time - and of an entropy that quantifies our uncertainty. Perhaps we belong to a particular subset of the world that interacts with the rest of it in such a way that this entropy is lower in one direction of our thermal time.*"

2b. The $H_{-1/2}$ Hilbert space and a new ground quantum state $H_0^{\perp}$ & ground quantum energy $H_1^{\perp}$ model

The Schrödinger (differentiation) operator is not bounded with respect to the norm of L_2, i.e. only on a dense subspace of L_2 a corresponding spectral representation of this operator can be defined. The not vanishing constant Fourier term of the baseline Hermite polynomial (which is the Gaussian function) leads to mathematical challenges with respect to the creation and annihilation operators of the related Hamiltonian operator of the quantum oscillator model. The Hilbert transform of a function f has always vanishing constant Fourier terms. As a consequence, the Hilbert-transformed Schrödinger operator form with extended domain $H_{-1/2}$ is bounded (with respect to the norm of L_2) leading to a bounded Hermitian operator with corresponding spectral form representation.

Based on the newly defined common Hilbert space domain spectral theory can be applied, while

- the (physical) test space keeps the same, i.e. $L_2 = H_0$
- the current domains of the considered operators are extended to enable a (convergent) energy norm $\|x\|_{1/2}$ and a corresponding weak variation representation of the considered operator equations with respect to the inner product $(x,y)_{-1/2}$.

The corresponding notions from variation theory are "energy norm" and "operator norm" with correspondingly defined minimization problems ("energy" resp. "action" minimization problems). The corresponding eigenvalue problem of an operator T is then related to the inner product $(Tx,x)_{-1/2}$.

With respect to the newly proposed Pseudo-differential and Fourier multiplier operators with extended fractional Hilbert scale domain we note the following:

- The Maxwell equations are represented by differential equations or integral equations. Both representations are considered as equivalent.
- The Lagrange ("force") and the Hamiltonian ("energy") formalisms are considered as equivalent. The mathematical proof is based on the Legendre transform, i.e. the equivalence is only valid if the assumptions of the Legendre transform are fulfilled.

In both cases, corresponding (mathematical) regularity assumptions are required to enable those propositions. A restriction of the domain regularity of the considered operators leads to no longer well-defined classical differential equations resp. to no longer valid Lagrange formalism. In other words, the provided consistent model in the distributional framework represents the mathematical/transcendental view of the considered physical world, while the corresponding classical solutions of the several differential equations are mathematical approximations to those physical models. This concept also overcomes the "physical interpretation" challenge of the "Neumann PDE" representation of the pressure p in the NSE model.

As a consequence there is only a "one-energy" (field) concept and corresponding (PDE specific) manifestations/ forms of considered "Nature forces".

The proposed variation Hilbert space frame is built on the space-time frame with dimension $n = m + 1 = 4$. Therefore the Huygens' Principle (which is also valid for the initial value problem of the wave equation) is valid for all considered "wave" PDE, overcoming e.g. the $n > 10$ requirement of the string theory. At the same time, the characteristics roles of a space-time dimension = 4 is also underlined by the specific role of undistorted spherical travelling waves (Courant-Hilbert, "*methods of mathematical physics*", II, VI, §10.3).

Schrödinger's "purely quantum wave" vision is about half-odd integers, rather than integers to be applied to wave-mechanical vibrations which correspond to the motion of particles of a gas resp. the eigenvalues and eigen-functions of the harmonic quantum oscillator still governed by the Heisenberg uncertainty inequality. The alternatively proposed $H_{1/2}$ energy space enables Schrödinger's vision ((ScE) (7.23) ff):

let ω denotes the angular frequency, h the (h-bar) Planck constant and

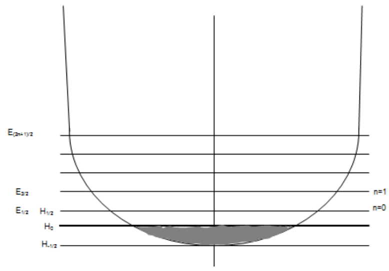
$$e := \frac{\omega h}{2}$$

Then Schrödinger's "half-odd integer vision" is about the following replacement:

$n = 0$	$E_0 = e$	$\rightarrow$	$E_{1/2} = 1 * e$
$n > 0$	$E_1 = 1 * \omega h$	$\rightarrow$	$E_{3/2} = 2 * e$
	$E_2 = 2 * \omega h$	$\rightarrow$	$E_{5/2} = 3 * e$
...			
	$E_n = n * \omega h$	$\rightarrow$	$E_{(2n+1)/2} = (n+1) * e,\ n = 0,1,2,...$
...			

As a consequence the corresponding eigenvalue and eigenfunction solutions of the number operator (i.e. the product of generation and annihilation operators) start with index $n = 1$, not already with $n = 0$.

With respect to the ladder operators of the harmonic quantum 0scillator the proposed alternative quantum state and related energy Hilbert scales can be visualized by



(Hilbert scale: $H_{-\frac{1}{2}}, H_0, H_{\frac{1}{2}}$)

A successfully applied least action principle (being interpreted as a maxime of Kant's reflective judgment) results into appropriate consistent mathematical-physical models, those models can be declared as law of natures. The above is related to the three "forces of nature" as modelled by the SMEP. The nature of those elementary particles and the way they move, is described by quantum mechanics, but quantum mechanics cannot deal with the curvature of space-time. Space-time are manifestations of a physical field, the gravitational field. At the same time, physical fields have quantum character: granular, probabilistic, manifesting through interactions. The to be defined common mathematical solution framework needs to provide a quantum state of a gravitational field, i.e. a quantum state of space. The crucial difference between the photons characterized by the Maxwell equations (the quanta of the electromagnetic field) and the to be defined quanta of gravity is, that photons exists in space, whereas the quanta of gravity constitute space themselves ((RoC2) p. 148). The proposed mathematical framework provides a common baseline to integrate quantum mechanics & thermodynamics with gravity & thermodynamics. From a physical model problem perspective this is about a common mathematical framework for black body radiation ((BrK4) remark 2.6, Note O55, O71, O72) and black hole radiation ((RoC3) p. 56, 60 ff)). The thermodynamics is the common physical theory denominator with the Planck concept of zero point energy of the harmonic quantum oscillator (BrK), (BrK1), and the Boltzmann entropy concept. An integrated model needs to combine the underlying Bose-Einstein and the Dirac-Fermi statistic. In this context already Schrödinger suggested half-odd quantum numbers rather than integers. "*From the point of analogy one would very much prefer to do so. For, the "zero point energy" of a Planck oscillator is not only borne* *out by direct observation in the case of crystal lattices, it is also so intimitely linked up with the Heisenberg uncertainty relation that one hates to dispense with it. On the other hand, if we adopt it straightaway, we get into serious trouble, especially on contemplating changes of the volume (e.g. adiabatic compression of a given volume of black-body radiation*" ((ScE) p. 50).

In the context of the newly proposed "energy-space" $H_{1/2} = H_1 + H_1^{\neg}$ we also refer to the Bose-Einstein condensation, where below the critical temperature T_C BEC "normal gas" particles coexist in equilibrium with "condensed" particles. Unlike a liquid droplet in a gas, here the "condensed" particles are not separated in space $(H_{-1/2} = H_0 + H_0^{\neg})$ from normal particles. Instead they are separated in momentum space. The condensed particles all occupy a single quantum state of zero momentum, while normal particles all have finite momentum.

2c. The $H_{-1/2}$ quantum state space and a new (plasma) Landau damping and space-time state/energy model

The Hilbert space $H_{-1/2}$ is suggested as physical quantum (Hilbert) state space model accompanied by correspondingly defined variational (Differential, Pseudo Differential or singular integral operator) equations. Beside the NSE and the YME problem areas the following other related areas are considered: (1) plasma dynamics and (2) space-time quantum geometrodynamics.

Concerning (1) we consider the model problem of the Landau damping phenomenon (which is about "*wave damping w/o energy dissipation by collision in plasma*") proposing a proof of this phenomenon, based on a $H_{-1/2}$(weak) problem-adequate (variational) Landau equations representation.

Concerning (2) we propose a $H_{-1/2}$(weak) quantum gravity model, which overcomes the current „quantum state stage background dependency" problem (as a consequence of key principles of GRT and all related „philosophical" aspects of „space-time and the transcendental external world" ((AnE), (BoD), (RoC0-3), (ScE1-2), (SmL), (WeH3), (WhJ0-2)). The proposed quantum space-time gravity model ensures „quantum state stage background indepencency", going along with non-linear elasticity PDO equations (note: the origin of the notion „tensor" came for elasticity theory), in our case, embedded in a weak $H_{-1/2}$ variational PDO framework, governed by the least action principle.

Concerning the problem area (1) we note that based on the classical Vlasov (partial differential) equation (describing the time evolution of the distribution function of plasma consisting of charged particles with long-range interaction) the non-linear Landau damping phenomenon has been proven in (MoC). The central element of the proof is about analytical (!) norm estimates in sync with the underlying Gaussian distribution function regularity. At the same point in time, the Vlasov equation in its classical PDE representation overlooks the important physical phenomenon of "*electrons travelling with exactly the material speed and the wave speed*" ((ShF) p. 392). The not physical problem adequate (analytical) norm estimates in combination with the physical modelling gap on plasma collision level show that the Vlasov equation is a not appropriate mathematical model for the non-linear Landau damping phenomenon. In (BrK6) an alternative model (based on the original Boltzmann-Landau (collision) equations; (LiP) (LiP1)) is proposed. The non-linear Boltzmann-Landau collision operator is given by

$$Q(f,f)=\frac{\partial}{\partial v_i}\left\{\int_{R^N} a_{ij}(v-w)\left[f(w)\frac{\partial f(v)}{\partial v_j}-f(v)\frac{\partial f(w)}{\partial w_j}\right]dw\right\}$$

with

$$a_{ij}(z)=\frac{a(z)}{|z|}\left\{\delta_{ij}-\frac{z_i z_j}{|z|^2}\right\}=\frac{a(z)}{|z|}P(z)\coloneqq\frac{1-[1-a(z)]}{|z|}[Id-Q](z)\quad Q(z)\coloneqq(R_iR_j)_{1\le i,j\le N}$$

and $a(z)$ symmetric, non-negative and even in z and R_i denote the Riesz operators, and with an unknown function f corresponding at each time t to the density of particle at the point x with velocity v. It can be approximated by a linear Pseudo Differential Operator (PDO) of order zero with symbol

$$b_{ij}(z)=z\cdot a_{ij}(z)=\frac{z}{|z|}\left\{\delta_{ij}-\frac{z_i z_j}{|z|^2}\right\}=\frac{z}{|z|}P(z)\coloneqq\frac{z}{|z|}[Id-Q](z)$$

whereby $a_{ij}(z)$ denotes the symbol of the Oseen kernel (LeN). Corresponding Hilbert space norm estimates are provided to build a problem adequate proof of the Landau damping phenomenon.

An appropriate plasma collisions (dynamics) model is a central building block for the related geometrodynamics problem/solution area (2). The proposed framework is also suggested to be applied to build a unified quantum field and gravity field theory based on the conceptual thoughts of Wheeler/deWitt (CiI), and and the related Loop Quantum Theory (LQT), which is a modern version of the theory of Wheeler and deWitt, where "*the* *variables of the theory describe the fields that form matter, photons, electrons, other components of atoms and the gravitational field - all on the same level*" ((RoC1) section 8, "dynamics as relation").

(CiI) 2.8: *Einstein's "general relativity" or ""geometric geometry of gravitation" or "geometrodynamics", has two central ideas: (1) Space-time geometry "tells" mass-energy how to move, (2) mass-energy "tells" space-time geometry how to curve. The concept (1) is automatically obtained by the Einstein field equations, (CiI) (2.3.14), basically as the covariant divergence of the Einstein tensor is zero. At the same point in time there are multiple tests of the geometrical structure and of the geodesic equation of motion, e.g. gravitational deflection and delay of electromagnetic waves, de Sitter and Lense-Thirring effect, perihelion advance of Mercury, Lunar Laser Ranging with its relativistic parameters: time dilation or gravitational redshift, periastron advance, time delay in propagation of pulse, and rate of change of orbital period*, (CiI) 3.4.
(CiI) 3.5: "*Hilbert used a variational principle and Einstein the requirement that the conservation laws for momentum and energy for both, gravitational field and mass-energy, be satisfied as a direct consequence of the field equations. ... Einstein geometrodynamics, ..., has the important and beautiful property the the equations of motion are a direct mathematical consequence of the Bianchi identities*."

With respect to the overall conceptual idea of this homepage a Hilbert space based geometrodynamics is proposed to be built on "space-time states", which are represented by elements of $H_{-1/2}$, while their corresponding "space-time energy" elements are represented by the corresponding "dual" (wavelets) elements in $H_{1/2}$. The Einstein field equations are proposed to be re-formulated as a weak (!) least action minimization problem by correspondingly defined variational equations representation. With respect to the Bianchi identities we emphasis that if $(u,v)_{-1/2}$ denotes the inner product of $H_{-1/2}$ the following relationships hold true: $(div(u),v)_{-\frac{1}{2}} \sim (u,\nabla v)_{-\frac{1}{2}} \sim (u,\nabla v)_0$. The methods of functional analysis are basically the same as those in the elasticity theory (MaJ1). The building principles for an appropriately defined variational representation is about that the way, (1) how "*Space-time geometry "tells" mass-energy how to move*", can be obtained by those representation and that the multiple tests (observed phenomena) of the geometrical structure and of the geodesic equation of motion ((2) "*where mass-energy "tells" space-time geometry how to curve*") is modelled (as a kind of symmetry break down) as approximation solution in the compactly embedded sub-spaces H_0 resp. H_1 of $H_{-1/2}$ resp. $H_{1/2}$.

2d. The $H_{-1/2}$ quantum state space replacing the Dirac distributions space

With respect to the below we note that the Dirac theory with its underlying concept of a Dirac "function" is proposed to be replaced by (fluid/quantum/... state) "elements" of the distributional Hilbert space $\mathrm{H}_{-1/2}$. We note that the regularity of the Dirac distribution "function" depends from the space dimension, i.e. it is an element of $\mathrm{H}_{-1/2-\varepsilon}$ ($\varepsilon > 0$, n = space dimension). Therefore, the alternative $\mathrm{H}_{-1/2}$ quantum state concept avoids space dimension depending regularity assumptions for quantum mechanics "wave packages" / "eigen-functions" / "momentum functions" with corresponding continuous spectrum.
We note that for signals on R the spectrum of the Hilbert transform is (up to a constant) given by the distribution $\mathrm{v.p.}(\frac{1}{x})$, whereby the symbol "v.p." denotes the Cauchy principal value of the integral over R. Its corresponding Fourier series is given by $-\mathrm{i} \cdot \mathrm{sgn}(\mathrm{k})$ with its relationship to "positive" and "negative" Dirac "functions" and the unit step function $Y(x)$. The $\mathrm{H}_{-1/2}$ framework, replacing the Dirac "function" concept, enables a generalization to dimensions n>1 without any corresponding additional regularity requirements.

3. NSE, YME and plasma/geometrodynamics problem/solution areas

The common Hilbert scale is about the Hilbert spaces H_α with $\alpha = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$ with its corresponding inner products $(*,*)_\alpha$. The proposed mathematical concepts and tools are especially correlated to the names of Plemelj, Stieltjes and Calderón.

The newly proposed "fluid/quantum state" Hilbert space $H_{-1/2}$ with its closed orthogonal subspace of H_0 goes also along with a combined usage of L_2 waves governing the H_0 Hilbert space and "orthogonal" wavelets governing the $H_{-1/2} - H_0$ space. The wavelet "reproducing" ("duality") formula provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets, where the "*microscope observations*" of *two* wavelet (optics) functions can be compared with each other (LoA). The prize to be paid is about additional efforts, when re-building the reconstruction wavelet.

We propose modified Maxwell equations with correspondingly extended domains according to the above. This model is proposed as alternative to SMEP, i.e. the modified Maxwell equation are proposed to be a "Non-standard Model of Elementary Particles (NMEP)", i.e. an alternative to the Yang-Mills (field) equations. The conceptual approach is also applicable for the Einstein field equations. Mathematical speaking this is about potential functions built on correponding "density" functions. The source density is the most prominent one. Physical speaking the source is the root cause of the corresponding source field. Another example is the invertebrate density (=rotation) with its corresponding rotation field. The Poincare lemma in a 3-D framework states that source fields are rotation-free and rotation fields are source-free. The physical interpretation of the rotation field in the modified Maxwell equations is about rotating "mass elements w/o mass" (in the sense of Plemelj) with corresponding potential function. In a certain sense this concept can be seen as a generalization of the Helmholtz decomposition (which is about a representation of a vector field as a sum of an irrotational (curl-free) and a solenoidal (divergence-free) vector field): it is derived applying the delta "function" concept. In the context of the proposed distributional Hilbert space framework, the Dirac function concept (where the regularity of those "function" depends from the space dimension) is replaced by the quantum state Hilbert space $H_{-1/2}$. The solution $u \in H_{-1/2}$ of the Helmholtz equation in terms of the double layer potential is provided in ((LiI), 7.3.4). From the Sobolev embedding theorem it follows, that for any space dimension n>0 the modified Helmholtz equation is valid for not continuous vector fields.

3a. The related NSE problem/solution area

This section is about a straightforward solution of the NSE Millenium problem by closing the Serrin gap provided that the $H_{-1/2}$ Hilbert space is the accepted underlying fluid state model.

The Navier-Stokes Equations (NSE) describes a flow of incompressible, viscous fluid. The three key foundational questions of every PDE is existence, and uniqueness of solutions, as well as whether solutions corresponding to smooth initial data can develop singularities in finite time, and what these might mean. For the NSE satisfactory answers to those questions are available in two dimensions, i.e. 2D-NSE with smooth initial data possesses unique solutions which stay smooth forever. In three dimensions, those questions are still open. Only local existence and uniqueness results are known. Global existence of strong solutions has been proven only, when initial and external forces data are sufficiently smooth. Uniqueness and regularity of non-local Leray-Hopf solutions are still open problems.

Basically the existence of 3D solutions is proven only for "large" Banach spaces. The uniqueness is proven only in "small" Banach spaces. The question of global existence of smooth solutions vs. finite time blow up is one of the Clay Institute millennium problems.

The existence of weak solutions can be provided, essentially by the energy inequality. If solutions would be classical ones, it is possible to prove their uniqueness. On the other side for existing weak solutions it is not clear that the derivatives appearing in the inequalities have any meaning. Basically all existence proofs of weak solutions of the Navier-Stokes equations are given as limit (in the corresponding weak topology) of existing approximation solutions built on finite dimensional approximation spaces. The approximations are basically built by the Galerkin-Ritz method, whereby the approximation spaces are e.g. built on eigenfunctions of the Stokes operator or generalized Fourier series approximations. It has been questioned whether the NSE really describes general flows: The difficulty with ideal fluids, and the source of the d'Alembert paradox, is that in such fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbing on each other, provided they are separated by a streamline. It is clear that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

The question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the NSE blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost. Given a smooth datum at time zero, will the solution of the NSE continue to be smooth and unique for all time?

The NSE are derived from the (Cauchy) stress tensor (resp. the shear viscosity tensor) leading to liquid pressure force. In electrodynamics & kinetic plasma physics the linear resp. the angular momentum laws are linked to the electrostatic (mass "particles", collision, static, quantum mechanics, displacement related; "fermions") Coulomb potential resp. to the magnetic (mass-less "particles", collision-less, dynamic, quantum dynamics, rotation related; "bosons") Lorentz potential.

We note that the solution of the Navier-Stokes equation are related to the considered degenerated hypergeometric functions by its corresponding integral function representation (PeR1).

With respect to the open Millenium 3D non-stationary, non-linear NSE problem we note that the alternatively proposed "fluid state" Hilbert space $\mathrm{H}_{-1/2}$ with corresponding alternative energy ("velocity") space $\mathrm{H}_{1/2}$ enables a (currently missing) energy inequality based on existing contribution of the non-linear term. In the standard weak NSE representation this term is zero, which is a great thing from a mathematical perspective, avoiding sohisticated estimating techniques, but a doubtful thing from a physical modelling perspective, as this term is the critical one, which jepordized all attempts to extend the 3D problem based on existing results from the 2D case into the 3D case. The corresponding estimates are based on Sobolev embedding theorems; the Sobolevskii estimate provides the appropriate estimate given that the "fluid state" space is $\mathrm{H}_{-1/2}$ in a corresponding weak variational representation.

There is no uniqueness proof for weak solutions except for over small time intervals. The simplest possible model example how a singularity can appear, is the ODE

$$y'(t) = y^2(t)\,,\ y(0) = y_0$$

with the solution

$$y(t) = \frac{y_0}{1 - t \cdot y_0}$$

which becomes infinite in finite time.

Based on the Hilbert scale $H(\alpha)$, $\alpha \in R$, defined by the orthogonal set of eigenpairs of the Stokes operator we present a global unique weak $\mathrm{H}_{-1/2}$ solution of the generalized 3D Navier-Stokes initial value problem

$$(\dot{u}, v)_{-1/2} + (Au, v)_{-1/2} + (Bu, v)_{-1/2} = 0 \qquad \text{for all } v \in H_{-1/2}$$

$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2} \quad .$$

resp.

$$-(\Delta u, v)_{-1/2} + (\nabla p, v)_{-1/2} \cong (\nabla u, \nabla v)_{-1/2} + (\nabla p, v)_{-1/2} \cong (u, v)_{1/2} + (p, v)_0 \,.$$

The pressure p can be expressed in terms of the velocity by the formula

$$p = -\sum_{j,k=1}^{3} R_j R_k (u_j u_k)$$

where (R_1, R_2, R_3) is the Riesz transform. The Leray-Hopf projector is the matrix valued Fourier multiplier given by

$$P(\xi) = Id - \frac{\xi \otimes \xi}{|\xi|^2} = (\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2})_{1 \le j,k \le n} \quad , \quad P = Id - R \otimes R =: Id - Q$$

whereby Q is an orthogonal projector, i.e. it holds $Q := R \otimes R = (R_j R_k)_{1 \le j,k \le 1} = Q^2$. As a result the Leray-Hopf operator

$$P = Id - R \otimes R =: Id - Q = Id - \frac{D \otimes D}{D^2} Id - \Delta^{-1}(\nabla \times \nabla)$$

is also an orthogonal projection.

The global boundedness is a consequence of the Sobolevskii-estimate of the non-linear term ([SoP]) enabling the generalized energy inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2+\|u\|_{1/2}^2\le|(Bu,u)_{-1/2}|\le c\cdot\|u\|_{-1/2}\|u\|_1^2\cdot$$

Putting $y(t):=\|u\|_{-1/2}^2$ one gets

$$y'(t)\le c\cdot\|u\|_1^2\cdot y^{1/2}(t)$$

resulting into the a priori estimate

$$\|u(t)\|_{-1/2}\le\|u_0\|_{-1/2}+\int_0^t\|u\|_1^2(s)ds\le c\left\{\|u_0\|_{-1/2}+\|u_0\|_0^2\right\}\quad .$$

A "3D challenge" like the NSE above is also valid, when solving the monochromatic scattering problem on surfaces of arbitrary shape applying electric field integral equations. From (IvV) we recall that the (integral) operators A and A(t): $\mathrm{H}_{-1/2}\to\mathrm{H}_{1/2}$ are bounded Fredholm operators with index zero. The underlying framework is still the standard one, as the domains are surfaces, only. An analog approach as above with correspondingly defined surface domain regularity is proposed.

The initial boundary value problem determines the initial pressure $p_0(x)$ by the Neumann problem

$$\Delta p_0=(f_0-u_0\cdot\nabla u_0 \qquad \text{in } \Omega$$
$$\frac{\partial p_0}{\partial n}=\left[\Delta u_0-u_0\cdot\nabla)u_0+f_0\right]\cdot n \quad \text{at } \partial\Omega$$

with$f_0:=\lim_{t\to 0}f(\cdot,t)$. Applying formally the div-operator to the classical NSE the pressure field must satisfy the following Neumann problem ((GaG))

$$\Delta p=(u\cdot\nabla)u-f \qquad \text{in } \Omega$$
$$\frac{\partial p}{\partial n}=\left[\Delta u-(u\cdot\nabla)u+f\right]\cdot n \quad \text{at } \partial\Omega$$

where n denotes the outward unit normal to $\partial\Omega$. As it holds that

$$\left[\Delta u-(u\cdot\nabla)u+f\right]\cdot n|_{\partial\Omega}\to\left[\Delta u_0-(u_{.0}\nabla)u_0+f_0\right]\cdot n|_{\partial\Omega} \text{ in } H_{-1/2}(\partial\Omega)$$

and

$$\nabla\cdot\left[f-u\cdot\nabla)u\right]_{\partial\Omega}\to\nabla\cdot\left[f_0-u_0\cdot\nabla)u_0\right]_{\partial\Omega} \text{ in } H_{-1/2}(\partial\Omega)$$

the pressure p tends to p_0 in the sense that $\|\nabla(p(\cdot,t)-p_0\|\to 0$ as $t\to 0$.

As a consequence the prescription of the pressure at the boundary walls or at the initial time independently of u, could be incompatible with and, therefore, could retender the NSE problem ill-posed.

With respect to the relationship to the considered Hilbert space $H_{-1/2}$ we emphasis that the Prandtl operator with domain $H_{1/2}$ and range $H_{-1/2}$ is bounded and coercive and the corresponding exterior Neumann problem admit one and only on generalized solution (BrK2).

A $H_{-1/2}$ (fluid state) Hilbert space framework is also applied to derive optimal finite element approximation estimates for non-linear parabolic problems with not regular initial value data (BrK2).

Kolmogorov's turbulence theory is a purely statistical model (based on the H_0 (observation/test) Hilbert space), which describes (only!) the qualitative behavior of turbulent flows. There is no linkage to the quantitative fluid behavior as it is described by the Euler or the Navier-Stokes equations. The physical counterpart of his low- and high-pass filtering Fourier coefficients analysis is a "local Fourier spectrum", which is a contradiction in itself, as, either it is non-Fourier, or it is nonlocal ((FaM)). WE propose to combine the wavelet based solution concept of (FaM) with a revisited CLM equation model in a physical $H_{-1/2}$ Hilbert space framework to enable a turbulent $H_{-1/2}$ signal which can be split into two components: coherent bursts and incoherent noise. The model enables a localized Heisenberg uncertainty inequality in the closed (noise) subspace $L_2^{\neg} = H_0^{\neg} = H_{-1/2} - H_0$, while the momentum-location commutator vanishes in the (coherent bursts) test space H_0.

3b. The related YME problem/solution area

This section is about a straightforward solution of the YME mass gap problem provided that the $H_{-1/2}$ Hilbert space is the accepted underlying quantum state model.

We propose an alternative mathematical framework for the Standard Model of Elementary Particles (SMEP), which replaces gauge theory and variational principles: The underlying concepts of exterior derivatives and tensor algebra are replaced by (distributional) Hilbert scales and (purely Hamiltonean) variational principles. As a consequence, the vacuum energy becomes an intrinsic part of the variational principles, i.e. it is identical for all considered Lagrange resp. Hamiltonian mechanisms of all related differential equations, while the corresponding "force" becomes an observable of the considered (Hamiltonean) minimization problem.

In some problem statements of the YME there are basically two assumptions made:

1. the energy of the vacuum energy is zero

2. all energy states can be thought of as particles in plane-waves.

As a consequence the mass gap is the mass of the lightest particle.

Our challenge of proposition 1 is about the measure of the vacuum energy, which gives the value "zero". While the energy norm in the standard H_1 Hilbert space might be zero, the value of the quantum state with respect to the energy norm of the sub-space $\mathrm{H}_{1/2}$ still can be >0.

Our challenge of proposition 2 is going the same way: a particle with mass can be measured (condensed energy), i.e. it is an element of the test space H_0, while there still can be "wavelets" in the closed complementary space $\mathrm{H}_{-1/2} - \mathrm{H}_0$, where the test space is "just" compactly embedded. Those "wavelets" might be interpreted as all kinds of today's massless "particles" (neutrinos and photons) with related "dark energy". As a consequence there is no YME mass gap anymore, but there is a new concept of vacuum energy (wave packages, eigen-differentials, rotation differential) governed by the Heisenberg uncertainty principle. This is about an alternative harmonic quantum energy model enabling a finite "quantum fluctuation = total energy", while replacing Dirac's Delta function by $\mathrm{H}_{-1/2}$ distributions enabling and an alternative Schrödinger's momentum operator (BrK7).

A physical interpretation could be about "rotating differentials" ("quantum fluctuations"), which corresponds mathematically to Leibniz's concept of monads. The mathematical counterparts are the ideal points (or hyper-real numbers). This leads to non-standard analysis, whereby the number field has same cardinality than the real numbers. It is "just" the Archimedean principle which is no longer valid.

The proposed mathematical concepts and tools are especially correlated to the names of Schrödinger and Weyl (e.g. in the context of "half-odd integers quantum numbers for the Bose statistics" and resp. Weyl's contributions on the concepts of matter, the structure of the world and the principle of action (WeH), (WeH1), (WeH2)). It enables an alternative (quantum) ground state energy model embedded in the proposed distributional Hilbert scale frame of this homepage covering all variational physical-mathematical PDE and Pseudo Differential Operator (PDO) equations (e.g. also the Maxwell equations).

The electromagnetic interaction has gauge invariance for the probability density and for the Dirac equation. The wave equation for the gauge bosons, i.e. the generalization of the Maxwell equations, can be derived by forming a gauge-invariant field tensor using generalized derivative. There is a parallel to the definition of the covariant derivative in general relativity. With respect to the above there is an alternative approach indicated, where the fermions are modelled as elements of the Hilbert space H_0, while the complementary closed subspace $H_0^{\perp}$ is a model for the "interaction particles, bosons". For gauge symmetries the fundamental equations are symmetric, but e.g. the ground state wave function breaks the symmetry. When a gauge symmetry is broken the gauge bosons are able to acquire an effective mass, even though gauge symmetry does not allow a boson mass in the fundamental equations. Following the above alternative concept the "symmetry state space" is modelled by H_0, while the the ground state wave function is an element of the closed subspace $H_0^{\perp}$of $\mathrm{H}_{-1/2}$ (BrK).

When one wants to treat the time-harmonic Maxwell equations with variational methods, one has to face the problem that the natural bilinear form is not coercive on the whole Sobolev space. On can, however, make it coercive by adding a certain bilinear form on the boundary of the domain (vanishing on a subspace of H_{-1}), which causes a change in the natural boundary conditions.

In SMEP (Standard Model of Elementary Particles) symmetry plays a key role. Conceptually, the SMEP starts with a set of fermions (e.g. the electron in quantum electrodynamics). If a theory is invariant under transformations by a symmetry group one obtains a conservation law and quantum numbers. Gauge symmetries are local symmetries that act differently at each space-time point. They automatically determine the interaction between particles by introducing bosons that mediate the interaction. $U(1)$ (where probability of the wave function (i.e. the complex unit circle numbers) is conserved) describes the elctromagnetic interaction with 1 boson (photon) and 1 quantum number (charge Q). The group $SU(2)$ of complex, unitary (2x2) matrices with determinant I describes the weak force interaction with 3 bosons (W(+), W(-), Z), while the group $SU(3)$ of complex, unitary (3x3) matrices describes the strong force interaction with 8 gluon bosons.

Reformulated Maxwell or gravitation field equations in a weak $\mathrm{H}_{-1/2}$-sense leads to the same effect, as dealing with an isometric mapping g --> H(g) in a weak H_0 sense (H denotes the Hilbert transform) alternatively to a second order operator in the form $x \cdot P(g(x))$ in a weak $\mathrm{H}_{-1/2}$ sense. This results into some opportunities as

- the solutions of the Maxwell equations in a vaccum do not need any callibration transforms to ensure wave equation character; therefore, the arbitrarily chosen Lorentz condition for the electromagnetic potential (to ensure Lorentz invariance in wave equations) and its corresponding scalar function ((FeR), 7th lecture) can be avoided

- enabling alternative concepts in GRT to e.g. current (flexible") metrical affinity, affine connexions and local isometric 3D unit spheres dealing with rigid infinitesimal pieces, being replaced by geometrical manifolds, enabling isometrical stitching of rigid infinitesimal pieces ((CiI), (ScP)).

The gauge invariance is the main principle in current SMEP theory.

(BlD) 10.3: "*It is fine that the gauge field of electromagnetism has zero mass because there the force is mediated by photons, which are massless. However, Yang-Mills type forces must arise from the exchange of massive particles because of the observed short range of these forces. The Higgs mechanism helps in two ways. First, gauge fields can acquire mass by the symmetry breaking. Second, the undesirable Goldstone bosons (which arise in the symmetry-breaking process) can be usually gauged away.*"

The Higgs effect (or mechanism) builds on an extended from global to local $U(1)$ transformations symmetry group of the underlying Lagrangian. It explains the mass of the gauge W- and Z- (weak interaction) bosons of the weak "nuclear-force".

(HiP): "*Within the framework of quantum field theory a "spontaneous" breakdown of symmetry occurs if a Lagrangian, fully invariant under the internal symmetry group, has such a structure that physical vacuum is a member of a set of (physically equivalent) states which transform according to a nontrivial representation of the group. This degeneracy of the vacuum permits non-trivial multiplets of scalar fields to have nonzero vacuum expectation values (or "vacuons"), whose appearance leads to symmetry-breaking terms in propagators and vertices. … When the symmetry group of the Lagrangian is extended from global to local transformations by introduction of coupling with a vector gauge field the original scalar massless boson as a result of spontaneous breakdown of symmetry then becomes the longitudinal state of a massive vector (Higgs) boson whose transverse state sare the quanta of the transverse gauge field. A perturbative treatment of the model is developed in which the major features of these phenomena are present in zero order*."

The Higgs boson is supposed to be a heavy elementary particle (with non-zero rest mass of about 125 GeV with spin 0). The Higgs field is supposed to fill the whole universe interacting with each particle, which "moves" through it by a kind of frictional resistance, i.e. which has kinetic energy. Therefore, the Higgs effect (i.e. generating mass particles) requires a Higgs field with not vanishing amplitudes in the ground state.

3c. The related plasma/ geometrodynamics problem/solution area

This section is about a space-time quantum model, enabling a space-time stage background independency, provided that the $H_{-1/2}$ Hilbert space is the accepted underlying quantum state model.

Replacing the affine connexion and the underlying covariant derivative concept by a geometric structure with corresponding inner product puts the spot on the

Thurston conjecture*: The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structure (ThW).*

This conjecture asserts that any compact 3-manifold can be cut in a reasonably canonical way into a union of geometric pieces. In fact, the decomposition does exist. The point of the conjecture is that the pieces should all be geometric. There are precisely eight homogeneous spaces (X, G) which are needed for geometric structures on 3-manifolds. The symmetry group $SU(2)$ of quaternions of absolute value one (the model for the weak nuclear force interaction between an electron and a neutrino) is diffeomorph to S^3, the unit sphere in R^4. The latter one is one of the eight geometric manifolds above (ScP). We mention the two other relevant geometries, the Euclidean space E_3 and the hyperbolic space H_3. It might be that our universe is not an either... or ..., but a combined one, where then the "connection" dots would become some physical interpretation. Looking from an Einstein field equation perspective the Ricci tensor is a second order tensor, which is very much linked to the Poincare conjecture, its solution by Perelman and to S^3 (AnM). The geometrodynamics provides alternative (pseudo) tensor operators to the Weyl tensor related to H_3(CiI). In (CaJ) the concept of a Ricci potential is provided in the context of the Ricci curvature equation with rotational symmetry. The single scalar equation for the Ricci potential is equivalent to the original Ricci system in the rotationally symmetric case when the Ricci candidate is nonsingular. For an overview of the Ricci flow regarding e.g. entropy formula, finite extinction time for solutions on certain 3-manifolds in the context of Prelman's proof of the Poincare conjecture we refer to (KlB), (MoJ).

The single scalar equation for the Ricci potential (CaJ) might be interpreted as the counterpart of the CLM vorticity equation as a simple one-dimensional turbulent flow model in the context of the NSE.

The link back to a Hilbert space based theory might be provided by the theory of spaces with an indefinite metric ((DrM), (AzT), (DrM), (VaM)). In case of the L(2) Hilbert space H, this is about a decomposition of H into an orthonal sum of two spaces H1 and H2 with corresponding projection operators P1 and P2 relates to the concepts which appear in the problem of S. L. Sobolev concerning Hermitean operators in spaces with indefinite metric ((VaM) IV). For x being an element of H this is about a defined "potential" ((VaM) (11.1))

$$\varphi(x) := \big((x)\big)^2 = \|P_1 x\|^2 - \|P_2 x\|^2$$

and a corresponding "grad" potential operator $\boldsymbol{W}(x)$, given by

$$\boldsymbol{W}(x) = \frac{1}{2}\mathrm{grad}\varphi(x) := P_1(x) - P_2(x) \quad , \quad \text{(VaM) (11.4).}$$

The potential criterion $\varphi(x) = c > 0$ defines a manifold, which represents a hyperboloid in the Hilbert space H with corresponding hyperbolic and conical regions. The tool set for an apprpropriate generalization of the above "grad" definition is about the (homogeneous, not alway non-linear in h) Gateaux differential (or weak differential) **VF**(x,h) of a functional F at a point x in the direction h ((VaM) §3)). The appropriate weak inner product might be the inner product of the "velocity" space $\mathrm{H}_{1/2}$. We note the Sobolev embedding theorem, i.e. $\mathrm{H_k}$ is a sub-space of C^0 (continuous functions) for $k > n/2$, i.e. there is no concept of "continuous velocity/momentum" in the proposed Hilbert space framework, i.e. there is no Frechet differential existing ((VaM) 3.3). This refers to one of the several proposals, which have been made to drop some of the common sense notions about the universe ((KaM) 1.1), which is about continuity, i.e. space-time must be granular. The size of these grains would provide a natural cutoff for the Feynman integrals, allowing to have a finite S-matrix.

A selfadjoint operator B defined on all of the Hilbert space H is bounded. Thus, the operator B induces a decomposition of H into the direct sum of the subspaces, and therefore generates related hyperboloids ((VaM) 11.2). Following the investigations of Pontrjagin and Iohvidov on linear operators in a Hilbert space with an indefinite inner product, M. G. Krein proved the Pontrjagin-Iohvidov-Krein theorem (FaK).

In an universe model with appropriately connected geometric manifolds the corresponding symmetries breakdowns at those "connection dots" would govern corresponding different conservation laws in both of the two connected manifolds. The Noether theorem provides the corresponding mathematical concept (symmetry --> conservation laws; energy conservation in GRT, symmetries in particle physics, global and gauge symmetries, exact and broken). Those symmetries are associated with "non-observables". Currently applied symmetries are described by finite- (rotation group, Lorentz group, ...) and by infinite-dimensional (gauged $U(1)$, gauged $SU(3)$, diffeomorphisms of GR, general coordinate invariance...) Lie groups.

A manifold geometry is defined as a pair (X,G), where X is a manifold and G acts transitively on X with compact point stabilisers (ScP). Related to the key tool "Hilbert transform" resp. "conjugate functions" of this page we recall from (ScP), that Kulkarni (unpublished) has carried out a finer classification in which one considers pairs (G,H), where G is a Lie group, H is a compact subgroup and G/H is a simple connected 3-manifold and pairs (G_1,H_1) and (G_2,H_2) are equivalent if there is an isomorphism $G_1 \to G_2$ sending H_1 to a conjugate of H_2. Thus for example, the geometry S^3 arises from three distinct such pairs, (S^3,e), $(U(2),SO(2))$, $(SO(4),SO(3))$. Another example is given by the Bianchi classification consisting of all simply connected 3-dimensional Lie groups up to an isomorphism.

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