Ground State Energy

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This section applies the proposed solution concept of the YME section to the harmonic quantum oscillator. The today's 1-dimensional harmonic quantum energy model, which is the Helmholtz free energy, provides the simplest model of a ground state energy definition. The underlying defining function -log2sinh(x) plays also a key role in the vacuum energy of electromagnetic fields and the Planck body radiation law.

The total energy of the harmonic quantum oscillator model is represented by a divergent series. The corresponding "re-normalization technique" is by the argument that "*it can be justified that the infinite series over all integers n is finite with value = -1/12*".

Following the solution concept of the YME section the alternatively proposed H(1/2) (energy) Hilbert space is based on a corresponding variation equation Hilbert space H(-1/2). This H(-1/2) Hilbert space is defined by the one-dimensional G. T. Symm (singular integral operator in potential theory and elastostatics) operator A (e.g. (BrK)).

The solution concept in a nutshell

It is built on the representation of the Hilbert space H(-1/2) in the form = **V**+**W**, where **V**=H(0)=L(2) and **W** defines the complementary space of L(2) with respect to the inner product of H(-1/2). We note that the eigenfunctions of the Hermitian operator are the Hermite polynomials which span the Hilbert space **V.** The "lowest" eigenfunction related to the quantum ground state is given by the Gauss-Weierstrass function. Let H denote the Hilbert transform operator. The Hilbert transform of each Hermite polynomial is orthogonal to its origin with respect to the L(2) inner product i.e. they also built an eigenfunction basis of **V**=H(0). As a consequence of other properties of the Hilbert transforms the commutator (x,H) applied to this alternative eigenfunction basis of L(2) vanishes.
The transition from the classical harmonic oscillator model to a corresponding quantum mechanics description is enabled by the Schrödinger momentum operator P:=-i*h*d/dx. The challenge is that by this transition the commutator (x,P) no longer vanishes, as it is the case for the classical model (x,p).

The operator A (and the underlying definition of the inner product of H(-1/2)) relates to the Hilbert transform basically by the equation

A(d/dx)(u) = H(u) .

This enables an alternative Schrödinger momentum operator definition in the form

P(u):=h*H(u) .

As a consequence of the above the commutator for quantum states related to the space **V** vanishes (with a corresponding discrete spectrum) while the commutator related to the **W**-space component of a quantum state in H(-1/2) still not vanishes (with a corresponding continuous spectrum related to the quantum ground state).
We note that the Riesz operators are the n-dimensional analogues of the 1-dimensional Hilbert transform operator; those operators enable the analog approach for space dimensions n>1.

We further note the relationship of the wave equation with the Lalesco-Picard integral equation (TrF): let l and m denote the greek letters "lamda" and "mue" and let A, B denote arbitrary constants. Then the solutions of the wave equation a*exp(m*x)+B*exp(-m*x) solve the (homogenous) Lalesco-Picard integral equation with Parameter l. provided that the absolute value of Re(m):=Re(squar(1-2*l)) is smaller than 1 which, for real l, implies that l>0. For the fields of real numbers the spectrum of the L-P integral equation covers the infinite segment l>0. Each point of this segment is an eigenvalue of index 2 of the L-P equation.

For corresponding mathematical and philopophical conceptional thoughts and related tool sets published on internet between 2010 -2013 we refer to

http://www.quantum-gravitation.de

http://www.navier-stokes-equations.com

In the following we give some additional context to the related topics of the other sections of this homepage:

Subatomic particles and primes

(DeJ) 18, VII: "*The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?*"

Distributional Hilbert space and RH/NSE/YME problems

The today´s well accepted zero energy formula of the quantum oscillator is (just (!)) a divergent series. Nobody seems to be concerned about this. Sophisticated renormalization techniques were developed to overcome this homemade "issue", when building a quantum field theory. The free energy of a system of interacting oscillators to model the Planck blackbody radiation law contains same divergent series (Feynman R., P., Hibbs A. R., "Quantum Mechanics and Path Integrals", (10.85)).

At this point we just mention that from a purely mathematical logic point of view it holds that every assertion/conclusion from a wrong assumption is always true. Therefore, from the obvious wrong assumption above one could e.g. conclude that the universe is a stack of turtles, end of story. This recalls the story of "*the emperor's new clothes*".

The underlying still unsolved mathematical conceptual problem is similar to the non-vanishing constant Fourier coefficient of the Theta function for the RH duality problem. The above solution of the RH in combination with remarkable properties of the Hilbert/Riesz transforms enables an alternative mathematical ground state energy model.

A new Hilbert scale based truly infinitesimal small geometry is provided, which is proposed to replace the Semi-Riemannian (metrical space) manifold concept (and its underlying gauge theory to enable the Standard Model of Elementary Particles) by a Hilbert space based quantum (differential forms) ground state energy model, building on an intrinsic ground state energy scalar product. The latter one is consistent with existing quantum state within a Hilbert space framework. The quantization technique from quantum mechanics transfers a "real" world´s Partial Differential Equations (PDE) model to a "quantum" world Hilbert space model. The new model operates the other way around: out of the a priori "quantum/ mathematical" world model, which is built by Pseudo-Differential Equations (PDO) (in its variational form representation) within a distributional Hilbert space it creates the "real/physical" PDE world (in its variational representation form within a Hilbert space).

The link between P2 and an appropriate vacuum energy model is given by the Hilbert space H(-1/2). The extended tool set to the "quantization" operators are Pseudodifferential Operators with domains H(-a), a>0 ((AhM) II § 13). The key idea is to enable additionally to the discrete eigenvalues/eigenfunctions (which span the H(0) Hilbert space) a continuous spectrum, which spans the complementary space H(-1/2)-H(0). In this framework the Lagrange formalism (minimization of "sorces") is no longer valid (as the Legendre transform is no longer defined), but the Hamiltonian formalism (minimizing "energy") is still valid: the the quantum vacuum energy is defined by the continuous spectrum of the spectral decomposition of the distributional Hilbert space. Plemelj's "mass element" and "potential" definition" (modelled as Stieltjes integral) provides to linkage between the H(-1/2) Hilbert space and differential forms. By this, the concept of differentiable manifolds can be replaced by a (symplectic?) Hilbert space geometry:

the standard notation of the Hamiltonian formalism is about the momentum p, the location q, the Hamiltonian function H and the action S with

$$S = pdq - Hdt.$$

Within the framework of symplectic manifolds the symplectic formulation of the Hamiltonian equations is about the existence of a 2-form w, which generates the 1-form dH as a result of the "inner product" i(X)w of the Hamiltonian vector field X =X(H) and the 2-form w:

$$i(X)w = dH.$$

The approach above enables a (truly) Hilbert space inner product on the 1-forms.

We also note the thermodynamic potentials, e.g. the inner and free eneries modelled as differential equations

$$dE = T\,dS - P\,dV + c\,dN$$
$$dF = -S\,dT - c\,dV + c\,dN$$

The proposed solution concept for the NSE open question is about rotation-invariant fluids as elements of a Hilbert space with negative (distributional) Hilbert scale defined by the eigenpairs of the Stokes operator. It is enabled by the alternative "normal" derivative definition provided by J. Plemelj (which requires reduced regularity assumptions to the solution u in the size of a reduction from C(1) to C(0)), leading to a energy Hilbert space H(1/2), which does not create a Serrin gap for n=3 and H(1/2) energy Hilbert space.

The curl of a vector determines the rotation of the vector field. For a rigid body rotation, the curl of the velocity field is proportional to the angular velocity. The total flux of a vector field curl(A) through a surface S is (according to the Stokes' theorem) identical with a "total circulation" of the vector field. Plemelj's alternative concept of a normal derivative leads him to the concept of a "current", alternatively to the standard "current density". Therefore, Plemelj's concept leads to an alternative definition of a kind of "pointwise" "total circulation" on the surface.

In the proposed H(1/2) energy Hilbert space framework the non-linear NSE-term is governed by the corresponding inequality with respect to the energy norm H(1/2) according to the lemma 3.2 of (GiY). At the same time the balance to the standard energy norm of H(1) is achieved by two appropriately defined exponent values of the *time-variable factor multiplied with the two energy norms of H(1/2) resp. H(1). The exponential values are chosen in that way, that the time-weighted norms are in sync with the convergence behavior as t tends to zero of the corresponding semigroup estimates (lemma 2.1 and 2.2 of (GiY)).*

We notice that there is a well-established theory of vortex dynamics for incompressible fluid flows (e.g. (MaA). The link to the rotation invariant Riesz operators is given by a vorticity-stream formulation of the Euler and NSE. The celebrated Kelvin's conservation of circulation (*circulation around a curve moving with fluid is constant in time*) resp. Helmholtz's conservation of vorticity flux (*vorticity flux through a surface moving with the fluid is constant in time*) to the Euler equation provides the relationship to the alternatively proposed Plemelj normal derivative.

We claim that the provided NSE solution which is addressing the too strong/restrictive mathematical model assumption of the NSE can be also applied to the Maxwell equations, as well as to the zero state energy "problem" of the harmonic quantum oscillator. The d'Alembert paradox (fluid vs. velocity/momentum) is related to this too restrictive assumptions for the NSE, while the Schrödinger paradox (particle vs. wave/momentum) is related to the latter one.

We note that

- the Biot-Savart law, which provides the foundation to derive the Maxwell equations, has been derived empirically and cannot be proven strongly

- the Biot-Savart singular integral operator is a PDO of order -1

- the Biot-Savart formula enables the transformation between the velocity and the vorticity representation of the NSE

- the Hilbert transform plays a key role in the Constantin-Lax-Majda (CLM) one dimensional model for the 3D-Navier-Stokes vorticity equation.

We further note that there is a relationship between the confluent hypergeometric function (as applied in our RH solution) and an integral representation of the NSE, derived by Fourier coefficient analysis (Peralta-Fabi R., An Integral Representation of the Navier-Stokes Equations-I, Revista Mexicana de Física 31 No. 1 (1984) 57-67).

The mathematical area, which links "real world" (physical) partial differential equations (PDE) with "quantum world", is Variation theory, based on a weak (energy) Hilbert space inner product resp. a (action) operator norm minimization representation.

The momentum/energy measure is defined by the induced norm of the inner (energy) product. In case of "real world" PDE models this is enabled by the Green formulas, which link the Laplace operator (and the related potential theory) with the Dirichlet (energy) integral. The latter one defines the inner product of the energy Hilbert (Sobolev) space H(1) to guarantee the required regularity assumptions of the Green formulas.

J. Plemelj's alternative definition of a potential, based on a "*mass element*" du(y), alternatively to a mass density u(x) as domain of the Green integral operator and a corresponding alternatively defined normal derivative definition ("Strömung") lead to equivalent Pseudo differential operator (PDO) equations, which are Riesz transform operator **R**=R(i) equations (resp. Hilbert transform operator **H** equation for n=1) with reduced regularity assumptions of the corresponding domain for u(y).

J. Plemelj's alternative definition of a normal derivative leads to extended Green formulas, where the normal derivative is replaced by a differential of the conjugate potential function (in case of space dimension n=2, where the conjugate of a potential function u is characterized as the transform of u), with the following reduced regularity assumptions: the normal derivative of a potential function u needs no longer to be existing everywhere on the boundary, but only the conjugate function of this potential function u is required to be continuous on the boundary. This corresponds to an extension from differentiable functions to continuous functions, which corresponds basically to a shift on a Hilbert scale by -1 (.e. for weak (Pseudo) Differential Equation representations to a shift by -1/2). The extended Green formulas are (in a nutshell) given as follows ((PlJ) p. 9-13):

let D(u,v) denote the Dirichlet integral (inner product) with domain D and let <u,v> denote the related inner product on the boundary of D. Let further **u** denote the conjugate function of a potential u. Then it holds:

i) D(u,u)=<u,d**u**> is positive for all not constant potential functions u
ii) <v,d**u**>=<.u,d**v**>

Therefore, J. Plemelj's potential theory enables (weaker) variation representations with respect to the inner (energy) product of H(1/2). As the Legendre transform is no longer valid for the less regular operator domain, the Lagrange formalism is no longer valid, while the Hamiltonian formalism is still valid.

In case of the NSE problem for space dimension n=3 this gains additional scale factor to close the Serrin gap, which is a consequence of the Sobolev embedding theorem. This is claimed to enable already successfully applied (classical solution) proof techniques for case n=2.

By operating with "div" on both sides of the NSE the field p(x,t) can be formally obtained as a solution of a Neumann problem. Therefore, to describe the values of the pressure at the bounding walls or at the initial time independently of v, could incompatible with the NSE and, therefore, could render the problem ill-posed. The above extended Green formulas provide a framework to overcome the Neumann "pressure" incompatibility problem.

Inserted addition: The NSE and the Maxwell equation do have equivalent differential forms representation (see e.g. (FlH) Flanders H., Differential Forms with Applications to the Physical Sciences, Dover Publications, Inc., New York, 1989):
a 1-form w=Pdx+Qdy+Rdz may be identified with an arbitrary vector field (P,Q,R) ex R(3). The exterior derivative operator d then is a 2-form, which subsumes the ordinary gradient, curl or rotation, and divergence. The term d(dw)=0 (which is in essence the Poincare lemma) is the 3-space interpretation of curl(grad(f))=0=div(curl**v**). For the Maxwell field quantities **E**=electric field, **H**=magnetic field, **B**=magnetic induction, **J**=electric current density, **D**=dielectric displacement, q=charge density the Maxwell equations are given by curl(**E**)=f(**B**), curl(**H**)=f(**J**,**D**), div(**D**)=f(q), div(**B**)=0, where there is a corresponding differential form representation given by da=0, db+c=0,dc=0 (see (FlH) 4.6).

In essence the extended Green formulas enable differential form representation of the NSE and the Maxwell equations *including* appropriate boundary value conditions, i.e. enable well posed PDE differential form representations (of both equation systems) with corresponding weak variation equations representation with appropriate Hilbert space domains.

In case of the YME problem the concept above enables a spectral theory in the extended H(-1/2) Hilbert space, which retains discrete eigenvalues with corresponding eigenfunctions as basis of the space H(0), while generating an additional continuous spectrum for the complementary space H(-1/2)-H(0). The latter one is proposed as zero state energy with related (zero state) eigenfunction, which spans the complementary space.

In a weak H(-1/2)-sense (with inner product **((**u,v**))**) the Schrödinger momentum operator **P**=-ihgrad corresponds to the -ih**R** operator in a weak H(0)-sense (with inner product (u,v)), i.e. **((P**u,v**))**=(**R**u,v) for all elements u,v of the ("location" & test) Hilbert space H(0) (see e.g. (EsG) examples 3.1-3.5). We emphasis, that the Riesz operators are rotation invariant.

The above alternative quantum mechanics model enables "truly symmetric" (selfadjoint) creation and annihilation operators in case of the harmonic quantum oscillator: the eigenfunction of the zero state of the harmonic oscillator is given by the Gauss-Weierstrass function.

The Heisenberg uncertainty relation states that it is impossible to determine in a system all observable values simultaneously (at the same time) with any accuracy of observation. This principle is still valid with respect to the H(-1/2) system, while all possible observable values (the eigenvalues of the creation and annihilation operators) are determined in the test space H(0)=L(2). This is just and only not possible for the zero state energy value, which is anyway not being specified uniquely in a physical sense in all existing physical models.

Beside the special role of the fractional Hilbert spaces H(+/-1/2) one other reference to the above RH solution is the fact, that for a Hilbert transformed function v=H(w)=H(-Hv) with v,w elements of H(0), (whereby e.g. the Dawson function is the Hilbert transform of the Gauss-Weierstrass function), is the fact, that it holds (v,w)=0 and that the commutator (xH-Hx)(v)=0 vanishes.
At the end, we claim that Schrödinger was right with his not accepted statement (see M. Born, "The interpretation of quantum mechanics"), that "the only reality in physical world is "waves". The concept of "waves" in our proposal are generalized waves with respect to a Hilbert space H(-a), a>0. Particles and energy quanta h*v appear as (a kind of discrete) "projections" (/"collapse") from a "H(-a)-wave" into the subset Hilbert space H(0), enabling physical observation interpretations as "resonance phenomena of interfering "waves" in this subset Hilbert space.

The proposed approach to be solved the Yang-Mills equations and the mass gap problem is about rotation-invariant differentials as elements of a Hilbert space with negative (distributional) Hilbert scale defined by the eigenpairs of the Hamiltonian operator.

*The analogy to the above switch from H(1) to H(1/2) energy space becomes probably best visible by the model problem of the harmonic quantum oscillator, which goes along with the solutions of the wave equation for strings. Those are equations for oscillators with frequencies w(n), which increase in the size of "n" as n tends to infinite. The spectrum of an oscillator is about equidistant eigenvalues h*w*(n+1/2) with the key differentiator to the classical harmonic oscillator of a non-vanishing ground state energy E(0)=SUM(h*w(n)/2), which is a divergent series. Therefore the energy H(1)-norm is infinite (which requires identities in the form SUM(n)=-1/12 as the prototype for the renormalization technique), while an energy norm H(1/2) would support a finite energy per definition. The link to the RH solution is the divergent Fourier series representation of the cot(x)-function, which can be interpreted as the derivative of the log(sin(x/2))-function. The latter one is represented by a convergent Fourier series and an element of the H(0)=L(2) Hilbert space, i.e. cot(x) is an element of H(-1), resp. has a finite energy norm with respect to the H(1/2)-norm.*

The mean of a physical observable "(A)" in the state u (with state space inner product (u,v) and norm (u,u)=1) of an element u of the state space H is represented by a linear, Hermitian operator A in the form ("(A)")(u)=(u,Au). The link to the above is a replacement of H:=H(0) by H:=H(-1/2).

The proposed approach for a quantum gravity model is about rotation-invariant differentials as elements of a Hilbert space with negative (distributional) Hilbert scale defined by the eigenpairs of the Hamiltonian operator.

The link between the RH-proof-P2 and an appropriate vacuum energy model is given by the Hilbert space H(-1/2) with its corresponding energy Hilbert space H(1/2). The (to be extended tool set) of "quantization" operators are Pseudo-Differential Operators with domains H(-a), a>0 ((AhM) II § 13).

The link to the Yang-Mills theory (Coulomb potential, Yukawa potential, Dawson function, confluent hypergeometric functions) is given in

Braun K., "*An alternative trigonometric integral representation of the Zeta function on the critical line*", Note 34

Plemelj's alternative concepts of a "mass element" and its related potential enable 1-forms as domain of Pseudo-Differential Operators (PDO), defining graph and energy norms of corresponding Hilbert spaces. Same (resp. isometric) Hilbert spaces (with not positive Hilbert scale factor) build the framework of variational theory to solve Partial Differential or Pseudo Differential equations.

Our baseline proposition is to declare the weak representations of PDE or PDO equations as the "truly" physical (world) models (not the strong PDE model) and declare the strong PDE as related (macro world) approximations to the weak representations. The prize to be paid for the macro world approximation is a more restrictive regularity assumption than physically necessary. Then there is a consistent (Hilbert space) model (framework) between quantum & macro world equipped with the geometrical properties of a Hilbert space, providing e.g. an inner product, alternatively to the external product of differentiable manifolds, and enabling e.g. spectral theory.

Therefore, the proposed solution approach is b*a*sed on

1. a problem adequately defined, fractional scaled (energy) Hilbert space, singular integral operators and Plemelj's alternative "potential" definition.

2. a relationship between the asymptotic behavior of the imaginary parts of the Riemann zeta function on the critical line and the large complex zeros of the Jost function in the complex wave number-plane for s-wave scattering by truncated potentials.

ad 1: Plemelj's concept is about an alternative definition of a "mass element", which requires less regularity assumptions as the definition of a "mass density". At the same time Plemelj's alternative normal derivative definition, which he called a "current" ("*continuity*" instead of "*differentiability*" assumptions) ensures a valid Green formula. Plemelj's "mass element" is one-to-one linked to a differential, i.e. a (truly infinitesimal) Hilbert space framework can be designed for 1-forms. This solution concept is built on the solution concepts of P1 and P2.

ad 2: in (JoS) a variant of the Hilbert-Polya conjecture is proposed in this context and considerations about the Dirac sea as “virtual resonances” are discussed. We note that the regularity of the "Dirac sea" is about the distributional Hilbert space H(-n/2+a), a>0, while "our" distributional Hilbert space is about H(-1/2).

One key differentiator to standard theory framework is about the fact, that the Legendre transformation is no longer applicable. Therefore, the Lagrange and the Hamiltonian formalism are no longer equivalent; in fact, the Lagrange formalism is no longer defined in the infinitely small "area".

A new Hilbert scale based truly infinitesimal small geometry is provided, which is proposed to replace the Semi-Riemannian (metrical space) differentiable (!) manifold concept (and its underlying gauge theory to enable the Standard Model of Elementary Particles) by a Hilbert space based quantum (differential forms) ground state energy model, building on an intrinsic ground state energy scalar product.

We consider the 1-dimensional periodic L(2)-functions on the unit circle. Let T denote the normal derivative operator of the double-layer potential, which is a bounded, selfadjoint integral operator ((KrR), theorem 8.21). We apply Plemelj's alternative definitions of a "mass element" and his alternative definition of a potential (PlJ) to define an inner product for 1-form du, dv in the form

((du,dv)):=(Tu,Tv) .

The inner product is well defined for u, v being elements of H(0)=L(2), only. Plemelj's alternative normal derivative definition, which he called "current", ensures compatibility with the Green formula, even for less regular functions (L(2)-functions) u and v.

Based on the above the Yang-Mills functional can be formulated in a Hilbert space framework as (standard) minimization problem with respect to the energy or operator norm.

For u being an element of H(1/2) the energy norm (Tu,u) and operator norm (Tu,Tu) are equivalent to the norms of H(1/2) and H(0)=L(2). In case of the reduced regularity assumption that u is only an element of H(0)=L(2), this "generates" a energy norm and a operator norm, which are equivalent to H(0) and H(-1/2), only.

The linkage to quantum mechanics is given by the fact, that the "quantum elements" are represented as elements u and v of the Hilbert space H(0)=L(2).

In case of the standard regularity assumptions, i.e. if u and v are elements of H(1), the inner product is identical to the Dirichlet integral. This is due to corresponding properties of the Hilbert transform, i. e. it holds

((du,dv))=(Tu,Tv)=(H(dx)(u),H(d/dx)(v)=((dx)(u),(d/dx)(v)

i.e. T = H(d/dx).

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