A mathematical framework
for an integrated gravity and quantum model,
2011-2017

The conceptual idea

The Berry-Keating conjecture is about an unknown quantization ***H*** of the classical Hamiltonian H=xp, that the Riemann zeros coincide with the spectrum of the operator 1/2+i***H***. This is in contrast to canonical quantization, which leads to the Heisenberg uncertainty principle and the natural numbers as spectrum of the harmonic quantum oscillator. The Hamiltonian needs to be self-adjoint so that the quantization can be a realization of the Hilbert-Polya conjecture.

The central concept is about a proposed alternative harmonic quantum energy model which enables a finite "quantum fluctuation = total energy". The model is based on a fractional (distributional) Hilbert space framework, enabling a self-adjoin Hamiltonian operator. It provides a truly infinitesimal geometry overcoming current handicaps of the manifolds framework of Einstein`s field equations (*differentiable* (?) manifolds resp. exterior (differential) algebra), while, at the same time, the Hilbert space provides a closed subspace of the L(2) test space, which enables continuous spectra.

The provided solution goes along with other proposed solutions in the context of the Riemann Hypothesis and the Navier-Stokes Equations i related homepages. The key idea is to replace the L(2)=H(0)-based quantum state (Hilbert space) frame by the distributional H(-1/2) Hilbert space frame (while the theorem of Wigner keeps still valid). The H(0) Hilbert space still keeps to be the observation/test space, while the orthogonal closed sub-space H(-1/2)-H(0) provide an alternative model for the current Dirac "function" concept, enabling mathematically defined wave package, which still can be "tested" in a H(0) Hilbert space frame. This overcomes current challenges of the Dirac theory regarding the regularity of the Dirac "function", as the regularity of this "function" depends from the space dimension: the Dirac "function" is an element of the Hilbert space H(-n/2-e), e>0. At the same point in time, the H(-1/2) Hilbert space provides an alternative inner product for (weak) variational representations of PDE or Pseudo-Differential equations. Its corresponding energy norm minimization problems correspond to the Hamiltonian principle and can be applied to any classical field equations (i.e. the quantum mechanics "correspondence principle" is still valid). The corresponding Cauchy problems allow less than H(0)-regularity assumptions to the initial value "functions", i.e. "functions"/elements of H(-1/2), which are not part of the test space.

1. Current gravity model & its handicaps

The main characteristics of current gravity model and its related handicaps regarding the physical model requirements are

a. metric space, affine connexions
handicap: no scalar fields (vector fields, only), no (infinitesimal) geometry

b. differentiable manifolds
handicaps: physical justification is only about continuous manifolds, additional regularity requirements are purely mathematical model driven

c. exterior differential forms, exterior product, exterior algebra
handicaps: no geometry, gravitational collapse and space-time singularities not covered adequately; if physical singularities in space-time are not to be permitted (R. Penrose) inside such a collapsing object at least one of the following holds

- negative local energy occurs
- Einstein's equations are violated
- the space-time manifold is incomplete
- the concept of space-time loses its meaning at very high curvatures, because of quantum phenomena

2. Current quantum model & its handicaps

The main characteristics of current quantum model and its related handicaps regarding the physical model requirements are

a. separable Hilbert space
handicaps: location and momentum operator have different domains (separable Hilbert spaces) leading to non-vanishing related commutator and the Heisenberg uncertainty principle

b. the Dirac function
handicap: Dirac (delta) function regularity depends from the space dimension (due to the Sobolev embedding theorem)

3. An alternative mathematical framework

a. A separable distributional (quantum state) Hilbert space H(-1/2) with slightly better regularity than Delta function Hilbert space (independently from space dimension n and valid for all cases n), where L(2)=H(0) test space is a closed sub-space of it. In other words, Dirac's Location Operator is replaced by the orthogonal projection from H(-1/2) Hilbert space onto test space H(0)

b. The standard derivative definition (momentum operator) is replaced by a Calderon-Zygmund (convolution, singular integral) PDO of order 1. In other words, Schrödinger's momentum Operator is replaced the orthogonal projection of the Hilbert space H(1/2) onto the test space H(0)

c. The Dirac function concept (H(0)-inner product of a "function" and its related Fourier transform) is replaced by the inner product of an element of separable distributional Hilbert space and its dual element of the corresponding domain of the momentum operator.

Rational

The proposed mathematical framework above is supposed to provide a truly infinitesimal geometry (H. Weyl). A physical interpretation could be about "rotating differentials" ("quantum fluctuations"), which corresponds mathematically to Leibniz's concept of monads. Its mathematical counterpart is the ideal points (or hyper-real numbers). This leads to non-standard analysis, whereby the number field has same cardinality than the real numbers. It is "just" the Archimedian principle which is no longer valid. This looks like a cheap prize to be paid, especially as hyper-real numbers might provide at least a proper mathematical language for the "Big Bang" initial value "function" and its related Einstein-Hilbert action functional. Looking on hyper-real numbers from the "real" number perspective one must admit to classify the term "real" as a contraction in itself, if it is understood as *real*. Already the existence of each irrational number (not only the transcendental numbers; and the cardinality of the irrational numbers is different from the rational numbers) is ensured by an axiom, "only", i.e. the "empty space" between two rational numbers is filled with infinite irrational numbers with same cardinality as the field of real numbers itself, i.e. with multiple "universes". The difference of real numbers to hyper-real numbers is "just" the fact that there are additionally infinite small and large numbers "existent", ensured "just" by a second axiom.

Some more details

We propose a fractional (energy) Hilbert space H(1/2), which already plays an elegant role in universal Teichmüller theory. It is also related to the bounded variation functions. Ist dual space with respect to the L(2) Lebesgue Hilbert space is the H(-1/2) Hilbert space. The latter one is the proposed quantum state Hilbert space. While the Hermite polynomials (or it Hilbert transforms) build an orthogonal system of the Hilbert space L(2)=H(0) (and related discrete energy spectrum) the basis of the Hilbert space H(-1/2) requires an additional eigenfunction with continuous spectrum. This "eigen-pair" is proposed to be a model for the dark energy model, given by the common (additional) root operator of the ladder "symmetric" operators ("Erzeugungs- und Vernichtungsoperatoren"; "Bosonische und Fermionische Kletteroperatoren"). By this the "symmetry" theory is also anticipated, as the current particles zoo ("materialized" in H(0) Hilbert space) has all the time the same symmetry partner (field), i.e. the "eigen-function" with continuous spectrum, which spans the closed sub-space H(-1/2)-H(0).

The integral of all frequencies of the proposed harmonic quantum oscillator model is finite (which is not true for the current model (!)), while the Heisenberg uncertainty relation is still valid in the distributional Hilbert space, while allowing discrete momentum-location measurements in sub-space H(0). The "measurement" Hilbert space (which is identical to the statistics modelling Hilbert space) can be interpreted as projections from the H(-1/2) (state) space to its sub-space H(0).

At the end, this mathematical model is claimed to enable a new quantum gravity model replacing complex (not sufficient) mathematical models (e.g. differentiable manifolds with dimensions greater than 10, M-Theory, (Super) String theory, loop quantum gravity, all of them w/o any physical interpretations, by a (at the end, 4 dimensional Minkowski-like) Hilbert space providing not only a metric (exterior differentials), but also a geometry (with inner product). It's still beyond human imagination and open for corresponding physical interpretations, but providing a consistent mathematical model (and therefore an appropriate language) combining the "very large" with the "very small" (R. Penrose).

Along with the alternative Hilbert space H(-1/2), as a model for the quantum states two additional conceptual changes are proposed to apply the same Hilbert space alternatively to the current gravity theory framework (differentiable manifolds & affine connexions).

1. Already for the space dimension n=1 the Dirac Delta „function“ is not an element of the (newly proposed quantum state) Hilbert space H(-1/2). This is due to the Sobolev embedding theorem. Therefore, the H(-1/2) Hilbert space concept does not require the Dirac Delta „function“ concept anymore and the Hilbert space extention from H(0) to H(-1/2) enables an alternative wave package concept (with regularity requirements independently from the space dimension). At the same time the model also enables an alternative interpretation of the Neutrinos and their relationship/interaction with Fermions and Baryons. At the same time the concept enables an alternative to current symmetry breaking concept to explain the generation of matter in the early phase of the universe, where energy is required to generate matter w/o violating energy conservation laws out of (massless) photons. Mathematically speaking this is enabled by the compact embedding of the quantum state space H(-1/2) into H(0), which is compactly embedded into the energy space H(1/2).

2. Einstein's field equations are based on the concept of differentiable manifolds and physical terms are described (indirectly) by vector fields. In case scalar fields would exist this enables direct interpretation/verification of observations and measurements with the mathematical model. The standard scalar field are the real numbers, whereby the term „real“ includes also the irrational numbers and its subset, the transcendental numbers. Ordered fields that have infinitesimal elements are called non-Archimedian. As the distance of two real numbers cannot be infinitely small, G. W. Leibniz argued that the theory of infinitesimals implies the introduction of ideal numbers (monads). This leads to the Non-standard analysis resp. to the concept of hyperreal fields. The differentials (1-forms) can also be brought into relationship to the Hilbert space H(-1/2).

Nota bene

The classical Yang-Mills theory is a generalization of the Maxwell theory of electromagnetism where the *chromo*-electromagnetic field itself carries charges.

As a classical field theory it has solutions which travel at the speed of light so that its quantum version should describe massless particles (gluons). However, the postulated phenomenon of color confinement permits only bound states of gluons, forming massive particles. This is the mass gap. Another aspect of confinement is asymptotic freedom which makes it conceivable that quantum Yang-Mills theory exists without restriction to low energy scales. The problem is to establish rigorously the existence of the quantum Yang-Mills theory and a mass gap.

Identifying "fluids" or "sub-atomic particles" not with real numbers (scalar field, I. Newton), but with hyper-real numbers (G. W. Leibniz) enables a truly infinitesimal (geometric) distributional Hilbert space framework (H. Weyl) which corresponds to the Teichmüller theory, the Bounded Mean Oscillation (BMO) and the Harmonic Analysis theory. The distributional Hilbert scale framework enables the full power of spectral theory, while still keeping the standard L(2)=H(0)-Hilbert space as test space to "measure" particles' locations. At the same time, the Ritz-Galerkin (energy or operator norm minimization) method and its counterpart, the methods of Trefftz/Noble to solve PDE by complementary variational principles (A. M. Arthurs, K. Friedrichs, L. B. Rall, P. D. Robinson, W. Velte) w/o anticipating boundary values) enables an alternative "quantization" method of PDE models (P. Ehrenfest), e.g. being applied to the Wheeler-de-Witt operator.