Hardy Spaces, Hyperfunctions,
Pseudo-Differential Operators and Wavelets

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The Hardy Space and Hilbert Scales

Let

$$\Omega := \{ s = \sigma + it | \sigma > 1/2, -\infty < t < \infty \}$$

then the Riemann Hypothesis is the statement that $1/\zeta(s)$ is analytic on the half-plane Ω. The appropriate Hilbert space framework is the Hardy space $H^2(\Omega)$ of all analytic functions F on Ω such that

$$\|F\|^2 = \sup_{\sigma > 1/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\sigma + it)|^2 dt < \infty \; .$$

Any $F \in H^2(\Omega)$ has almost everywhere on the critical line a non-tangential boundary value function $F^*(t) := \lim_{\sigma \to 1/2} F(\sigma + it) \in L^2(R)$ (defined almost everywhere) such that

$$\|F\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F^*(\sigma + it)|^2 dt < \infty \; .$$

Thus the Hardy space $H^2(\Omega)$ may be identified via the isometric embedding $F \to F^*$ with a closed subspace of the L^2-space of the critical line with respect to the Lebesgue measure scaled by the factor $1/2\pi$. We note that the operator H of the previous section applied to a complex-valued function produces its conjugate complex function.

One defines the Fourier-Mellin transform $M : L^2((0,1]) \to H^2(\Omega)$ by:

$$M(f)(s) := \int_0^\infty x^{s-1} f(x) dx \quad , \quad f \in L^2((0,1]) \; , \quad s \in \Omega \quad ,$$

whereby M is an isometry.

The related Fourier-Hilbert scale theory is built on the Riemann mapping theorem. This asserts that any open region in the complex plane, bounded by a simple closed loop, can be mapped holomorphically to the interior of the unit circle

$$D := \{ z \| z | < 1 \} \quad ,$$

the boundary being also mapped accordingly.

Due to a result from Hardy the mean function

$$\mu_\delta(r) := \frac{1}{2\pi}\int_0^{2\pi} \left|u(re^{i\varphi})\right|^\delta d\varphi \quad , \quad \delta > 0$$

is increasing, i.e. it's either divergent or is bounded, as $r \to 1$ for $u(z)$ being a regular, analytical function to the interior of the unit circle, i.e. on the open disk $D := \{z \| z| < 1\}$. Then the Hardy space $H_2(D)$ consists of those functions, whose mean square value on the circle of radius remains bounded as $r \to 1$.

Let $H_2^*(D)$ be the Hardy space of L^2 functions on the unit cycle Γ with an analytical continuation inside the unit disk D. The inner product is defined as follows:

$$\langle u, v \rangle := \frac{1}{2\pi}\oint_\Gamma u(t)\bar{v}(t)dt \; .$$

For a point $z \in D$ let $e_z(t)$ a set of functions defined by

$$e_z(t) := \frac{1}{\bar{z}e^{it} - 1} \; .$$

Applying the Cauchy integral formula then the functions $e_z(t)$ define a linear continuous mapping

$$C : H_2^*(\Gamma) \to H_2^*(D)$$

of functions on Γ to an analytical function in D defined by:

$$\hat{u}(z) := [C(u)](z) := \langle u, e_z \rangle$$

whereby

$$\langle u, e_z \rangle = \frac{1}{2\pi}\oint_\Gamma u(t)\,\overline{(\frac{1}{\bar{z}e^{it} - 1})}dt = \frac{1}{2\pi}\oint_\Gamma \frac{u(t)ie^{it}}{z - e^{it}}dt = \frac{1}{2\pi i}\oint_\Gamma \frac{u(y)}{z - y}dy \; .$$

This mapping C is an isometry of the Hilbert spaces $H_2^*(\Gamma)$ and $H_2^*(D)$, where the inner product on $H_2^*(\Gamma)$ is defined by

$$\langle u, v \rangle := \lim_{r \to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{it})\bar{v}(re^{it})dt \; .$$

The mapping C could be inverted by an operator

$$C_*^{-1} : H_2^*(D) \to H_2^*(\Gamma) \quad , \quad u(t) = \lim_{r \to 1} \hat{u}(re^{it})$$

The reproducing operator $S_P := C_*^{-1}C$ on $H^2(\Gamma)$ is called the *Szegö singular integral operator*. Considered on $H_2^*(\Gamma)$ the Szegö operator S_P is an orthogonal projection on its closed sub space $H_2^*(\Gamma)$. For $H = L_2^*(\Gamma)$ and its closed vector subspace $H^* := H_2^*(\Gamma) \subset L_2^*(\Gamma) = H$, the following characterization holds true

$$u \in H_2^*(\Gamma) \quad \text{if and only if} \quad u_\nu = 0 \text{ for } \nu < 0 \quad .$$

Supposing that $\tilde{u} \in H_2^*(\Gamma)$, i.e. that $\tilde{u}$ has Fourier coefficients with $\tilde{u}_\nu = 0$ for $\nu < 0$, then the element u of the Hardy space associated to $\tilde{u}$ is the holomorphic function

$$u(z) = \sum_0^\infty u_\nu z^\nu \quad , \ |z| < 1 \ .$$

The properties of the Hilbert transform leads to

$$(H[u])(z) = \sum_1^\infty u_\nu z^\nu \quad , \ |z| < 1 \ .$$

Remark: For a complex valued function $2\pi -$periodic function $f(\varphi) = u(\varphi) + iv(\varphi)$ its conjugated function can be represented by

$$\bar{f}(\varphi) = -\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \oint_{\varepsilon,\pi} f(\varphi + \vartheta) - f(\varphi - \vartheta) \cot \frac{\vartheta}{2} d\vartheta = \frac{1}{2\pi i} \oint_{0,2\pi} f(\vartheta) \cot \frac{\varphi - \vartheta}{2} d\vartheta \ .$$

Let $a_0; a_n, b_n$ be the Fourier coefficients of f. Then $0; -b_n, a_n$ are the Fourier coefficients of its conjugate and it holds

$$\frac{1}{\pi} \int_0^{2\pi} f^2(\varphi) d\varphi = \frac{a_0^2}{2} + \frac{1}{\pi} \int_0^{2\pi} \bar{f}^2(\varphi) d\varphi \qquad \text{resp.} \qquad \frac{1}{\pi} \int_0^{2\pi} \bar{f}^2(\varphi) d\varphi = \sum_1^n a_n^2 + b_n^2 \ .$$

Remark: In the one-dimensional case hyperfunctions are the distributions of the dual space $C^{-\omega}$ of the real-analytical functions of a real variable C^ω, defined on some connected segment $\subset R$. In the one-dimensional case any complex-analytical function, as any distribution f on R, can be realized as the “jump” across the real axis of the corresponding in $C - R$ holomorphic Cauchy integral function

$$F(x) := \frac{1}{2\pi i} \oint \frac{f(t)dt}{t - x} ,$$

given by

$$(f, \varphi) = \lim_{y \to 0^1} \int_{-\infty}^{\infty} F(x + iy) - F(x - iy))\varphi(x) dx \quad .$$

Example: The principle value $P.v.(1/x)$ of the not locally integrable function $\frac{1}{x}$ is the distribution g defined by ([BPe])

$$(g,\varphi):=\lim\int_{x\geq\varepsilon}\varphi(x)\frac{dx}{x}=\int_{-\infty}^{\infty}\log|x|\varphi'(x)dx \qquad \text{for each } \varphi\in C_c^{\infty} .$$

The relation of this specific principle value to the Fourier transform is given by

$$\left[P.v.(\frac{1}{x})\right]^{\wedge}=-i\pi\,\mathrm{sgn}(t) \quad \text{and} \quad \left[P.v.(\frac{1}{x})\right]^{\wedge\wedge}=-2\pi P.v.(\frac{1}{x}).$$

Remark: The Dirac distribution "function" can be interpreted as the "jump" across the real axis of a corresponding holomorphic Cauchy integral function in $C-R$:

Lemma: If $\varphi\in C_c^{\infty}$ and $\rho>0$ then

$$-\int_{-\infty}^{\infty}\arg(x+iy)\varphi'(x)dx=\int_{-\infty}^{\infty}\frac{y}{x^2+y^2}\varphi(x)dx .$$

In the one-dimensional case hyperfunctions are the distributions of the dual space $C^{-\omega}$ of the real-analytical functions of a real variable C^{ω}, defined on some connected segment $\subset R$. Any real-analytical function is $\in C^{\infty}$, but not every function $\in C^{\infty}$ is analytical, e.g. it holds

$$e(x):=\begin{cases} e^{-\frac{1}{x^2}} & x>0 \\ 0 & x=0 \end{cases}\in C^{\infty} \qquad \text{but} \qquad e(x)\notin C^{\omega} .$$

From $e^{(n)}(0)=0$ for all n for the Taylor series it follows

$$\sum_0^{\infty}\frac{0}{n!}x^n=0,$$

what's different to $e(x)$ except at $x=0$, i.e. $e(x)\notin C^{\omega}$ is not an analytical function. The situation is different in case of complex-analytical functions, which are holomophic and analytical at the same time. This means that the dual (distribution) space $C^{-\omega}$ of the space of the real-analytical functions C^{ω} characterizes the so-called hyperfunctions.

A hyperfunction of one variable $f(x)$ on an open set $\Omega\subset R$ is a formal expression of the form $F_+(x+i0)-F_-(x-i0)$, where $F_{\pm}(z)$ is a function holomorphic on the upper, respectively lower, half-neighborhood $U_{\pm}=U\cap\{z|\,\mathrm{Im}(z)>0\}$, for a complex neighborhood $U\supset\Omega$ satisfying $U\cap R=\Omega$. The expression $f(x)$ is identified with 0 if and only if $F_{\pm}(z)$ agrees on Ω as a holomorphic function.

If the limits exist in distribution sense, the formula gives the natural imbedding of the space of distributions into that of hyperfunctions. Hyperfunctions can be defined on real-analytic manifolds. Fourier series are typical examples of hyperfunctions on a manifold:

(*) $\sum_{\nu\in Z} a_\nu e^{i\nu x}$ converges as a hyperfunction if and only if $a_\nu = O(e^{\varepsilon|\nu|})$ for all $\varepsilon > 0$.

Some examples of generalized functions interpreted as hyperfunctions are

i) Dirac's delta function $\delta(x) = -\frac{1}{2\pi i}\left[\frac{1}{x+i0} - \frac{1}{x-i0}\right] = \pi \lim \int_0^\infty e^{-ak}\cos kx dk$, $a \to 0$

ii) Heaviside's function $Y(x) = -\frac{1}{2\pi i}\left[\log(-x-i0) - \log(-x+i0)\right] = -\frac{1}{2\pi i}\log(-z)$.

The Heaviside function can be characterized ([BPe] B. E. Petersen, 1.16) by

$\lim \log(x+iy) = \log x + i\pi\hat{Y}$ for $y \to 0^+$ and $\hat{Y}(x) = Y(-x)$

iii) $x_\pm^\lambda = \frac{\pm(\mp z)^\lambda}{2i\sin\pi\lambda}$ for $\lambda \notin Z$

$x_\pm^m = \pm\frac{1}{2\pi i}\mp(z)^m \ln(\mp z)$ for $\lambda = m \in Z$

iv) the Feynmann propagator (Green's function) is the solution

$$\frac{1}{2\pi i}(S^\vee - S^\wedge)$$

of the distribution wave equation

$$(\frac{\partial^2}{\partial t^2} - \Delta)S(t,x) = \delta(t)\delta^m(x)$$

with

$$2\pi(2\pi)^m S^\wedge(t,x) = \int\int \frac{e^{-i\omega t + ikx} dk d\omega}{(\omega - |k| - i\varepsilon)(\omega - |k| - i\varepsilon)}$$

$$2\pi(2\pi)^m S^\vee(t,x) = \int\int \frac{e^{-i\omega t + ikx} dk d\omega}{(\omega - |k| + i\varepsilon)(\omega - |k| + i\varepsilon)} \quad .$$

Pseudo-Differential Operators

The class of distributions, which is defined by divergent integrals, is the class of ***oscillatory integrals*** leading to the concept of Pseudo-Differential operator. They are in the form

$$A(x) = \int e^{i\phi(x,\theta)} a(x,\theta) d\theta \ ,$$

where the phase function $\phi(x,\theta)$ is a suitable real valued function such that the integrand oscillates rapidly for large $|\theta|$ and the amplitude function $a(x,\theta)$ being allowed to have polynomial growth in θ. It would be too restrictive to require the integral to define a function. Therefore it's interpreted in the distribution sense. Thus one is actually be concerned with integrals of the type

$$\langle A, v \rangle = \int\int e^{i\phi(x,\theta)} a(x,\theta) v(x) dx d\theta \ .$$

The study of the Hilbert transform and the study of operational calculus for non-commuting operators in quantum mechanics contain some basic ingredients of the theory of pseudo-differential operators.The Hilbert transform is a classical pseudo-differential operator with symbol $-isign(s)$. Its salient features enabled the introduction of the algebra of singular integral operators.

Singular distributions can be generated by Hadamard's "finite part" of a divergent integral; a technique for extracting a finite part from a divergent part, building pseudo functions appying Cauch's principle value concept), where it turns out that this finite part defines a singular distribution. We note (if $\phi(0) \neq 0$) the "finite part" representation

$$Fp \int_{-\infty}^{\infty} \frac{\phi(t)}{|t|} dt = \lim_{\varepsilon \to 0} \left[\left(\int_{-\infty}^{\infty} + \int_{\varepsilon}^{\infty} \right) \frac{\phi(t)}{|t|} dt + 2\phi(0) \log \varepsilon \right] .$$

Holomorphic functions in the distribution sense are defined in the following way:

Definition : Let $z \to g_z$ be a function defined on a open subset $U \subset C$ with values in the distribution space. Then g_z is called a holomorphic function in $U \subset C$ (or $g(z) := g_z$ is called holomorphic in $U \subset C$ in the distribution sense), if for each $\varphi \in C_c^\infty$ the function $z \to (g_s, \varphi)$ is holomorphic in $U \subset C$ in the usual sense.

The phase function $\phi(x,\theta)$ of oscillatory integrals is a suitable real valued function such that the integrand oscillates rapidly for large $|\theta|$ and the amplitude function $a(x,\theta)$ being allowed to have polynomial growth in θ. It would be too restrictive to require the integral to define a function.

Wavelet

A wavelet is a function $\psi(x) \in L_2(R)$ with a Fourier transform which fulfills

$$0 < c_\psi := 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \ .$$

Classical Hilbert spaces in complex analysis are examples of wavelets, like Hardy space of L_2 functions on the unit circle with analytical continuation inside the unit disk.

The wavelet transform of a function $f(x) \in L_2(R)$ with the wavelet $\psi(x) \in L_2(R)$ is the function

$$W_\psi[f](a,b) := \frac{1}{\sqrt{c_\psi}} \int_{-\infty}^{\infty} f(t)\overline{\psi}_{b,a}(t)dt = \frac{1}{\sqrt{c_\psi}} \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{a}}\overline{\psi}(\frac{t-b}{a})dt \ , \quad a \in R - \{0\}, b \in R$$

For a wavelet $\psi(x) \in L_1(R)$ its Fourier transform is continuous and fulfills

$$0 = \hat{\psi}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(t)d\omega$$

The wavelet transform to the wavelet $\psi(x) \in L_2(R)$

$$W_\psi : L_2(R) \to L_2(R^2, \frac{dadb}{a^2}) \ ,$$

is isometric and for the adjoint operator

$$W_\psi^* : L_2(R^2, \frac{dadb}{a^2}) \to L_2(R)$$

$$W_\psi^*[g](a,b) := \frac{1}{\sqrt{c_\psi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t)\frac{1}{\sqrt{a}}\psi(\frac{t-b}{a})g(a,b)\frac{dadb}{a^2}$$

it holds $W_\psi^* W_\psi = Id$ and $W_\psi W_\psi^* = P_{range(W_\psi)}$.

The continuous wavelet transform is known in pure mathematics as Calderón's reproducing formula, i.e. for $\psi(x) \in L_1(R^n)$ real and radial with vanishing mean, i.e.

$$\int_0^{\infty} \frac{|\hat{\psi}(a\omega)|^2}{a} da \equiv 1 \ .$$

For

$$\psi_a(x) := \frac{1}{a^n}\psi(\frac{x}{a})$$

it holds Calderón's formula

$$f = \int_0^{\infty} \psi_a * \psi_a * f \frac{da}{a} \ .$$

Riemann-Stieltjes integral densities and Hyperfunctions

We briefly sketch the link between Riemann-Stieltjes integral densities and hyper functions and distributions in order to motivate the several following definition: Let $\sigma(\lambda) := \|E_{\lambda} x\|^2$ in $\lambda \in (-\infty, \infty)$ be a bounded variation spectral function, which builds according to the Green function

$$G(z) = \int \frac{d\sigma(\lambda)}{\lambda - z}$$

the two holomorph Cauchy-Riemann representation in $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) < 0$ by

$$G(x+iy) - G(x-iy) = \int \left[\frac{1}{\lambda - (x+iy)} - \frac{1}{\lambda - (x-iy)} \right] d\sigma(\lambda)$$

Then the Stieltjes inverse formula is valid for continuous points a and b, i.e.

$$\sigma(b) - \sigma(a) = \lim_{y \to 0} \frac{1}{2\pi i} \int_a^b G(x+iy) - G(x-iy) dx \quad .$$

If there exist a spectral density functions $\sigma'(\lambda)$, it holds

$$\sigma'(\lambda) = \lim_{\mu \to 0^1} \frac{1}{2\pi i} \left[G(\lambda + i\mu) - G(\lambda - i\mu) \right] \quad .$$

In the one-dimensional case any complex-analytical function, as any distribution f on R, can be realized as the "jump" across the real axis of the corresponding in $C - R$ holomorphic Cauchy integral function

$$F(x) := \frac{1}{2\pi i} \oint \frac{f(t)dt}{t - x},$$

given by

$$(f, \varphi) = \lim_{y \to 0^1} \int_{-\infty}^{\infty} F(x+iy) - F(x-iy))\varphi(x) dx \quad .$$