Quantum cosmology for pedestrians

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The application of quantum theory to the description of the universe as a whole is known as quantum cosmology. A brief, self-contained introduction to this field, accessible to an upper-level undergraduate physics student is presented. Perhaps the most remarkable quantum-cosmological idea—that the universe originated *ex nihilo* via a quantum-mechanical tunneling process—is discussed, and the probability for such a quantum cosmogenesis is calculated.

I. INTRODUCTION

"Why are we here?" "Where did it all come from?" These two questions are perhaps the most fundamental to be asked by our species. In recent years, a number of physicists and cosmologists have begun to develop a new description of the ultimate system, the universe as a whole. Although the first question still lies beyond the purview of physics, "quantum cosmology," an amalgam of quantum theory and classical relativistic cosmology, now offers a scientific answer to the second. Indeed, describing the Universe in terms of *both* Einstein's general theory of relativity and quantum theory leads to what is perhaps the most remarkable quantum-cosmological idea, that the universe originated via quantum-mechanical tunneling—the process behind nuclear alpha decay or the scanning tunneling microscope. Like any quantum system, the universe is to be described by a quantum-mechanical state vector which encapsulates all the "knowable" information about that system. At the heart of the quantum cosmological program lies a twofold goal; the discovery of the "correct" state vector—that which predicts (or, quantum-mechanically speaking, assigns a high probability to) a late universe with the features that we observe, and an understanding of the mechanism by which that particular state is selected from the set of all possible states. The foundations of this program were laid in the 1960s, with the work of DeWitt,[1] Wheeler,[2] and Misner.[3] Largely unexplored, the field underwent a rebirth in the early 1980s, when classical cosmology proved unable to explain fully the universe's very early history. The seminal idea that quantum processes were responsible for the birth of the universe was introduced by Tryon,[4] in 1973. The idea of quantum tunneling of the universe as a whole was first presented by Atkatz and Pagels,[5] and has been further developed by Vilenkin.[6]

Our universe is fundamentally quantum in nature; the "observed" classical laws governing systems are obtained from the underlying quantum laws in the limit $h \to 0$ (h is Planck's constant). This is the correspondence principle. For particles this means that the de Broglie wavelength $\lambda = h/p$, where p is the particle's momentum, is small in comparison to the characteristic length of the physical system. If the length scale of a system is much larger than the relevant wavelength, classical laws suffice for its description. Why, then, should we apply quantum theory to the study of the entire universe? The answer is provided in the standard hot big bang cosmological scenario—the universe is expanding, and so was smaller in the past. At very early times the universe was small indeed—for example, some current models predict that at the "Planck time,"[7] about 10^{-44} s after the big bang, the observable universe, whose size today is roughly 10^{28} cm, was no larger than about 10^{-53} cm. Compare this to a proton, of diameter 10^{-13} cm! Thus there was a brief but extraordinarily important period in the evolution of the universe in which the classical approximation may not be applied, and quantum effects dominated.

The observed expansion leads to another problem associated with the birth of the universe: the initial singularity, the state of zero size and infinite density which inevitably arises when running "big-bang-the-movie" backwards. At the singularity the classical laws of physics "break down," and lose their predictive power. And yet those very laws, when applied to the universe as a whole, imply the inevitability of the initial singularity—it cannot be avoided classically. Quantum cosmology provides a way out—the avoidance of the initial singularity, and the very origin of the universe, may well be due to quantum processes.

II. CLASSICAL VERSUS QUANTUM

The classical behavior of a system is determined by solving, subject to the appropriate initial conditions, the "equations of motion"—one second-order, or two first-order differential equations for each degree of freedom. These equations possess a "first integral," the total energy,

$$E = \frac{1}{2} m \left(\frac{d\mathbf{r}}{dt} \right)^2 + V(\mathbf{r}), \tag{2.1}$$

which is a constant of the motion. This is a first-order differential equation, and may be integrated immediately (for simplicity, we consider the one-dimensional case);

$$t(x) - t_0 = \sqrt{\frac{m}{2}} \int^x \frac{dx'}{\sqrt{E - V(x')}} . \tag{2.2}$$

The "turning points," at which the potential energy equals the total energy, $E = V(x)$, give the limits of the motion; there the velocity vanishes. Suppose a particle of total energy $E < V_0$ is incident from the left on a rectangular potential barrier of height V_0 (Fig. 1). The turning point occurs at the potential wall, and the particle is reflected; the barrier delineates a "forbidden region" which the particle cannot traverse. The particle will never be found in the classically allowed region to the barrier's right.

The quantum-mechanical analysis yields a different result. The particle is described in each region by a wave function $\psi(x)$, which satisfies the time-independent Schrödinger equation,

$$\left(\frac{d^2}{dx^2} + \frac{2m}{\hbar^2} [E - V(x)] \right) \psi(x) = 0, \tag{2.3a}$$

$$V(x) = \begin{cases} V_0 & |x| \le L \\ 0 & |x| > L \end{cases} . \tag{2.3b}$$

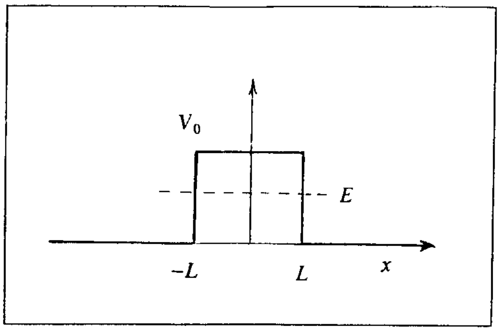

Fig. 1. A rectangular potential barrier in one dimension. A particle of energy $E<V_0$ is incident from the left-hand side.

In the left-hand region, the solution, ψ_L, represents an incoming plane wave (normalized to unit amplitude) and a reflected wave; $\psi_L=e^{ikx}+Re^{-ikx}$, while in the right-hand region there is only a transmitted wave, $\psi_R=Te^{ikx}$. In both cases the de Broglie wave number $k \equiv \hbar^{-1}\sqrt{2mE}$. Localized wave packets which provide a better quantum-mechanical description of a particle, can, of course, be constructed from these plane waves.

In the classically forbidden region, the wave function consists of growing and decaying exponentials, $\psi_F=Ae^{\kappa x}+Be^{-\kappa x}$, where the real parameter $\kappa \equiv \hbar^{-1}\sqrt{2m(V_0-E)}$.

Since the probability of finding the particle between x and $x+dx$ is given by the absolute square of the wave function, $|\psi(x)|^2$, quantum theory predicts a strikingly different behavior—there is a nonzero probability of finding the particle on the far side of the barrier—it may "tunnel" through the forbidden region. The probability of finding the particle on the far side (the probability of tunneling) is given simply by

$$P=\frac{j_{x>L}}{j_{\text{inc}}}=|T|^2, \tag{2.4}$$

where

$$j=\frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right) \tag{2.5}$$

is the probability current.

It is sufficient for our purposes to evaluate P in the limit $\kappa L\gg 1$; i.e., when $V_0\gg E$. After matching the values of the wave function and its first derivative at $\pm L$, straightforward algebra yields

$$P\propto e^{-4\kappa L}. \tag{2.6}$$

Suppose now the potential is no longer piecewise constant, but rather is of arbitrary shape. In general, the Schrödinger equation may be solved exactly only for a few special choices of the potential $V(x)$. For arbitrary potentials we must turn to some approximation scheme. If the potential is "slowly varying," such that its change over a de Broglie wavelength is much smaller than the kinetic energy,

$$\lambda\,\frac{dV}{dx}\,\frac{1}{|E-V(x)|}\ll 1, \tag{2.7}$$

the WKB or "semiclassical" approximation[8] is valid. In this case the (lowest order in $\hbar$) approximate solutions are

$$\psi(x)\propto\frac{1}{\sqrt{k(x)}}\exp\pm i\int^x dx'k(x'), \tag{2.8}$$

where $k(x) \equiv \hbar^{-1}\sqrt{2m[E-V(x)]}$. Near the classical turning points $|E-V(x)|$ is small and the WKB solution does not apply; there a linear approximation to the potential is used in order to relate the oscillatory and exponential wave functions which obtain to either side of these points. These are the familiar WKB "connection formulas." To the right-hand side of the barrier there is only a transmitted wave; the connection formulas generate the forbidden-region exponential wave function, and, applied again to that, the oscillatory wave to the barrier's left. Having obtained the incident and transmitted waves, a simple calculation of the tunneling probability follows; we obtain (again in the $V\gg E$ limit)

$$P\simeq\exp\left\{-\frac{2}{\hbar}\int_a^b dx\sqrt{2m[V(x)-E]}\right\}; \tag{2.9}$$

a and b are the classical turning points. This expression reduces to Eq. (2.6) for the rectangular potential barrier, Eq. (2.3b). The "tunneling probability of the universe" will be obtained from the expression (2.9).

III. RELATIVISTIC COSMOLOGY

In 1917, only one year after general relativity's introduction, Albert Einstein wrote a short paper called "Cosmological Considerations on the General Theory of Relativity."[9] This modeling of an entire universe as a four-dimensional (three spatial and one temporal) manifold, a "spacetime" whose geometry is determined by the matter distributed throughout, ushered in the era of "modern cosmology."[10] Spacetime geometry is encapsulated by the *line element*, ds, which gives the separation between two infinitesimally close spacetime points. The "flat" Minkowski line element of special relativity is simply

$$ds^2=-c^2\,dt^2+dx^2+dy^2+dz^2, \tag{3.1a}$$

or, in spherical polar coordinates,

$$ds^2=-c^2\,dt^2+dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2), \tag{3.1b}$$

and apart from the sign of the first term, is just a straightforward extension of the Pythagorean theorem to four, rather than two, dimensions. However, as Einstein showed, in the presence of matter the flat geometry of spacetime will be distorted—matter curves spacetime. Then the line element is more complicated, and in its most general form is written[11]

$$ds^2=g_{\mu\nu}(x)dx^\mu\,dx^\nu. \tag{3.2}$$

Here $g_{\mu\nu}(x)$, a second-rank tensor, is called the *metric*, the components of which are ten[12] independent functions of the four spacetime coordinates. It is important to understand the relationship between spacetime points and coordinates. Coordinates are merely *labels* assigned to the points. The intrinsic geometry of the spacetime—and hence the physics—depends only upon the infinitesimal separation between points, and not on the manner in which the points are labeled. While any choice of coordinate system is permissible, in dealing with a particular problem a particular choice of coordinates may be preferred—the mathematical formalism will be simpler. If spacetime is flat, there always exist Cartesian coordinates in which the line element has the form (3.1a). For example, it is a simple matter to write down the spherical-to-Cartesian coordinate transform which takes Eq. (3.1b) to Eq. (3.1a). If no such coordinate transformation exists, then the spacetime geometry is curved. The metric, and hence the curvature, is determined by solving the Einstein equations—ten coupled, nonlinear, partial differential equations—which relate the geometry of spacetime to the distribution of mass energy. Symbolically,

$$G_{\mu\nu}=8\pi G T_{\mu\nu}. \tag{3.3}$$

The *Einstein tensor*, $G_{\mu\nu}$, is a function of the metric and its first and second spacetime derivatives; the left-hand side thus represents the geometry. The source term, $T_{\mu\nu}$, is called the *stress-energy tensor*, and it represents the matter and energy distribution (G is the familiar Newton–Cavendish gravitational constant). These are the dynamical equations of cosmology, the "equations of motion" for the geometry of a universe.

It is believed that on large enough scales the universe is homogeneous and isotropic—that is, that all points are equivalent, and, viewed from any particular point, the universe "looks the same" in all directions. This symmetry simplifies the form of the line element considerably; the universe may then be described by the Friedmann–Robertson–Walker (FRW) line element,

$$ds^2=-c^2\,dt^2+a^2(t)\left(\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\right). \tag{3.4}$$

A comparison with Eq. (3.1b) shows only two changes: the part of the metric containing the squares of the spatial coordinate differences has been multiplied by the square of the *cosmic scale factor*, $a(t)$, and the dr^2 term's denominator now differs from unity. The scale factor is a measure of the universe's size, is a function only of time, and is the only undetermined function to appear. The universe's curvature is determined by the value of the *curvature parameter*, k, which may take any one of three values. A FRW universe is negatively curved, flat, or positively curved, as $k=-1$, $k=0$, or $k=1$. It is not known for certain which value describes our universe; the value must be inferred by direct astronomical observation. The coordinates r, θ, and ϕ are "comoving"—as the scale factor grows, the universe expands—the "proper" distance, Δs^2, between two spacetime points increases, although the *coordinates* of those points do not change. In a FRW universe, the distant galaxies have fixed coordinates; the galaxies recede not because they move *through* space—the shrapnel of the big bang "explosion"—but because spacetime itself is expanding, dragging the galaxies along for the ride. Viewed from any particular galaxy, any other appears to recede with a velocity proportional to its distance. The proportionality is given by Hubble's constant, H_0, named in honor of the discoverer of this cosmic expansion.

In FRW coordinates the four-dimensional spacetime has been "foliated" into a time sequence of "three-geometries"—one homogeneous, isotropic three-dimensional "space" exists at every instant of time. (This, of course, is the way we naturally view the world.) The "sheaf" of three-geometries so formed constitutes the spacetime. Even cosmologists cannot visualize four-dimensional universes—the best we can do is to picture two-dimensional surfaces curved through a nonphysical third dimension.

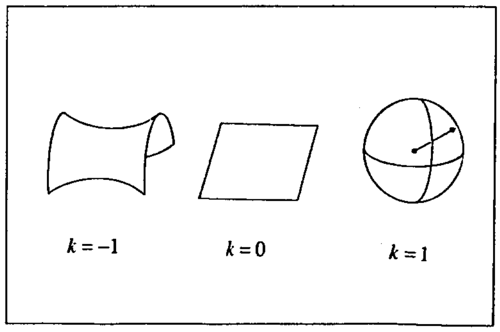

Fig. 2. Two-dimensional analogs of the FRW universes. Only the surfaces have meaning—the third dimension in which they are embedded is purely to aid visualization. The radius of the $k=1$ sphere is the scale factor $a(t)$.

Thus, we view the spatial geometry at a particular instant in time—a single "leaf" in the foliation—while suppressing one of its dimensions. These surfaces are the two-dimensional analogs of the FRW universes, and are shown in Fig. 2. The $k=-1$ saddle surface and the $k=0$ flat sheet are "open," or infinite in extent (a two-dimensional resident of these universes could travel forever in any direction), while the $k=1$ spherical surface is "closed" and finite (our traveler would eventually return to her starting point).

Some cosmologists find the spherical universe the most satisfying, perhaps because it is finite and may be pictured in its entirety. For this universe, the scale factor $a(t)$ is simply the radius of the sphere—as the scale factor grows, the (two-dimensional) universe blows up like a balloon. Keeping in mind that the universe is only the surface of the sphere, it is easy to see that the expansion happens everywhere—there is no "center of the universe," where the big bang occurred, and from which all the galaxies fly. (This is true for the open universes as well.) Although impossible to visualize, the three-dimensional spatial geometries behave in precisely the same manner as their lower-dimensional counterparts.

IV. CLASSICAL EVOLUTION OF THE UNIVERSE

We now turn to the dynamics of the FRW cosmology. The isotropy and homogeneity of the FRW universe require that the stress-energy tensor be that of a "perfect fluid," which has a particularly simple form; matter is parametrized completely by two functions, its density, $\rho(a)$, and its pressure, $p(a)$. The FRW line element contains a single metric function, the scale factor, which depends only on time; there are in this case two independent Einstein equations (here and in what follows we work in natural units[13] such that $\hbar=c=1$, and an overdot denotes differentiation with respect to time),

$$\dot{a}^2=\frac{8\pi G\rho}{3}\,a^2-k, \tag{4.1a}$$

and

$$\ddot{a}=-\frac{4\pi G}{3}\,(\rho+3p)a. \tag{4.1b}$$

Eliminating $\ddot{a}$ between them yields the *continuity equation*,

$$\frac{d\rho}{da}=-\frac{3}{a}\,(\rho+p). \tag{4.2}$$

It is usual to take Eqs. (4.1a) and (4.2) as the dynamical equations of big-bang cosmology. These two equations are insufficient to determine the three unknown functions, $a(t)$, $\rho(a)$, and $p(a)$, and must be supplemented by a third equation, the matter *equation of state*,

$$p=p(\rho). \tag{4.3}$$

Two matter configurations are of particular importance; electromagnetic radiation, and a pressureless dust. Their equations of state may be concisely written

$$p(\rho)=\frac{\gamma}{3}\rho, \quad \gamma=\begin{cases} 1 & \text{radiation,} \\ 0 & \text{matter (dust),} \end{cases} \tag{4.4}$$

and upon substitution into the continuity Eq. (4.2), we find the behavior of the energy density in a FRW universe

$$\rho(a)=\rho(a_i)\left(\frac{a_i}{a}\right)^{\gamma+3}, \tag{4.5}$$

where a_i is the value of the scale factor at an arbitrary reference time. We insert Eqs. (4.4) and (4.5) into the Einstein Eqs. (4.1a) and (4.1b), to find

$$\dot{a}^2=\frac{\text{const}}{a^{\gamma+1}}-k, \tag{4.6a}$$

and

$$\ddot{a}=-\frac{\text{const}}{a^{\gamma+2}}. \tag{4.6b}$$

It is a relatively simple matter now to determine the time evolution of a FRW universe. We consider the three cases, $k=-1$, $k=0$, and $k=1$. Notice that Eq. (4.6b) implies the deceleration of the universal expansion, independent of the value of k. For the negatively curved universe, the right-hand side of Eq. (4.6a) never vanishes, $\dot{a}^2$ is always greater than zero, and (since we know $\dot{a}>0$ now) the universe expands forever. For the flat universe, $\dot{a}^2$ vanishes only as $a\rightarrow\infty$; again, the universe expands eternally. The spherical, closed universe expands until $\dot{a}=0$ at which time the negative definiteness of $\ddot{a}$ ensures that collapse begins. The behavior of the scale factor for the three cases is shown in Fig. 3.

Since our goal is to treat this (relativistic) cosmology quantum mechanically, we will describe matter in the language of relativistic quantum field theory. We will treat the simplest possible quantum cosmogenesis—the spontaneous birth from "nothing" of an empty, closed universe. As has been discussed by Guth,[14] an *empty* newborn universe is all that is required—the "inflationary scenario"[15,16] can explain the creation of matter early on in the universe's evolution. Atkatz and Pagels[5] showed that only a *closed* universe can arise via quantum tunneling. Thus we set $k=1$, and take as the only contribution to ρ the *constant* vacuum energy density, ρ_{vac}. Lorentz invariance of the vacuum implies the equation of state $p=-\rho_{\text{vac}}$; the continuity Eq. (4.2) is then trivially satisfied. It is customary to define the cosmological term,

$$\Lambda\equiv 8\pi G\rho_{\text{vac}}, \tag{4.7}$$

and the Einstein equation finally becomes

$$\dot{a}^2+\left(1-\frac{\Lambda}{3}a^2\right)=0. \tag{4.8}$$

So long as $\rho(a)$ is constant or slowly varying, the universe expands exponentially—the solution of Eq. (4.8) is

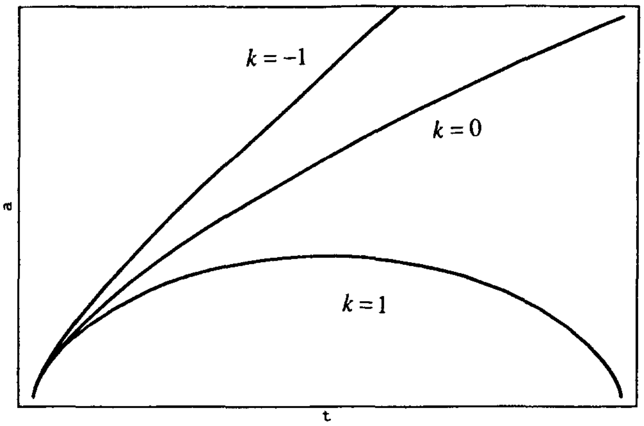

Fig. 3. Time dependence of the FRW scale factor $a(t)$.

$$a(t)=a_0\cosh(a_0^{-1}t), \tag{4.9}$$

where $a_0\equiv\sqrt{3/\Lambda}$. This is Guth's inflationary epoch, an extraordinarily brief period of extraordinarily rapid expansion—when the Universe is about 10^{-34} s old, for a period of 10^{-30} s or so, the density is sufficiently constant for the universe to expand by a factor of roughly 10^{50}. (Aside from this brief "glitch" extremely early on in the universe's history, inflationary models are identical to the standard big bang cosmology.)

V. QUANTIZATION AND THE WHEELER–DEWITT EQUATION

Nature does not "quantize;" it is intrinsically quantum. "Quantization" is a formal mathematical activity reserved for theoretical physicists who, through no fault of their own, have first developed their physical intuition observing the world in the limit of large quantum numbers. Thus we do not know how to write down a quantum theory *a priori*, but the classical theory provides the framework for discovering the underlying quantum theory. How, then, do we quantize a particular cosmology? How is any classical system quantized? So long as the potential depends only on the coordinates, the Hamiltonian, $H(p,x)$, is just the total energy expressed as a function of the coordinates and the canonical momenta. Consider for simplicity a one-dimensional system, for which the Hamiltonian takes the form

$$H(p,x)=\frac{p^2}{2m}+V(x)=E. \tag{5.1}$$

By replacing the coordinate, x, and its canonically conjugate momentum, p, by their corresponding (configuration space) operators,

$$p\rightarrow\hat{p}=-i\frac{\partial}{\partial x}, \tag{5.2a}$$

$$x\rightarrow\hat{x}=x, \tag{5.2b}$$

we construct the Hamiltonian operator, $\hat{H}$. The energy, E, is replaced by the energy operator,

$$E\rightarrow\hat{E}=i\frac{\partial}{\partial t}, \tag{5.3}$$

and the quantum equation of motion—the time-dependent Schrödinger equation—obtains by allowing these operators to act on a wave function, $\psi(x,t)$; viz.

$$\hat{H}\psi(x,t)=\left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right)\psi(x,t)=i\frac{\partial}{\partial t}\psi(x,t). \tag{5.4}$$

This procedure, known as "canonical quantization," is the one we follow when quantizing the FRW cosmology.

The Einstein Eq. (4.8) has precisely the same form as the Hamiltonian for a zero-energy particle whose position is described by a coordinate a, i.e., the left-hand side consists of two terms, a "kinetic energy" term proportional to the square of the "velocity," $\dot{a}$, and the potential energy term—a function of the coordinate only. To quantize we must replace the momentum conjugate to a by its corresponding operator, according to

$$p_a\to\hat{p}_a=-i\frac{\partial}{\partial a}. \tag{5.5}$$

The Einstein equations may be obtained via Hamilton's principle—the principle of least action. The Hilbert–Einstein action—the gravitational action—is that function of the metric and its first and second derivatives whose variation yields the Einstein equations; they are the Euler–Lagrange equations for that action. In the case of the closed FRW universe the action takes the form[17]

$$S_{\text{grav}}\equiv\int dt\,L_{\text{grav}}=\frac{3\pi}{4G}\int dt\left[-\dot{a}^2a+a\left(1-\frac{a^2}{a_0^2}\right)\right], \tag{5.6}$$

from which the canonical momentum, p_a, is easily obtained

$$p_a\equiv\frac{\partial L_{\text{grav}}}{\partial\dot{a}}=-\frac{3\pi}{2G}\dot{a}a. \tag{5.7}$$

The Einstein Eq. (4.8) may be written[18]

$$p_a^2+\left(\frac{3\pi}{2G}\right)^2a^2\left(1-\frac{a^2}{a_0^2}\right)=0, \tag{5.8}$$

or, upon quantization,

$$\left[\frac{\partial^2}{\partial a^2}-\left(\frac{3\pi}{2G}\right)^2a^2\left(1-\frac{a^2}{a_0^2}\right)\right]\psi(a)=0. \tag{5.9}$$

This is the Wheeler–DeWitt equation,[1,2] and its solution, $\psi(a)$, is known as the "wave function of the universe."[19] In our derivation of the Wheeler–DeWitt equation we have restricted ourselves to the $k=1$ FRW universe, the spatial geometry of which is a three-sphere whose radius is the scale factor, a. In this simple case, then, the Wheeler–DeWitt wave function depends upon that single *number*, a. It is important to keep in mind, however, that a here completely characterizes a three-geometry. In general the argument of the Wheeler–DeWitt wave function is an entire three-geometry, which must be characterized by a *set of functions*, the metric. Had we not restricted ourselves to this simple, maximally symmetric three-geometry, the Wheeler–DeWitt wave function would be a functional[20] on the infinite-dimensional space of all possible three-geometries, a manifold known as *superspace*; every point on superspace represents a three-geometry. If we restrict, via symmetry constraints, for example, the three-geometries under consideration, the wave function becomes a functional over a subspace of superspace—a *minisuperspace*. If the class of three-geometries is sufficiently restricted, the minisuperspace may be finite dimensional; thus, in our simple model $\psi(a)$ is a function defined on the one-dimensional minisuperspace, the half-line $0<a<\infty$.

In classical physics a particle's trajectory is determined through a knowledge of both its position, $x(t)$, and the canonically conjugate momentum, $p(t)$. In the quantum domain, x and p are replaced by noncommuting operators, implying an uncertainty relation between those observables. Quantum fluctuations in position and momentum forbid a precise knowledge of both. The classical trajectory becomes fuzzy and ill defined. The notion of trajectory loses its meaning—it is purely a classical concept.

In a like manner, for our canonically quantized FRW cosmology, we have chosen operators such that $[\hat{p}_a,\hat{a}]=-i$, thus the uncertainty relation,

$$\Delta p_a\Delta a\geqslant\tfrac{1}{2}, \tag{5.10}$$

is satisfied, and it is impossible to simultaneously specify, to arbitrary precision, both the scale factor a and (since $p_a\propto\dot{a}a$) its rate of change, $\dot{a}$. As a knowledge of both is required in order to determine the entire foliation of spacetime discussed previously, the notion of spacetime itself vanishes. This is true in general, not just for our minisuperspace example. A path through superspace, each point of which represents a three-geometry, is a four-geometry—a spacetime. Quantum mechanics forbids the simultaneous specification of the superspace coordinate and its canonically conjugate momentum—the time rate of change of the three-geometry. At the Planck scale, quantum fluctuations in the three-geometry and its rate of change forbid us to speak of a precise trajectory through superspace—it, too, becomes fuzzy and ill defined, and has been given the name "spacetime foam."[21] As the concept of a trajectory in superspace is without meaning, spacetime itself is purely a classical (and thus approximate) concept. As John Wheeler has written[2]

> These considerations reveal that the concepts of spacetime and time itself are not primary but secondary ideas in the structure of physical theory. These concepts are valid in the classical approximation. However, they have neither meaning nor application under circumstances when quantum-geometrodynamical effects become important. Then one has to forgo that view of nature in which every event, past, present, or future, occupies its preordained position in a grand catalog called "spacetime." There is no spacetime, there is no time, there is no before, there is no after. The question what happens "next" is without meaning.

VI. BOUNDARY CONDITIONS AND THE MINISUPERSPACE WAVE FUNCTIONS

The Wheeler–DeWitt Eq. (5.9) is identical to the one-dimensional time-independent Schrödinger equation for a one-half unit mass particle of zero energy subject to the potential

$$V(a)\equiv\left(\frac{3\pi}{2G}\right)^2a_0^2\left(\frac{a^2}{a_0^2}-\frac{a^4}{a_0^4}\right). \tag{6.1}$$

This potential is shown in Fig. 4, where, for convenience, we have chosen $a_0^2=G$. The quantized FRW universe is thus mathematically equivalent to a simple one-dimensional problem in nonrelativistic quantum mechanics—the "particle" at

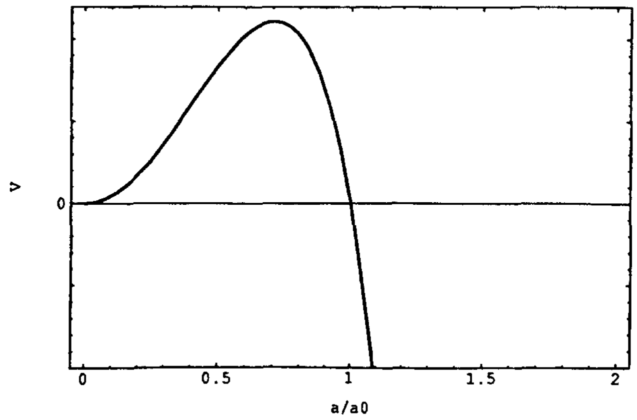

Fig. 4. The potential $V(a/a_0)$ appearing in the Wheeler–DeWitt equation.

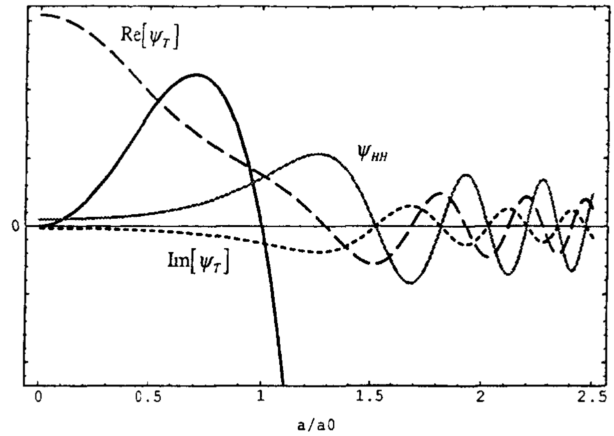

Fig. 5. The tunneling and Hartle–Hawking wave functions, superimposed on the potential. The solid line is $\psi_{\rm HH}$, the broken line is Re[ψ_T], and the dotted line is Im[ψ_T].

position a represents a universe with that value scale factor. The region beneath the barrier, $0<a<a_0$, is classically forbidden to the zero-energy particle; the region $a \geqslant a_0$ is classically allowed.

In order to determine the minisuperspace wave function—to select a particular solution of the Wheeler–DeWitt equation—the boundary conditions must be specified. We wish to consider the birth of an expanding universe via quantum tunneling; thus we will require that in the classically allowed region the solution represents only an outgoing wave; this choice of boundary condition defines the so-called "tunneling wave function," ψ_T.[22] The semiclassical solution describing such a wave is

$$\psi_T(a>a_0) \propto \frac{1}{\sqrt{k(a)}} \exp\left(-i\int_{a_0}^{a} da'\, k(a') + i\,\frac{\pi}{4}\right), \quad (6.2)$$

where $k(a) = \sqrt{-V(a)}$. The factor $\pi/4$ has been explicitly written for convenience in applying the WKB connection formulas; it is a phase factor, and can be absorbed in the wave function normalization. Notice that we have chosen the solution corresponding to negative momentum; as $p_a \propto -\dot{a}$ it is that solution that represents an expanding universe. Applying the connection formulas and performing the integrations, the underbarrier solution is found to be

$$\psi_T(0<a<a_0) \propto \frac{1}{\sqrt{\kappa(a)}} \left\{ \frac{1}{2} \exp\left[-\frac{\pi}{2G}\, a_0^2 \left(1-\frac{a^2}{a_0^2}\right)^{3/2}\right] + i \exp\left[\frac{\pi}{2G}\, a_0^2 \left(1-\frac{a^2}{a_0^2}\right)^{3/2}\right] \right\}, \quad (6.3a)$$

where

$$\kappa(a) = \frac{3\pi}{2G}\, a \left(1-\frac{a^2}{a_0^2}\right)^{1/2}, \quad (6.3b)$$

while the solution in the classically allowed region is

$$\psi_T(a>a_0) \propto \frac{1}{\sqrt{k(a)}} \exp\left[-i\,\frac{\pi}{2G}\, a_0^2 \left(\frac{a^2}{a_0^2}-1\right)^{3/2}\right], \quad (6.4a)$$

where

$$k(a) = \frac{3\pi}{2G}\, a \left(\frac{a^2}{a_0^2}-1\right)^{1/2}. \quad (6.4b)$$

Choosing equal amounts of ingoing and outgoing wave in the classically allowed region leads to the "no boundary" wave function, $\psi_{\rm HH}$, of Hartle and Hawking[19]

$$\psi_{\rm HH}(a>a_0) \propto \frac{1}{\sqrt{k(a)}} \cos\left[\frac{\pi}{2G}\, a_0^2 \left(\frac{a^2}{a_0^2}-1\right)^{3/2}\right], \quad (6.5a)$$

and

$$\psi_{\rm HH}(0<a<a_0) \propto \frac{1}{\sqrt{\kappa(a)}} \exp\left[-\frac{\pi}{2G}\, a_0^2 \left(1-\frac{a^2}{a_0^2}\right)^{3/2}\right]. \quad (6.5b)$$

In both cases the wave function in the classically allowed region is oscillatory, while the underbarrier solution is exponential. The tunneling wave function, which is complex, contains only an outgoing component, and represents an expanding FRW universe, while the Hartle–Hawking wave function, which is real, consists of both ingoing and outgoing pieces, and thus assigns a nonzero probability to a collapsing universe. Such a newborn universe does not survive long enough, however, to produce sentient observers who discover the collapse.

These wave functions, shown in Fig. 5, are selected by what are perhaps the two most widely discussed and easily interpreted boundary conditions. The tunneling and Hartle–Hawking boundary conditions simply restrict the modes present in the classically allowed region. Thus they are applicable to a more general class of quantum-cosmological models than the simple one presented here. In particular, they are applicable as well to nonempty models, in which various types of quantized matter are present. It has been argued, based on these models, that the tunneling wave function predicts a universe in which sufficient inflation has taken place, while the Hartle–Hawking wave function does not.[23] There is, as yet, no consensus on this matter. Indeed, these are by no means the only possible choices. Gibbons and Grishchuk, for example, have investigated a more general set of boundary conditions.[24]

VII. QUANTUM COSMOGENESIS AND THE EMERGENCE OF CLASSICAL SPACETIME

The "particle" at $a=0$—a quantum FRW universe of zero size, or, indeed, our cosmological "nothing"—may quantum-mechanically tunnel through the potential barrier, to appear at $a=a_0$. This tunneling event represents a FRW universe of size (scale factor) a_0 that has quantum mechanically popped into existence; i.e., a universe that has been created spontaneously, and nonsingularly (since it is of finite size). Choosing the tunneling wave function, it is now a simple matter to calculate the probability with which this occurs. If we denote the amplitude for the quantum creation of such a FRW universe by $\langle \mathrm{FRW}(a_0)|\mathrm{nothing}\rangle$, then Eqs. (2.9) and (6.1) yield

$$|\langle \mathrm{FRW}(a_0)|\mathrm{nothing}\rangle|^2 \equiv P$$

$$\simeq \exp\left[-\frac{3\pi}{G}\int_0^{a_0} da\ a\left(1-\frac{a^2}{a_0^2}\right)^{1/2}\right]. \tag{7.1}$$

Evaluating the integral and substituting back for a_0 we find

$$P(\rho_{\mathrm{vac}}) \simeq \exp\left(-\frac{3}{8G^2\rho_{\mathrm{vac}}}\right). \tag{7.2}$$

How are we to interpret this expression? We have only one universe—what does the concept of probability mean for a single system? In fact, the problem is more fundamental than that. The usual "Copenhagen" interpretation of quantum mechanics divides the world into the "observer," a classical system, and the "observed," a quantum system. The defining process is then *measurement*, when the interaction between the classical and the quantum causes the discontinuous "collapse of the wave function," and leaves the quantum system in a particular eigenstate of the observable being measured. When the entire universe is treated as a quantum system, however, there is no longer a place for the classical observer. This problem was first addressed by Everett[25] who developed a formulation in which the wave function evolves continuously, without collapse. Recent work by Gell–Mann and Hartle[26,27] has led to the idea of "decoherent histories," in which quantum mechanics assigns probabilities to *possible* histories of the universe. They have shown that Copenhagen quantum mechanics emerges as a limiting case of their more general framework, when the approximation of dividing the system into classical observer and quantum observed can be taken to be exact, e.g., at sufficiently late times, when the universe is "large."

It is in that light that we shall interpret Eq. (7.2). The history to which it assigns a probability is that of a closed FRW universe of vacuum energy density ρ_{vac} (or, since $\rho_{\mathrm{vac}} \propto a_0^{-2}$ of size a_0) nucleating from the eternally existing nothing, and then proceeding to evolve quantum mechanically. It is most probable that the universe is born with the maximum permissible energy density, or least size. The allowable values of the density depend upon the details of the quantum field theory describing matter and its interactions. However, in order that the semiclassical approximation remain valid, the vacuum energy density must satisfy $\rho_{\mathrm{vac}} \ll \rho_{\mathrm{Pl}}$, where the "Planck density," $\rho_{\mathrm{Pl}} \equiv m_{\mathrm{Pl}}/l_{\mathrm{Pl}}^3$. Thus, we will write $\rho_{\mathrm{vac}} = \alpha\rho_{\mathrm{Pl}}$, and require $\alpha \ll 1$.

The quantum picture of cosmological evolution emerges. The initial singularity at $a=0$, the entire underbarrier region, and a region surrounding a_0 reside purely in the quantum domain. At what point in the universe's development does a well-defined classical spacetime emerge from the spacetime foam? When does a Wheeler–DeWitt wave function Ψ "predict" classical spacetime? In quantum mechanics, of course, predict, means "assign a high probability to," and we are led again to the question of the meaning of probability in quantum cosmology. Hartle[28] has put forth a simple and straightforward "rule" for applying quantum mechanics to a *single* system, rather than, as in the Copenhagen interpretation, an ensemble of systems:

> If the wave function Ψ is sufficiently peaked about some region in configuration space, we predict that we will observe the correlations between the observables which characterize this region. If Ψ is small in some region, we predict that observations of the correlations which characterize this region are precluded. Where Ψ is neither small nor sufficiently peaked, we don't predict anything.

For example, given the measured value of the Hubble constant and mass density, we would like a "good" wave function for the universe to be peaked around a distribution of galaxies consistent with that which is observed. It is crucial to recognize that the wave function does not predict a specific value for H_0, or specific locations for the galaxies, but rather a *correlation* between these observables. Halliwell[29] has shown that an oscillatory wave function of the form e^{iS} predicts a correlation between the canonical coordinate q, and momentum p, of the form

$$p = \frac{\partial S}{\partial q}. \tag{7.3}$$

In the classically allowed region the semiclassical approximation to the Wheeler–DeWitt wave function yields just such an oscillatory solution. The ansatz made for the wave function is

$$\psi(a) = \exp\left(\frac{i}{\hbar} S(a)\right), \tag{7.4}$$

where the phase is a slowly varying function of the scale factor. Since we wish to investigate the classical limit, $\hbar \to 0$, we forego natural units, expand S as a power series in $\hbar$,

$$S = S_0 + \frac{\hbar}{i} S_1 + \left(\frac{\hbar}{i}\right)^2 S_2 + \cdots, \tag{7.5}$$

and insert the ansatz into the Wheeler–DeWitt Eq. (5.9). Equating the coefficients of $\hbar^n$ yields a coupled set of differential equations which may be solved sequentially. The semiclassical approximation to the wave function obtains by working only to first order in $\hbar$. We obtain

$$S_0 = \int^a da'\ k(a'), \tag{7.6a}$$

$$S_1 = \frac{1}{2} \ln\left(\frac{\partial S_0}{\partial a}\right). \tag{7.6b}$$

Thus the oscillatory semiclassical wave function $\psi \propto e^{iS_0}$ is peaked about a region of minisuperspace (every point of which represents a closed FRW universe) in which the correlation between the coordinate and momentum (scale factor and expansion rate), $p_a = \partial S_0/\partial a$, holds good. Using the expression (5.7) for p_a, the correlation reduces to

$$\dot{a}=\left(\frac{a^2}{a_0^2}-1\right)^{1/2}, \tag{7.7}$$

which we immediately recognize as Eq. (4.8), the Einstein equation satisfied by a FRW spacetime. Thus, in our simple example, *in the region of minisuperspace in which the wave function is oscillatory, a classical FRW spacetime, obeying the (classical) Einstein equation, emerges.*

We may arrive at this same conclusion in a somewhat different manner. It is a simple matter to show that to order $\hbar$ in the expansion (7.5) the probability density, $P(x)=\psi(x)^*\psi(x)$, and the probability current $j(x)$, are those appropriate to a statistical ensemble of classical particles of velocity $\nu(x)$, in which the probability of finding a particle in a region of width dx is proportional to the amount of time spent there, i.e., inversely proportional to the particle's velocity. We find

$$P(x)dx=\frac{dx}{\hbar k(x)}\propto\frac{dx}{\nu(x)}, \tag{7.8a}$$

and

$$j(x)=P(x)\nu(x). \tag{7.8b}$$

The nonclassical aspects of the motion—the spreading of the wave packet, the nonexistence of a well-defined trajectory—first appear at order $\hbar^2$. Thus we may say that classical spacetime emerges when terms $S_{n\geqslant 2}$ are small, a condition that will be met provided

$$|d^2S_0|\ll(dS_0)^2. \tag{7.9}$$

Writing $a=na_0$, and remembering that $\alpha\ll1$, Eq. (7.9) implies that a description in terms of classical spacetime will be a good approximation when

$$\frac{9}{16}n^2\frac{(n^2-1)^{3/2}}{2n^2-1}\sim1, \tag{7.10}$$

which is satisfied when the universe has evolved to the point where $a\approx1.75a_0$. A glance at Fig. 5 shows that it is roughly at this point that the wave functions become oscillatory. A classical spacetime, whose evolution is determined by the Einstein equations, has crawled from the quantum ooze.

In presenting the salient features of the quantum cosmological program I have utilized the simplest possible model. There are, of course, many issues I have only hinted at, or, in fact, ignored completely, e.g., the validity of the minisuperspace approximation,[30] the lack of an intrinsic time variable in the theory (the Wheeler–DeWitt equation corresponds to a time-*independent* Schrödinger equation),[31,32] and indeed, the feasibility of quantum mechanically creating a universe in the laboratory![33] The interested reader is encouraged to consult the more extensive review articles of Hartle,[34] and Halliwell,[35] and the references contained therein.

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[7]The "Planck scale," at which classical gravitational theory is no longer a good approximation to the full, underlying quantum theory of gravity, is determined by the fundamental constant, $\hbar$, which sets the scale of quantum effects, and Newton's constant, G, which sets the scale of the gravitational interaction. These, together with the speed of light, c, may be combined to yield the Planck length, Planck time, and Planck mass: $l_{\rm Pl}\equiv(\hbar G/c^3)^{1/2}=1.6\times10^{-33}$ cm, $t_{\rm Pl}\equiv(\hbar G/c^5)^{1/2}=5.4\times10^{-44}$ s, $m_{\rm Pl}\equiv(\hbar c/G)^{1/2}=2.2\times10^{-5}$ g.

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[9]A. Einstein, "Kosmologisch betrachtungen zur allgemeinen relativitätstheorie," translated by H. A. Lorentz *et al.*, *The Principle of Relativity* (Dover, New York, 1952).

[10]See, for example, J. N. Islam, *An Introduction to Mathematical Cosmology* (Cambridge University, Cambridge, 1992).

[11]There is an implied summation over repeated indices; $\mu,\nu=0,1,2,3$; $dx^0=c\ dt$, and dx^1, dx^2, and dx^3 are the three spatial coordinates.

[12]Since the metric is symmetric, i.e., $g_{\mu\nu}=g_{\nu\mu}$, it has ten, rather than sixteen, independent components.

[13]In natural units all physical quantities of interest have dimensions which can be expressed as powers of mass. Thus, for example, length and time have dimension mass^{-1}, and, since $c=1$, energy has dimension mass^1. In these units all Planck scale quantities above are powers of Newton's constant G; e.g., $m_{\rm Pl}=G^{-1/2}$.

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[17]Using the relation $H_{\rm grav}=p_a\dot{a}-L_{\rm grav}$, it is easy to verify that the Einstein equation (3.7) is, indeed, simply the vanishing of the gravitational Hamiltonian.

[18]In obtaining Eq. (5.8) from Eq. (4.8), I have glossed over a subtle point—when is the substitution $p_a\rightarrow\hat{p}_a$ to be made? We could, for example, have quantized $(p_a/a)(p_a/a)+[3\pi/(2G)]^2(1-a^2/a_0^2)=0$, rather than Eq. (5.8), in which case the Wheeler–DeWitt equation would be $\{\partial^2/\partial a^2-1/a\,\partial/\partial a+[3\pi/(2G)]^2a^2(1-a^2/a_0^2)\}\psi(a)=0$. This "operator-ordering ambiguity" does not affect any semiclassical calculations, and so we are free to make the simplest choice.

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Comparing problem solving performance of physics students in inquiry-based and traditional introductory physics courses

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Performance of students in an introductory inquiry-based physics class is compared with that of students in three other introductory physics courses on two different examination problems. One problem is a qualitative problem, typical of those used in inquiry-based physics. The second problem is a quantitative problem, similar to those found in a standard introductory physics text. The students in the inquiry-based physics course were all elementary education majors. They performed significantly better than the engineering students and as well as the honors physics students on the two problems used.

I. INTRODUCTION

Research has shown that many students taught introductory physics in the standard lecture-recitation format learn to solve quantitative problems well (as indicated by good course grades) but do not develop an understanding of physics concepts different from their initial common sense (mis)conceptions.[1–6] If the goal of introductory physics is both teaching students to solve quantitative problems and inducing a correct conceptual understanding, then our courses are not successful.

Courses have been designed which explicitly focus on changing the conceptual understanding of students.[7–12] However, comparing physics courses taught in the traditional format with nontraditional courses is difficult because often the goals, the emphasis, and, therefore, the problems used to assess understanding and knowledge in the two types of courses differ significantly. It is not possible to assess a student's qualitative understanding of physics by examinations which contain only problems requiring quantitative solutions. In order to compare traditionally taught students with those taught nontraditionally, it is desirable to use both quantitative and qualitative problems in assessing both groups of students.

This type of assessment has been done in the context of the calculus-based introductory course at the University of Washington, as part of the evaluation of a lecture-based course with tutorials.[8,13] The course differed from a traditional course only by the replacement of one of the lectures with a tutorial session each week. Unlike traditional recitation sessions, which emphasize quantitative problem solving, the tutorials focus on deepening student conceptual understanding. Students in the course with tutorials performed markedly better than students in a traditional lecture-based course (without tutorials) on both qualitative and quantitative problems.

Previous studies[4,12,13] have compared courses which emphasize quantitative problem solving but are taught in ways designed to improve conceptual understanding as well. Our study compares traditional courses with an inquiry-based course which differs completely from other courses in format, content, and design. *Physics by Inquiry*,[7] a laboratory-based set of modules, uses research on student understanding of physics to guide curriculum design[14] and emphasizes con-

