A Mathematical Framework for a Quantum Gravity Theory: Non-standard Analysis, Differential Forms and Variational Theory

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Abstract

Terms: quantum theory vs. quantum physics [RPe4], quantization of the Riemann Zeta-Function [IAr], Hilbert-transformed Zeta-Fake-Distribution function in the critical stripe [IKBr] and black body radiation, hyperbolic 3-non-Riemannian manifold, semi-Riemmanian manifold, metricity (Ricci identity) & torsion freedom, covariant vs. exterior derivative, Christoffel bracket connexion coefficients vs. non-standard affine connection, minimal surface, torsion and coercive inner geometry principles, (non-ideal) point particles vs. fields/wave, minimal surface vs. torsion-freedom, isometric and geodesic mapping, isotropic curves, imaginary surfaces, eigendifferentials, Non-Standard Analysis, Archimedian principle (i.e. the set of natural numbers is not bounded by a real number).

This note is about a newly proposed physical principle based on Leibniz's philosophy in combination with Heidegger's "World Picture". The mathematics is about Hyperreals and Non-standard Analysis in combination with a newly proposed physical principle which overcomes current issue of (massless) particles having no extentions but energy. By this approach current particle-wave dualism paradoxon should be able to be overcome. A remaining task of the mathematical physics is to model the term "force" in the terms "hyperreal numbers", which then would correspond to Leibniz' vis viva. The "divergence" and "energy flux" have to be adapted with reference to "energized" monads. In a to-be-quantum gravitation model this might be modelled as eigendifferentials. The k-forms are the appropriate concept to build PDE.

In [RPe3] a characterization of the n=4 dimensional space-time structure is mentioned in the context of connections and Riemann geometry and differential forms (the complex rotation group SO(n,C) is not simple for n=4, as for all other n>2, [RSe], 7.6).

In a hyper-real world it might be allowed to consider "2nd order infinitesimals". This would impact the relation between the two equivalent principles: the Lagrange and the Hamiltonian formalism (variation in the direction of the curve), reflecting the physical principles of causality and purpose. The equivalence in current "real" world is proven by the Legendre transformation

$$g := g(x,y) := \psi y - f = y\psi(x,y) - f(x,y)$$

which is built on the Leibniz formula

$$d(xy) = xdy + ydx + dxdy = xdy + ydx$$

A similar relation is used to model infinitesimal displacements in the context of affine connections ([HWe] §12). In the relation above the term "dxdy" is neglected as infinitely small of second order. In a hyper-real world the above might lead to an alternative expression in the form

$$d(g) = (y+dy)d\psi - \frac{\partial f}{\partial x}dx \quad .$$

This means that in such a world the equivalence of both principles is no longer valid. In a hyperreal world the remaining adequate action principle would be the Hamiltonian ("purpose/"energy" minimizing")principle only. This could lead to an appropriate (purely Hamiltonian formalism) hyperreal world model beyong the Heissenberg uncertainty border.

A proper hyperbolic, hyperreal 3-manifold (which seems to be required to become consistent with Huygens' principle) in combination with k-forms is proposed to model an appropriate quantum gravitation model within a variational framework. The manifolds are not built on metrization in combination with torsion-free conditions for co-variant differentials (therefore no semi-Riemannian manifold), but on metrization in combination with appropriate minimal surface defining attributes. The later one is built on a variation argument in direction of the normal. As a side effect this hyperbolic framework might generate specific spherical non-standard waves, replacing current wave models in quantum or gravitation theory.

The principle of general covariance of space-time continuum leads to the algebra of tensors. To enable proper analysis a definition of proper derivative concept is required (then called covariant derivative), which needs an agreement how to compare tangential spaces of two point with infinitesimal small "distance". This leads to the concept of affine connexions in current tensor theory [ESc1], [HWe]. We propose alternatively the framework non-standard analysis and differential forms to overcome current handicaps. In combination with the "future looking term" (y+dy) this might enable a modified field equations enabling geodesic equations where inertia and gravity of masses in movement are both covariant elements.

One of the first to be tackled questions in the proposed framework is, how to model radiation out of "ideal points" (monads).

In a nutshell

Currently there are two one-way roads ending up in quantum theory and gravitation theory. Both together are inconsistent, i.e. back "the road to reality" (R. Penrose) there must have been a branch, where underlying common sense assumptions lead to the two different ways forward ending up in a paradodon (of an empty, filled space). Our idea is based on the following assumptions:

1. "... *modern physics is already itself mathematical...*" (M. Heidegger [MHe2])

2. "... *general relativity and quantum mechanics can be derived from a small set of postulates, one or more of these postulates (continuity, causality, unitarity, locality, point particles) must be wrong*" (Michio Kaku [KaM])

3. ". . . *we observe that the non standard analysis is presented naturally, within the framework of contemporary mathematics, and thus appears to affirm the existence of all sorts of infinitely entities. . . . it appears to us today that the infinitely small and infinitely large numbers of a non-standard model of Analysis are neither more nor less real than, for example, the standard irrational numbers..."* (A. Robinson, 1966)

4. "..."*We may say a thing is at rest when it has not changed its position between now and then, but there is no 'then' in 'now', so there is no being at rest. Both motion and rest, then, must necessarily occupy time...."* Aristotle, 350 BC

5. "...*It is probably the last remaining task of the theoretical physics to show us how the term "force" is completely absorbed in the term "number"..."* (R. Taschner, [RTa])

Point particles (=real numbers) are the wrong physical postulate and should be replaced by ideal point (=Hyperreals), which exist in mathematics [ARo2]. In the field of Hyperreal numbers the Archimedian principle (the set of natural numbers is not bounded by a real number) is no longer valid.

One way of constructing a system incorporating non-standard reals is to define "numbers" as infinite sequences of reals (or equivalence classes thereof). The stae-of-the-art mathematicaly existing quantum objects are the elements of von Neumann's Hilbert space l(2).

6. ... n*othing stops us to turn over the ration and say that Non-Archimedean adding of quantities is the first cause, and Pseudo-Euclidean space is the model, which reflects this more fundamental ration. ..."* (P.V. Polyan, [PPo]). P. Polyan's question

> "... d*o the hyperreal numbers exist in the quantum-relative universe*?"

got an answer by M. Heidegger in the form

> "... *in the mathematical physics, ... yes*".

For this answer it holds by B. Russell [BRu2], chapter 2:

> "... *the only reason to abandon a conviction can be another conviction".*

0. Introduction

From Michio Kaku [MKa] we recall:

Because general relativity and quantum mechanics can be derived from a small set of postulates, one or more of these postulates must be wrong. The key must be to drop one or more of these assumptions about Nature on which we have constructed general relativity and quantum mechanics. Over the years several proposals have been made to drop some of our commonsense notions about the universe: continuity, causality, unitarity, locality, point particles.

We will focuss on the later commonsense notation, which is about "point particle" and move back to Newton and Leibniz where the branch seems to be. Before that we recall from Abraham Robinson (Non-standard Analysis, Amsterdam: North-Holland, 1966**):**

"... it appears to us today that the infinitely small and infinitely large numbers of a non-standard model of Analysis are neither more nor less real than, for example, the standard irrational numbers."

From M. Heidegger [MHe2] we recall the „World Picture"

that „***modern physics*** *is called mathematical because, in a remarkable way, it makes use of a quite specific mathematics. But it can proceed mathematically in this way only because, in a deeper sense, it* ***is already itself mathematical***"

The existence of mathematical framework unifying the concepts "particle & wave", already exist, as non-standard models of arithmetic has been proven by Th. Skolem in 1939. Also non -zero infinitesimal small numbers exist in the mathematical world. Ordered fields (like the real numbers, which provides the framwork for standard analysis, including and accepting ir-rational numbers), that have infinitesimal small elements, are called non-Archimedean, i.e. they do not fulfill the Archimedean principle (i.e. the set of natural numbers is not bounded by a real number).

Those numbers are called *Hyperreals* and the related analysis is the **Non-standard Analysis** [MDa].

The question is about the consequences to current physical principles, especially the causality principle, if a (massless) particlewith energy (a "photon", modelled as real number) will be replace by hyperreals, i.e. Leibniz monads.

„Purpose" and „causality" are building principles in philosophy (utiliarism, rationalism) and physics (Hamiltonian, Lagrange minimization principle) to give explanations for phenomena. A phenomenon is seen as the result of a change from some original status to the phenomeneon. The change itself is thereby driven by an (accepted transcendental) concept called „force". While a „purpose" oriented explanation of a phenomeneon does not need a final route cause, while a causality oriented explanation leads to the question oft he origin cause of everything. In Biology Darwin´s principle is sufficient to explain our human being phenomena just by applying the „purpose" principle. A „causality" principle to explain the existence of our human species is not needed and all existing explanations there are finally not consistent with the „causality“ principle itself, but just religiously to be justified/explained.

We propose to rename the term "purpose" by "convenience", which was introduced by M. Heidegger [MHe2] (*). He used the term "Dienlichkeit" (convenience, subservience) in order to describe the feature from which "das Seiende" (be-ings) "is looking to us" The "be-ings" he defined as the entity/unity of substance and form (object and subject, ..., whereby (to M. Heidegger´s opinion) the meaning how both terms are used in modern languages were already the result of a not complete accurate translation from Greek to Latin. The "thing" is a formed substance, i.e. a synthesis of substance and form. The form is assigned/ related to "rationality" and the substance is assigned/ related to "Ir-rationality".

(*) concerning to the above see also our highlighted (blue/bold) text passages of M. Heidegger [MHe] in the reference document (.doc).

"… *modern physics is called mathematical because, in a remarkable way, it makes use of a quite specific mathematics. But it can proceed mathematically in this way only because, in a deeper sense, it is already itself mathematical.* ..."

"... *Mathematical research into nature is not exact because it calculates with precision; rather it must calculate in this way because its adherence to its object-sphere has the character of exactitude. …"*

The existence of non-standard models of arithmetic was discovered by Th. Skolem in 1938/1938, one year after Heidegger´s publication of „The Age oft he World“. In the mathematical world non-zero infinitesimal small numbers exist, as well. Ordered fields (like the real numbers) that have infinitesimal small elements do not fulfill the Archimedean principle. Such fields are called non-Archimedean. The Non-Archimedean extention of real numbers are the Hyperreals (monads, ideal points) and the related analysis is the Nonstandard Analysis [ARo1/2]. This then closes back the loop to M. Heidegger [MHe2].

1. Causality vs. Purpose

From B. Russell [BRu1], chapter 4 we recall:

"*Activity is to be distinguished from what we mean by causation. Causation is a relation between two phenomena in virtue of which one is succeeded by the other. Activity is a quality of one phenomenon in virtue of which it tends to cause another. Activity is an attribute corresponding to the relation of causality; it is an attribute which must belong to the subject of changing states, in so far as those states are developed out of the nature of the subject itself. It is an actual quality of a substance, forming an element in each state of the substance*."

In current mathematical physics the Hamiltonian and the Lagrange principles are equivalent due to the Legendre transformation. The corresponding mathematical (world) framework are n-manifold. To keep all known forces consistently modeled within this framework it needs to be that each added force into that model increases the dimension of the manifold "world". Current Superstring theory requires the dimension 11. Unfortunately the existing mathematics doesn´t work anymore. For the Superstring theory a to-be-developed mathematic is required. This is a fairly high prize to be paid just to get the 4 nature forces modelled build on current quantum field theory.

Therefore: „why not allow for a moment the following statements:

„Causality" is a specific concept, especially developed by human being only due to Darwin´s principle („survival of the fittest"); „causality" is not at the same level as „learning". Standard Analysis is the appropriate mathematical framework for mathematical physics, which reflects current way of describing human world. „Causality" and physics require experiments and measurements. Measurements applying mathematical arithmetics require the **Archimedean principle**.

„Purpose" is a more general Nature concept in the (transcendental) sense of M. Heidegger ("Dienlichkeit", [MHe2]). It allows also „learning", and not only specifically for human beings. Non-standard Analysis is the appropriate mathematical framework for mathematical, non-Archimedean fields in physical models (which defines the underlying mathematical (and therefore the mathematical physics) existing "thing" of phenomena).

Relations between different physical model formalism describing same or similar physical models are mathematically described by transformations. The probably most known example are the Lagrange and Hamiltonian formalism, which are equivalent..... BUT, the Legendre transformation, which proves this equivalence, depends from the mathematical framework. In this case (as in mostly all current physical models), a key physical concept and model assumption is that of a "particle" (with no physical meaning at all) with its mathematical counterpart of a "real" number. As a "particle" has no physical meaning, nearly all "real" numbers are irrational. On top of this, there is a linkage of the (meta-physical) particle "object" to physical (experimental) observation to describe or/and explain concepts like force, momentum, etc. respectively continuity, differentiability, etc. On the other side, the "transcendental" Dirac "function", which is no function at all, is a well defined accepted object by the physicists, which is (in case of dimension n=1) a distribution of the Hilbert space H(-1/2-e), for e>0. Funny enough to be mentioned, that no physicist seems to have any problem to accept, that the area spanned by the Dirac "function" (which is the x-axis and the positive y-axis) is equal to 1.

All in all the language used by physicists (which is about using "appropriate mathematical terminologies" put into context to describe/explain "observations"), which is given in a certain mathematical framework ("a given stage"), is already anticipating the expected observations resp. the to be observed or described "objects" ("the played scenes of actors on the "given" stage"), before the experiment/observation happens (i.e. "before the act starts"). The gravitation field equations are THE classical example for such a situation.

As all physical core conceptional assumptions are anyway transcendental we propose to re-build the framework, changing

- physical particles, real points replaced by monades, ideal points

- continuous, differentiable functions replaced by Distributions and Hyperfunctions

- 4-dim. vector space replaced by quaternions

- 4-dim. manifolds,11-dim superstrings space replaced by 4-dim. (or 4.5 dim.) integral currents

- Lagrange formalism replaced by Hamiltonian formalism.

It is therefore basically about replacing Newton`s massless (real number) point/particle concept by Leibniz`s (non-standard number) "living" ideal point concept and putting the latter one into the context of k-forms with its relation to partial differential equations and field models. This enables the definition of an appropriate Hilbert space and Hermitean operator framework, which leverage current location & momentum and corresponding eigen differential (with its relation to quantum wave packages and (continuous) spectral theory due to John von Neumann and Paul Dirac)) convergence issues. Werner Heissenberg`s uncertainty principle marks the borderline between standard vs. non-standard Hilbert space, i.e. between the "real" causality oriented and the "ideal" purpose oriented perceived or assumed "world". The objective of this new framework is to enable contiguity in quantum gravitation by an appropriately define "force".

For a first touch and feel related to "Hyper-complex differential form calculus, function theory in quaternions and Clifford algebra (generalized complex analyticity to quaternions by Cauchy-Riemann approach) with weaken classical conditions compared to the classical M-differentiability and complex-analyticity" we refer to R.S. Kraußhar, "A characterization of conformal mappings in R(n) by a formal differentiability condition"

"The exterior derivative d is a closed densely-defined unbounded operator in an appropriate Hilbert space setting, which is connected to the de Rham complex. Due to the Hodge decomposition and the Poincare inequality the mixed (*), weak formulation of the Hodge-Laplacian is well-posed. Leveraging proofs for well-posed PDE in the framework of the calculus of variations might enable an appropriate definition of a mixed formulation of hyperbolic (differential form) gravitation equations, where e.g. well known concepts modelling shock wave singularities/observations can be applied."

As a first conclusion for state-of-the-art mathematical models to describe gravitation theory all „manifolds“, where the Legendre transformation shows the equivalence of „purpose“ and „causality“ are excellence to explain human being world phenomena, but has to fail, when trying to explain quantum gravitation phenomena via „particle-wave dualism“or just simple „time“. The „in-between“ between „particle-wave“ or just „two points in time“ seems to fulfill „its“ purpose quite successful (at least in the sense of Darwin). There is no real reason for a causality of fit. And mathematics provides the framework for a corresponding „World Picture“ with the concept of Hyperreals and Nonstandard Analysis.

Today we think of the set of real numbers as equivalent to the set of points of the real line - a sort of ruler extending endlessly in both directions from the point corresponding to zero. To the ancient Greeks, there were only points corresponding to rational numbers (ratios of whole numbers, e.g., 2/5) and between any two points on a line there were only a finite number of such rational "points". When irrational numbers were discovered, they were deemed "incommensurable", meaning they could not be expressed as such ratios and, in a sense, were non-measurable.

From a mathematical point of view the difference between the field of Hyperreal numbers and real numbers is the missing *Archimedian principle*, which is, that the set of natural numbers is not bounded by a real number. This (missing commosense assumption) might already indicate an apropriate modification of Heissenberg/von Neumann quantum mechanics Hilbert space framework, just by replacing the summation indices out of N by indices out of N*.

As mathematical sophistication increased since Leibniz, the ideas of Cauchy, Weierstrass and others took hold, and monads and moments - in their original guise - faded away. In the Standard Analysis that derived from their work, all real numbers were either rational or irrational, and "infinitesimal" came to mean simply very, very small, but real.

There is an effective limit to the measurability of distances between points that are extremely close together. So, in a sense, there are "spaces" around points in which infinitesimals might reside. Perhaps aspects of logic break down, as they seem to in quantum mechanics, when dealing with microcosmic worlds.

Leibniz’s differential calculus, which enabled the mathematical modelling of continuity and continuous functions in combination with Newton’s massless particle model in the context of his mechanics models (also to describe gravitation) got a great success. At the very end this approach in combination with Einstein’s gravitation theory lead to massless particle, which are purely energy, but still acting as particle, which can only be influenced by independent acting forces on it. On the other side, forces are only measured in such a way, that massless particles are assumed and observed due its behaviour to such forces. In other words: the forces are transcendental, and the massless particles are “real”. Applying continuity is in such a framework at the very end not possible, which is in line with Heissenberg’s uncertainty relation.

Why not turning the whole thing around, gonig back to the complete idea of Leibniz, which lead him to his differential calculus?

Leibniz proposed monads, which are completely independent with its own “internal” living force. The relations between monads are somehow “self-organizing” via a proposed “pre-defined harmony” concept. In modern terminology this could mean, that the first one describes a real continuity beyong our world, where the border is given by Heissenberg’s uncertainty relation (which is valid in the Hilbert-space $L_2 \cong l_2$). The later one proposes instead of “causality” a “purpose”, which is responsible for an appropriate “inter-relation” of the living forces (=monads). The definition of non-standard numbers ([Ro]) is very much related to Leibniz’s monad concept. It is based on infinite series of numbers with certain properties. Trying to make a link already from this to l_2 might indicate relations of non-standard (hyper-real) numbers to quantum theory “particle” modelling.

Another relation to some basic concepts of standard quantum mechanics might be, roughly speaking, moving from a discrete eigenvalues producing bounded, hermitian and positive definite operator to a bounded, hermitian operator only. The Leibniz concept of the monad is per definition transcendental, i.e. additionally, if one would accept a living force as physical “reality” a continuous eigenvalue spectrum becomes physical relevant (from a non empirical perspective), i.e. the spectral theorem (developed by Hilbert, von Neumann, Dirac) can be applied to (see also §18). This would put an alternatve light for instance on modelling the zero point energy.

1. Gravitation and Quantum Theory

Both, gravitation theory and quantum theory provide an accurate model to describe “reality” within their own domain. The theories become inconsistent, when linking them together, i.e. a quantum gravitation theory, which unites both, is missing. In a jounalistic way the current unsolved answers and inconsistencies got the branding “particle-wave dualism”. From a philosophical point of view this is very much in line with Descartes’ philosophy.

The key to solve current inconsistencies must be to drop or change one or more of the commonsense assumptions about Nature (see [Ka] Kaku M., 1.2) i.e. continuity, *causality*, unitarity, locality, point particles, based on which both theories have been constructed.

From a philosophical perspective ([Fi] Fischer K.) the “particle-wave” “duality” is related to the contradictionary concepts of Plato (“ideal” world) and Aristoteles (“empirical” world). The later one provides our today’s state of the art to see and define science. Descartes accepted both “worlds”, but strictly separated between matter and spirit as two different “worlds” without any interacting relations.

Leibniz united and solved Descartes’ “splitted matter and spirit world” into one consistent philosophical model, building on the concept of “monads” ([Fi] Fischer K.). The monads lead also to the “birth” of the mathematical calculus concept, in parallel to Leibniz driven from a physicist’s perspective by Newton leading to the great concept of classical mechanics. The success of the classical mechanics model, built on the concept of an existing “infinitesimal distance”, resulted into a widely acceptance of “real number” as “real” numbers, even those include irrational numbers and an only small part are rational numbers. In combination with the mathematical concept of a “function” related this enabled the definition of terms like “speed” and “momentum”, which lead the mathematical concept of “**continuity**”.

To describe classical mechanics there are the two great (mathematically equivalent) formalisms: it’s the Lagrange (particle mechanics) and Hamiltonian (energy; analytical mechanics) formalism. Both formalisms are from a mathematical point of view “least action principles” or “variational principles” to model (continuous!) motions of particles.

The key observation and “property” of the Hamiltonian formalism is, that the underlying “philosophical” principle of the “least action principle” is not about ***causality***, but about “***<u>purpose</u>***”, i.e. minimizing action along an infinitesimal small distance.

The “ dx ”-concept of differential calculus is only part of Leibniz’s solution of Descartes’ dualism concept. His “full” solution model, the “monad” concept, got in the meantime a mathematical description ([Ro] A. Robinson et. al), which is the non standard analysis, based on non-standard numbers (hyper-reals), which includes the real numbers. The concept of continuity (as most of all other standard mathematical concepts and theories) keeps being valid in that sense, that the restriction to only real numbers gives the “standard” continuity.

Our proposal to solve current particle-wave dualism in a hyperbolic Maxwell and Einstein world is the following:

to apply Leibniz's concept completely and consistently to the Hamiltonian formalism, based on an appropriate extension from "real" to "hyper real" world and based on a new physical principle, which is in line with the key proposition of Leibniz's philosophy:

1. to extent "particle"/"continuity" to the (trancendent) "hyper real" case

Replace real number by "hyper real" number to model a "particle", still without any physical extension. As a consequence the definition (!) of "continuity in a standard sense" (which is to the author's opinion not a philosophical principle, but only a term resp. definition) has to be replaced by "continuity in a non-standard sense".

2. only assume "purpose", not "causality" for the (trancendent) "hyper real" case

modelled by hyperbolic PDE using variational principles (with its underlying concepts of actual and virtual displacements), i.e. "least action principle".

This concept proposes

A. a new (explicit transcendental) physical principle ...

There is only one force in a 4 dimensional (non-standard) space-time continuum, which unites the known existing 4 forces in the nature. The underlying source, which is at the same time the "basic" entity=quantum=inflaton is the "monad", which has its "own" force (Leibniz' "vis viva", which is a transcentendal form). The measurable and realized force in our "real" world is a radiation out of the "monad", mathematically described as a specific sherical wave as solution of a hyperbolic radiation equation in a non-standard analysis framework. All, what's within the monade, i.e. beyond the border from "real" to "hyper-real", is "transcendent" in the sense of Kant, but there is still a principle of "purpose" valid, which garantuees the linkage to our world's causality; as this "purpose" get its realization on this side of the border, both is valid, the same purpose, which now is equivalent to causality, i.e. Lagrange formalism becomes valid, too.

B. ... a mathematical program for a consistent model combining quantum (particle) and gravitation (field) theory

which neccessarily has to unite Maxwell and Einstein equation consistently, overcoming current inconsistencies about nesseccary singular behavior of electronic particle, solving a hyperbolic (radiation) differential equation described by "least action principles" (as Einstein's gravitation equations) in a non-standard analysis framework, modelling a radiation acting out of "monades", which fits to Maxwell's and Einstein's equations. In case this would be successful there then will be a direct link from Maxwell's electron ($U(1)$) (in 4 dimension time-space continuum) to Einstein's graviton (in a 4 dimensional

Einstein space), without having to go via $SU(2) \otimes U(1)$ and potential $SU(2) \otimes O(10)$ models, in which case for each step the number of dimensions has to be increased to ensure consistency until eleven, at least. *The field of real numbers R can be interpreted as the Lie algebra of the compact, one-dimensional group $U(1)$, in which the only relevant attribute of an electron within the Maxwell theory – i.e. the electric charge - is modelled. For more complex particles, which requires additional defining attributes (e.g. color, taste), the "standard modelling approach" requires a higher dimensional Lie group (a Lie group is a group, which is at the same time a manifold to allow differential calculus, Lie derivatives are tensor fields which keep invariant under symmetry operations).*

In [RSe], 7.6, some arguments are given, which characterize the space-time continuum with dimension n=4 in relation to the above.

Affine connections

In standard (tensor) analysis it requires an agreement, how to describe the change behavior of a tensor, when moving from one point P to another point Q. This leads to the definition of the (covariant) so-called covariant derivative. This definition requires athe concept of an affine connected manifold (affine connexions, affinity).

From [Esc1], 5, we recall the remark, that the name for the concept is missleading, and not optimal choosen for two reasons:

- each manifold without an affinity has already with the general coordination transform as affine transformation

- a given Γ provides exactly not a arbitrary affine relation between two neighborhood points, but classifies a specific one in all cases.

An affinity has the following properties:

- it has linear, but not homogen trnasformation behavior

- the difference of two affinities in same continuum is a tensor; therefore the $\delta\Gamma^{k}_{lm}$ of an infinitesimal variation of an affinity is a tensor

- the sum of an affinity and a tensor is an affinity

- there is an infinitesimal scalar ds defined along the geodics.

The special choice/assumption to combine the coordinate differentials of the SRT with the scalar ds leads to the only symmetric affinity, representated by the Christoffel bracket ,

$$\Gamma_{lm}^{k} = \left\{ \begin{matrix} s \\ ik \end{matrix} \right\}$$

which is the only *symmetric affinity* with relation to the fundametal tensor g_{ik} ([Esc1], 9, 8, [HWe]).

The corresponding concepts for k-forms leads to ([RSe] 7.6) the assumptions

- metricity

- rotation freedom.

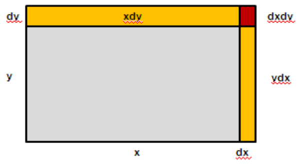
Symmetry ist not the same as duality; our thesis is the following:

duality (e.g. build on appropriately defined non-Riemannian manifolds formulated in a Hilbert scale framework, .. exterior derivative, ... duality fomula for the co-derivative and the "star" operator) is the proper mathematical concept for a (mathematical) ***quantum theory*** and that "***symmetry***" is not a principle, but the outcome of "observing and interpreting the nature" in a (measurable) ***quantum physics*** ([RPe4].

Legendre transformation

Invarant vector and tensor fields, the flow of vector fields, Lie derivative and its geometric interpretation as divergence of vector fields as its infinitesimal volume deformation of its flow require an appropriate non-standard extension in the context of Leibniz's "vis viva". Leibniz formula is already giving non trivial differential calculus in the form

$$d(x+y) = dx + dy \qquad d(xy) = xdy + ydx + dxdy = xdy + ydx \ .$$



In standard analysis the term $dxdy$ *is neglected as infinitely small of second order (!). This might be a first opportunity, when extending k-forms into a non standard framework:*

Lagrange --> Hamilton: $L(x,y) \rightarrow H(x, \frac{dL}{dy})$

The **Legendre transformation** (Lagrange --> Hamilton) of $f(x,y)$ is defined by

$$g := g(x,y) := \psi y - f = y\psi(x,y) - f(x,y)$$

and

$$d(g) = yd\psi - \frac{\partial f}{\partial x}dx + (d\psi dy)$$

The product $d\psi dy$ is neglected to be zero in the standard theory as infinitesimal small of second order compared to dx. If one would neglected this and calculate in a non-standard way it would result into

$$d(g) = (y + dy)d\psi - \frac{\partial f}{\partial x}dx \quad .$$

Does this change the equivalence of Lagrange and Hamilton formalism and therefore enable an appropriate model, when "moving" from the real into the hyper-real world?

Proof:

Putting $\psi := \psi(x, y) := \frac{\partial f(x, y)}{\partial y}$ the differential of $f(x, y)$ gives

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial x}dx + \psi dy$$

As holds $\frac{\partial[y\psi(x, y)]}{\partial\psi} = y$ and $d(\psi y) = \frac{\partial(\psi y)}{\partial\psi}d\psi + \frac{\partial(\psi y)}{\partial y}dy = yd\psi + \psi dy$

it follows $d(g) = d(\psi y) - df = yd\psi + \psi dy - \left[\frac{\partial f}{\partial x}dx + \psi dy\right] = yd\psi - \frac{\partial f}{\partial x}dx + (d\psi dy)$.

The product $d\psi dy$ is neglected to be zero in the standard theory as infinitesimal small of second order compared to dx. If one would neglected this and calculate in a non-standard way it would result into

$$d(g) = (y + dy)d\psi - \frac{\partial f}{\partial x}dx \ .$$

Black-body radiation, Riemann Zeta function

A famous usage of Dirichlet's series is in the context of Planck's black-body radiation function

$$\frac{dR(\lambda,T)}{d\lambda}=\frac{c_1}{\lambda^5}\frac{1}{e^{c_2/\lambda T}-1}=\frac{c_1}{\lambda^5}\sum_1^{\infty}e^{-nc_2/\lambda T}$$

with $c_1=2\pi hc^2$ and $c_2=hc/k$. The relation to the Zeta function

$$\zeta(s)\Gamma(s)=\int_0^{+\infty}\frac{x^s}{e^x-1}\frac{dx}{x}$$

is given by

$$\frac{\pi^4}{90}=\zeta(4)\Gamma(4)=\int_0^{+\infty}x^4(\sum_1^{\infty}e^{-nx})\frac{dx}{x}=\int_0^{+\infty}x^{-4}(\sum_1^{\infty}e^{-\frac{n}{x}})\frac{dx}{x} .$$

This describes the total radiation and its spectral density at the same time, i.e.

$$g(x)dx=\frac{x^{-4}}{e^{1/x}-1}\frac{dx}{x}=\frac{x^4}{e^x-1}\frac{dx}{x}=g(\frac{1}{x})dx .$$

The weak formulation of the Riemann Zeta function [KBr], built on the Hilbert-transformed Theta function (with the related positive Berry conjecture answer based on the weak duality equation of the Riemann duality equation) could enable an alternative model for the total radiation and its spectral density. This in combination with [IAr] could enable a quantum theory model based on a radiation density model allowing singularities.

In [JPl] §8, it is suggested to replace the potential

$$(*)\quad v(s)=-\frac{1}{\pi}\oint\log|\zeta(s)-\zeta(t)|u(t)dt$$

(whereby $\zeta(s)$ is the complex parametrization of $\partial\Omega$) by

$$(**)\quad v(s)=-\frac{1}{\pi}\oint\log|\zeta(s)-\zeta(t)|du(t)$$

with the following argument :

"*Bisher war es üblich, für das Potential die Form(*) zu nehmen. Eine solche Einschränkung erweist sich aber als eine derart folgenschwere Einschränkung, dass dadurch dem Potentiale der grösste Teil seiner Leistungsfähigkeit hinweg genommen wird. Für tiefergehende Untersuchungen erweist sich das Potential nur in der Form (**) verwendbar.*"

This argument might indicate a relation between potentials defined by differentials and the Hilbert transformation (see below), which plays also a key role in [KBr]. (! note that in case of $\partial\Omega=S^1$ it holds

$$\log|\zeta(s)-\zeta(t)|=c\log\sin\frac{s-t}{2} \quad \text{and} \quad \frac{d}{dt}\log\sin\frac{s-t}{2}=-\frac{1}{2}\cot\frac{s-t}{2}).$$

Why a Non-Riemmanian manifold?

Riemannian manifolds

- are smooth, i.e. there exists in every point the 1st and 2nd derivatives

- have a metric g_{ik} and the g_{ik} depend from the coordinates

- locally in every point the manifold is similar to an euclidian space

Some general comments

In the context of an appropriate vector and tensor analysis for monads the eigendifferentials and wave packages, which are key concepts in quantum theory, would become new physical interpretations.

The new concept above would be consistent with Huygens' principles, putting another physical interpretation about the pointwise radiation along front lines and the model of shock waves, providing alternative interpretations of observed diffraction and scattering behaviors.

The new physical principles might overcome current inconsistencies between obervations and quantum field theory projections ging the following explanations:

The zero point energy is proposed to be the radiation out of the "monad", which is the smallest entity, without any extension and relation to other monads (but "more" than an ideal/real point or a string). Using the word "quantum" for such a "monad" the energy density of such a quantum is called in other context as "quantum vacuum". It contains all information and all patterns of dynamic energies of the universe.

The Casimir effect shows a zero point radiation. As a request to an appropriate mathematical model the total energy of such a quantum vacuum should not be divergent.

A photon does not realize any time; how it can act in such a case? As a request to the appropriate mathematical model the asymmetry of "time" in a non-standard hyperbolic world should come out of the specific non-standard spherical "radiation" wave out of the monad.

Hamiltonian principle is linked to skew symmetric bilinear forms. In case such bilinear forms are of maximal rank it's called as symplectic form.

Für die vergangene Singularität, den Urknall, muss in etwa die Bedingung WEYL = 0 gelten, für die zukünftigen Singularitäten, schwarze Löcher, grosses Zermalmen, muss WEYL gegen unendlich gehen. Die herkömmlichen und wohlbekannten Verfahren der Quantisierung lassen keinen Weg erkennen, eine zeitlich asymmetrische, quantisierte Theorie zu erzeugen, wenn die klassische Theorie, auf die diese Verfahren angewendet werden (die etablierte allgemeine Relativitätstheorie oder ihre entsprechenden Modifikationen) ihrerseits zeitlich symmetrisch ist.

Einstein ist auf sein Konzept gekommen aufgrund der folgenden Frage, die er sich gestellt hat: „wenn er sich genau neben einem Lichtstrahl bewegt, wie sieht dann aus seiner Sicht der Lichtstrahl aus?" Es müsste doch eigentlich so sein, dass der Lichtstrahl eine Folge von stationären Wellen gleicht, in der Zeit erstarrt, m.a.W. der Lichtstrahl müsste bewegungslos erscheinen. Wenn man ihm nacheilt, sollte man ihn als ruhendes, räumlich oszillierendes, elektromagnetisches Feld wahrnehmen. So etwas scheint es aber nicht zu geben, weder aufgrund der Erfahrung noch aufgrund der Maxwell-Gleichungen, die keine stationären, erstarrten Wellen zulassen. Aufgrund der Maxwell-Gleichungen müssten Elektronen ihre Energie in Sekundenbruchteilen verlieren.

In any relativistic theory the vacuum, the state of lowest energy, if it exists in „reality", has to have the energy zero.

Ebenso muß für freie Teilchen mit Impuls $\vec{p}$ und Masse m die Energie $E=\sqrt{m^2c^4+\vec{p}^2c^2}$ sein.

Es gibt eine durch die ganze Literatur durchgängige Wahl, dem harmonischen Oszillator die Grundzustandsenergie $\hbar\omega/2$ zuzuordnen. Das wußte Planck bei der Ableitung seiner Strahlungsformel, mit der im Jahr 1900 die Quantenmechanik begann, schon besser: er ordnete Zuständen mit n Photonen die Energie $n\hbar\omega$ zu und nicht den Wert $(n+1/2)\hbar\omega$, der mit relativistisch kovarianter Beschreibung der Photonen unverträglich ist.

Die Grundzustandsenergie ist grundsaetzlich unmeßbar, man verfügt daher über sie so, daß Berechnungen einfach und insbesondere daß sie endlich sind. Energien frei zusammengesetzter Systeme sollten additiv sein. Es gibt in jedem Hohlraum unendlich viele Frequenzen ω_i für Photonen. Ordnet man jeder Frequenz einen Beitrag $\hbar\omega_i\ /2$ zur Grundzustandsenergie zu, so hat schon der Grundzustand ohne Photonen unendlich viel Energie $\sum_i \hbar\omega_i\ /2=\infty$.

Das Mißverständnis, die Grundzustandsenergie liege fest, beginnt in der klassischen Physik. Die Wahl der Hamiltonfunktion

$$H=\frac{p^2}{2m}+\frac{1}{2}\omega^2x^2$$

verfügt über die klassisch nicht meßbare Grundzustandsenergie so, daß der Zustand niedrigster Energie, der Punkt $(x=0, p=0)$ im Phasenraum, die Energie 0 hat. Diese Wahl macht den algebraischen Ausdruck für die potentielle Energie $V(x)$ einfach, man hätte aber genauso gut $V(x)=\frac{1}{2}\omega^2x^2-\hbar\omega/2$ wählen können.

In the following sections we put together several mathematical concepts. There is nothing new compared to e.g. the great book of R. Penrose, but there is some emphasize on specific mathematical aspects to underline the ideas from above. The intention is to give some guidance to move some next steps forward to the “road to reality”, developing the appropriate mathematical model (B). A validation procedure and proof of concept of it the model has to explain/”produce” the fundamental physical constants, which are somehow the gatekeepers to Leibniz’s (and Plato’s) transcendental (i.e. ideal) world.

The framework of the super-gravitation theory is a (4+n)-dimensional space. A unification model of the three elementary forces $SU(3), SU(2), U(1)$ are proposed to be by e.g. $SU(5)$ (which can be interpreted also as Kaluza-Klein theory or $O(10)$ or $E(6)$) replacing the combined $SU(3)\otimes SU(2)\otimes U(1)$ symmetry group. This “GUT trial” requires 24 Yang-Mills fields.

The framework of the super-string theory is an 11 dimensional space. Its key concept is modelling the different particles as real extension (with some distance) of virtual points to a string, having different possible vibration modi (which is basically the same, than proposing different energy levels) to differentiate different possible particle types.

Both approaches do not allow a consistent match to the graviton. Both approaches already use implicitely transcendental objects, which are the mathematical points, either as necessary concept of the underlying mathematical model (which is being seen as proof of concept for the physical concept itself) or as neccessary, imaginary point in a physical field, just refer to, in order to desribe the action of a transcendental force of a field.

Instead of this, following Leibniz, why not proposing that there is only one "true" entity (instead of $photons, gluons, leptons, W^{+}, W^{-}, Z^{0} bosons, gravitons..$), which combines both attributes, "particle" and "living force (vis viva)" "within" one substance, which is the **monad** of Leibniz.

G.W. Leibniz, *Monadologie*, Philipp Reclam Jun., Stuttgart.

From [Fi] Fischer K. we recall a few statements:

A substance (=monad=points de substance; monas, i.e gr. "entity", "unique entity", "Einheit", "das Eine") is a meta-physical point (points de métaphysiques).

A physical point requires "extension" to explain mechanics (which is related to the mathematical concept of a metric)*; a mathematical point has no extension, but its real existence is missing* (which is related the mathematical concept of a field)*. The monad fulfills both requirements. The force has to be seen as substance and the substance can only be thought as force. The force is transcendent, i.e. can not be observed and measured; only the action of force is an observable variable. In a pure world of bodies everything is mechanically. Force is therefore a term, which goes beyond this word (fons mechanismi).*

This concept has to be emdedded in the existing (gravitation) field concept, which is based on Einstein's space-time struce, i.e. a 4 dimensional oriented and time-like oriented Lorentz manifold with a twofold, covariant $C^{\infty}-$tensor field g. For every time-like vector x of such a Lorentz space (i.e. for every vector x of the space-time structure, which fulfils $g(x,x)>0$) the sub-space

$$x^{\perp} := \{y \in E : g(x,y) = 0\} ,$$

equipped with the metric $-g$, is euclidical. For two future-oriented vectors x and y of a Lorentz space with $g(x,x) = g(y,y) = 1$ it holds $g(x,y) \geq 1$.

Two types of "standard" forces

1. There are forces which maintain the given kinematical conditions:

Let W denote the work, $d\vec{s} = \vec{v}dt$ and the momentum (impulse) $\vec{p} = m\vec{v}$. The work of forces to a body is the same as the increase of body´s kinetic energy:

$$dW = \vec{F}d\vec{s} = (\vec{F}d\vec{v})dt = ((m\dot{\vec{v}})d\vec{v})dt = d(\frac{m}{2}\vec{v}^{2}) = (m\vec{v})dt = \vec{p}dt =: dT$$

resp. $$W = \int_{t_1}^{t_2} \vec{F} d\vec{s} = \frac{m}{2}(\vec{v}_2^{\,2} - \vec{v}_1^{\,2}) = T_1 - T_2$$

I.e. W gives all work, which all forces act in the time interval $[t_1, t_2]$.

The analytical treatment of mechanics does not require knowledge of these forces.

The total differential of the action as function of the coordinates and the time is a 1-form

$$\omega = dS = \sum p_i dq^i - Hdt \ .$$

Its differential

$$d\omega = \sum dp_i dq^i - dHdt$$

is integral invariant, which is relevant for corresponding "Erhaltungsgrössen", leading to the Lagrange identities.

2. There are forces which come from an external field or from the mutual interaction of particles:

The quantity of prime importance for the analytical treatment of mechanics is not a force, but the work done by impressed forces for arbitrary infinitesimal displacements. They are analytically defined as the coefficients of an invariant differential from of the first order, which gives the total work of all the impressed forces for an arbitrary infinitesimal change of the position of the system.

A mathematical framework is required to bridge from Leibniz's "monad" concept to monads as physical entities (with differential forms as mathematical counterpart). With that the monads would not only solve/unite Descartes' matter-spirit issue/separation, but also the current inconsistencies between quantum and gravitation models. This would enable an unified quantum and gravitation field theory. There already exists a mathematical framework dealing with "monads", which are the hyper-real numbers building the foundation of the non-standard analysis. Therefore the required mathematical framework is likely to be built in a hyper-real (non standard analysis) variational theory framework, being consistent with Hamiltonian principle.

"Duality" is likely to be THE guiding principle to build an adequate mathematical model,

- enabling a characterization of the 4 dimensional hyper-real Einstein space as THE valid space-time continuum with appropriate hyper-real sherical waves

- explaining the assymmetry of the time arrow in an only real 4-dim. Einstein space

- explaining physical constants and "observed" particles as observations and measurerable quantities, when crossing the border from the "hyper-real" to the "real" (world) state.

A standard physical principle for the Nature for our **real** world is “**purpose**” and “**causality**”, which are both from a mathematical point of view **equivalent**. This physical principle remains valid for the **hyper-real** world concerning “**purpose**”, while “causality” is out of scope of any measureable experience and observation capability, which is beyond the velocity of light (*the latest experimental results about “superluminal barrier traversal” are explained and modelled by “virtual” photon on top of “real” photons” (!!!*).

The variational theory is likely also to be the appropriate framework for scattering and variance modelling.

The Harmonic (Quantum) Oscillator and the Hilbert transformation

In the one-dimensional case the concept of hyperfunctions enables a link between distributions and a holomorphic, i.e. a complex-analytical function, as any distribution f on R can be realized as the "jump" of the corresponding in $C-R$ holomorphic Cauchy integral function

$$\hat{f}(x) := F(x) := \frac{1}{2\pi i}\oint \frac{f(t)dt}{t-x}$$

across the real axis, given by

$$(f,\varphi) = \lim \int_{-\infty}^{\infty} F(x+iy) - F(x-iy))\varphi(x)dx \quad \text{for } y \to 0^+ \ .$$

A real function and its Hilbert transform together create a so called strong analytical signal. This signal can be written with an amplitude and a phase where the derivative of the phase can be identified as the instantaneous frequency. A function and its Hilbert transform are orthogonal. A function and its Hilbert transform have the same energy and therefore the energy can be used to measure the calculation accuracy of the approximated Hilbert transform. The Hilbert transform defined in the time domain is a convolution between the Hilbert transformer $\frac{1}{\pi t}$ and a function $f(t)$.

The simplest version of the harmonic oscillator is the Hamiltonian system with Hamiltonian

$$H(p,q) = \frac{1}{2}(p^2 + \omega^2 q^2) \quad \text{and} \quad \dot{q} = p \ , \quad \dot{p} = -\omega q \ , \ddot{q} = -\omega^2 q$$

Identifying $R^2 \cong C$ by putting $z = p + i\omega q$ a solution to $H(p,q) = \frac{1}{2}|z|^2$ is given in the form

$$z(t) = Ce^{i\omega t} \ .$$

The Hermite polynomials are used to model the energy states of the harmonic quantum oscillator. A complex function is called Hermitian if its real part is even and its imaginary part is odd. If $g(t)$ is a real function, then e.g. $\hat{g}(\xi)$ is Hermitian and therefore $|\hat{g}(\xi)|^2$ is even. A complex signal u is called a strong analytical signal if it holds $Hu = iu$. For strong analytical signals u it holds $H(\mathrm{Re}(u)) = \mathrm{Im}(u(x))$, i.e.

$$z(t) = u(t) + iH(u(t))$$

is a strong analytical signal. From this, the combination of Hermite polynomials with its Hilbert transforms in the form

$$z_n(t) = \varphi_n(t) + i\hat{\varphi}_n(t) \ ,$$

define an alternative orthogonal system for the solution space of the harmonic quantum oscillator. This might provide an alternative model for the zero point energy of the harmonic quantum oscillator, which might overcome current inconsistencies between the Casimir effect i.e. existing radiation at absolute zero point of the temperature) and the calculated infinite energy density from the harmonic quantum operator model. The orthogonal system $z_n(t)$ can be calculated by the following recursion formulas:

using the abbreviation

$$a_n := \sqrt{\frac{2(n-1)!}{n!}} \qquad b_n := \sqrt{\frac{(n-2)!}{n!}}$$

the weighted Hermite polynomials (see section 9)

$$\varphi_n(x) := \frac{e^{-\frac{x^2}{2}} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}} \quad \text{with} \quad H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \ , \ H_0(x) = 1 \, , \ H_1(x) = x \, ,$$

fulfill the recursion formula

$$\varphi_n(x) := a_n x \varphi_{n-1}(x) - (n-1) b_n \varphi_{n-2}(x) \, , \quad \varphi_0(x) := \pi^{-1/4} e^{-\frac{x^2}{2}} \, , \quad \varphi_1(x) := 2^{-1/2} \pi^{-1/4} x e^{-\frac{x^2}{2}} \quad .$$

The corresponding recursion formula for the Hilbert transforms of the Hermite polynomials is given by

$$\hat{\varphi}_n(x) := a_n \left[x \hat{\varphi}_{n-1}(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi_{n-1}(y) dy \right] - (n-1) b_n \hat{\varphi}_{n-2}(x)$$

$$\hat{\varphi}_0(x) = \pi^{1/4} \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} \sin(\omega x) d\omega \ \cdot$$

In any relativistic theory the vacuum, the state of lowest energy, if it exists in „reality“ (with its ir-rational numbers as model for physical points), has to have the energy zero (or has to have an infinitely small energy in a hyper-real world!!?!).

3. Hyperreal Numbers (Ideal Points) and k-Forms

"tertium datur"=Leibniz`"vis viva" living force of quantums and fields

The "tertium non datur" principle (or the "principium exclusi tertii sive medii inter duo contradictoria") is in contradiction to one of the most successful pieces of the mathematics, the differential calculus (due to Leibniz) modelling changes over time, movements, momentum, Maxwell equations, etc. quite successfully:

in order to see this, a standard example is considering two points on a curve, which can either be the same or they are different. If the "law of the excluded middle" is accepted as true, this are the only two options:

In case the points are different, they define a unique line between each other, which crosses the curve at both points. Therefore this line can never be a tangent to the curve, as a tangent touches a curve at one point only.

In case both points are the same, they define not only one, but infinitely many lines at that point. But which of those should now be the uniquely defined tangent to the curve?

Leibniz solved this contradiction, first starting with the status of two different points and then, at some point in time during his argumentation, they get closer and closer together to become finally identical: this behaviour is explained by Leibniz as an "somehow" existing "infinitely small distance" between both of them.

The mathematican and philosoph L. E. J. Brouwer criticized especially propositions, derived from the law of the excluded middle, of the form

If it holds for no x: not A(x), then for all x: A(x)

Alternatively he proposed an intuitional calculus of logic, out of it the law of the excluded middle can not derived.

The negotiation of the law of the excluded middle gets relevant for all propositions related to the infinity and related to past or future events, assuming that truth is ensured knowledge. A popular example for such an proposition is: "Either the world exists without any starting date or it started at some point in time in the past" [LBr]

Core elements of the Cartesian philosophy are matter & mind, extension & idea, life & force. Each pair is complementary, which gets also visible in the today´s wave-particle dualism. Leibniz solved the underlying inconsistency by the concept of "substance", which is active. The activity of substances is metaphysically necessary. A substance is a being capable of actions.

[BRu1] (chapter 4): "*Activity is to be distinguished from what we mean by causation. Causation is a relation between two phenomena in virtue of which one is succeeded by the other. Activity is a quality of one phenomenon in virtue of which it tends to cause another. Activity is an attribute corresponding to the relation of causality; it is an attribute which must belong to the subject of changing states, in so far as those states are developed out of the nature of the subject itself. It is an actual quality of a substance, forming an element in each state of the substance."*

[KFi] (chapter 4): "*Mechanical activity follows from the principle of matter; living activity follows from the principle of mind (=Plato´s form). Matter and mind are the both forces, which define the being of each monad (Leibniz´s substance). The ratio of mind/matter (=soul/body) is always the same in all entities and corresponds to the ratio of final cause to acting cause*."

The later proposition might enable a first hint to a corresponding mathematical handling (ratio=?=(x+dx)/x), as Leibniz´s monad has its counterpart in non-standard numbers in the form a=x+i, x real and i=infinitesimal) and the Lagrange and Hamiltonian formalism reflect causality vs purpose caused action principles.

The set of non-standard numbers *R is a non-archimedean ordered field (the set of complex numbers is an archimedean not-ordered field). *R contains non-trivial infinitely small (infinitesimal) numbers. R* and R have the same cardinality. The continuum hypothesis states that the set of real numbers has minimal possible cardinality which is greater than the cardinality of the set of integers.The elements *N-N are the infinite natural numbers. The elements *R-R are called hyper-real numbers. An infinite number is greater than any finite number. If a is any finite real number then there exists a uniquely determined standard real number x=st(a), called standard part of a such that r-a is infinitesimal. Differentials dx or dy are infinite elements, while dy/dx is a finite element; the standard number of dy/dx=st(dy/dx) corresponds to the derivative of y(x).

The mathematical building principle for the set of non-standard numbers is based on (internal) set theory and algebra. The core building elements are Frechet filter, ultrafilter, maximal ultrafilter/ideal to construct a field. The physical interpretation of that building principle is that a space-time interval can be divided into a finite number of infinitesimal distances.

The smallest set of propositions (which are analytic) to define real numbers include the Archimedian principle. This principle gives the relation of "real" numbers to the natural numbers. The latter one can be interpreted as a measurement, created by human beings to enable the counting of observed phaenomena. Without that proposition the set fits to the non-standard numbers.

A journalist might ask: "Did I understood right, that building analytical objects can only be done axiomatically? And the way we build those numbers, which we used to call "real" and which are and have to be analytic, is based on a measurement concept, which we used to call natural numbers, because they are based on purely empirical observations of human beings?"

4. Tensors and k-forms

From R. Taschner, [RTa] we recall:

"*It is probably the last remaining task of the theoretical physics to show us how the term "force" is completely absorbed in the term "number".*

An alternative mathematical approach for a quantum gravitation theory is proposed embedded in the calculus of variations and the calculus of exterior differential forms based on Robinson`s calculus of hyper-real numbers. The physical rational behind this approach is basically replacing Newton´s transcendental mass point and particle charges by Leibniz´s transcendental ideal points (monads) and living forces, enabling to embed quantum ("particle") theory consistently in a gravitation ("field") theory.

In classical tensor analysis, one never knows what is the range of applicability simply because one is never told what the space is with the known consequences concerning the solution finding of Einstein`s field equations. A analogue picture describing this situation might be a stage and actors, where the stage determines actors` behaviour, but also the other way around. The mathematics of PDE distinguishes between well-posed and not well-posed problems. In this sense the current gravitation field PDE are just shaky.

The difficulties caused by the tensor concept has been overcome in modern times by the theory of differentiable manifolds. Tensor fields do not behave themselves under mappings. With exterior forms one have a really attractive situation in this regard. There is an important inner consistency of the differential calculus, i.e. the exterior derivative of a differential form is independent of the coordinate system in which it is computed. At the same time the differential calculus links perfectly back to Leipniz`s infinitesimal "differentials". Those are transcendental, but nevertheless with a precise mathematical meaning, i.e. those entities are elements of a certain dual space.

How to build now such a space enabling a Hilbert space structure and which dimension it should have?

The first idea is to look at the well known theory of elliptic elasticity problems with its known PDE solution concepts (coerciveness, Korn`s inequality and Garding inequality) and key physical attributes (e.g. torsion).

The basic equations of elasticity are direct consequences of the extended Hamilton's extended principle. A deformable solid body is considered under the influence of two sets of force distributions:

a) so-called body forces F1, F2, F3
b) so-called surface forces T1, T2, T3.

The most usual example for a body-force distribution is the influence of a gravitational field. Surface forces are in operations, whenever a body is subject to contact with external agencies at its surface.
Playing the role of the generalized-force components are the body- and surface-force distributions, which act upon the solid as influences of external agencies. The generalized

coordinates are the components of the displacement u1, u2, u3. The strain potential energy per unit volume is usually described by W. The Hooke's law gives the relation between appropriate derivatives of W to derivatives to T. The second Korn's inequality (which is a Garding type inequality) enables to formulate well-posed elasticity equation problems.

The second idea refers to the geometric interpretation of mass gravitation of the Einstein equations (with its relation to the tensor theory). Those field equations for the empty space-time arises from setting the first variation (with respect to the space-time metric) of the integral of the scalar curvature equal to zero. E. Cartan extended Einstein`s theory by considering torsion and put it into relation with the spin of matter fields. The idea is to work in the framework of the calculus of differential forms, (adapted) moving frames (by which the differential invariants Gauss curvature K and mean curvature H can be derived) and to replace the "scalar curvature minimization" problem in a semi-Riemannian (Lorentz-) manifold (which is always torsions-free) by building an appropriate hyperbolic manifold allowing torsion and moving from a scalar constant-curvature to a scalar constant-mean-curvature. This lead to the concept of minimal surface in the frameworks of the calculus of differential forms.

The classical geometry of a structure of a 4-manifold is Kaehler geometry, the geometry of a complex manifold with compatible Riemannian metric. In contrast to it, the symplectic geometry is the geometry of a closed skew-symmetric form. It is a 2-dimensional geometry that measures the area of complex curves instead of the length of real curves.

There is an important difference between **Kaehler and symplectic manifold**:

- a Kaehler manifold M has a fixed complex structure built into its points; M is made from pieces of complex Euclidean space C(n) that patched by holomorphic maps. One adds a metric g to this complex manifold and then defines the sympectic form.

- a symplectic manifold first has the form w, and then there is a family of automorphism J:TM-->TM, J*J=-Id, that turns TM into a complex vector bundle, imposed at the tangent space level (not on the points).

The relation to Functional Analysis is as follows: the exterior derivative d is a closed densely-defined unbounded operator (which can be interpreted as the counterpart of the momentum operator in quantum mechanics) in an appropriate Hilbert space setting, which is connected to the de Rham complex. Due to the Hodge decomposition and the Poincare inequality the mixed (*), weak formulation of the Hodge-Laplacian is well-posed. Leveraging proofs for well-posed PDE in the framework of the calculus of variations might enable an appropriate definition of a mixed formulation of hyperbolic (differential form) gravitation equations, where e.g. well known concepts modelling shock wave singularities/observations can be applied.

In relation to the quantum mechanics we mention eigen differentials, which correspond to the formalism of wave package modelling. Von Neumann and Dirac developped the corresponding Spectral theory, where eigen differentials are orthogonal wave packages.

The Problem of Plateau (raised by J.-L. Lagrange) is to prove the existence of a minimal surface bounded by a given contour, e.g. an arbitrary Jordan curve in a n-dimensional Euclidean space ([JDo]). Naturally, an arrangement of knots in the contour will produce coresponding complications in the minimal surface, such as self-intersections and branch points. In this context there is an interesting characterization of the dimension n: the set of all inner branch points is empty for n=2 (Riemann mapping theorem) and n=3, and it is never empty for n>3 ([ROs]). This is one of several evidence arguments (see our paper in the corresponding section), that an n=3 manifold is more likely the appropriate dimension of the solution space for quantum gravitation field equations than the string theory requires to integrate all 4 Nature forces.

It´s proposed to replace current (commonsense) transcendental assumptions about „physical" elements in mechanics and field models by proper alternatives. This is basically about moving from Newton´s mechanics´ mass points x and particle charges (without any physical meaning) concept to Leibniz´s ideal point (monads, verae unitates, ~"x+dx") and living forces concept.

Currently there are two one-way roads ending up in quantum theory and gravitation theory. Both together are inconsistent, i.e. there must be a branch somewhere back the road, when underlying different (commonsense) assumptions lead to different ways forward to go.

Differential calculus is about modelling „continuity" and no „action between distances", which is valid for the gravitation (field) theory, but not for the quantum theory. This discrepancy leads to the particle-field dualism interpretation, which is a polite description of the fact that both theories are just wrong.

To the author´s opinion the particle-field incompatibility is due to the fact of not appropriate transcendental „assumptions" and/or due to the missunderstanding, that Newton´s mechanical model are without any transcendental elements at all, i.e. purely „empirically" and therefore „true", because the theory is based on „real" numbers. Who has ever measured an irrational number value or who can sketch a link to Mandelbrot´s fractale model or how „close" is „close" in a PDE based „Nahwirkungstheorie" or why a ir-rational number is supposed to be more „real" than a hyper-real number? It´s just the missing Archimedian principle (i.e. that the set of natural numbers is unbounded above), which makes both fields different.

The set of non-standard numbers *R is a non-archimedean ordered field (the set of complex numbers is an archimedean not-ordered field). *R contains non-trivial infinitely small (infinitesimal) numbers. The elements *N-N are the infinite natural numbers. The elements *R-R are called hyper-real numbers. An infinite number is greater than any finite number. If a is any finite real number then there exists a uniquely determined standard real number x=st(a), called standard part of a such that r-a is infinitesimal.

In a hyper-real (instead of a today´s ir-rational) world there is no need for a mechanical mass point based (field) model, but it is still possible to explain observed forces modeling action following the "purpose" principle. Our idea is to exchange the transcendental mass point concept (introduced by Newton, modelling observed forces indirectly) by a transcendental „living forces“ concept (introduced by Leibniz, enabling a consistent model of the current 4 known forces of the Nature directly with extending the space dimension per each modelled force.

Of course, in the same way as Newton´s physical ideal mass point cannot be measured, Leibniz´s living force (vis viva) can not be measured, too. An appropriate „hyper-real“ model has to explain current Nature constants, i.e. when forces are becoming real(ized) at the „toll gate“ between purely hyper-real (ity) and real(ity).

The new concept is about proposing Leibniz´s concept of forces (e.g. the "vis viva") as physical principle, replacing Newton´s mass point concept, which is basically replacing today´s concept of (transcendental) particle charges (without any physical meaning) by the living force.

At the same time in a purely hyper-real (ity) the acting forces cannot be explained/ modelled by a "causality" principle (Lagrange formalism) and has to be replaced by a purely "purpose" principle (Hamiltonian´s "action" formalism). In real (ized) observed forces both principles are equivalent (Legendre transform).

Differential equations are used to construct models of reality. Sometimes the reality we are modeling suggests that some solutions of differential equation need not be differentiable. For example the “vibration string” equation (which is the wave equation in an one dimensional space dimension) has a solution u(x,t)=f(x-kt) for any function of one variable f, which has the physical interpretation of a “traveling wave” with “shape” f(x) moving at velocity k. There is no physical reason for the “shape” to be differentiable, but if it is not, the differential equation is not satisfied at some points. But one do not want to throw away physically meaningful solutions because of technicalities.” The mathematical framework, which enables such solutions, is the distribution theory.

Building an appropriate mathematical variational theory (minimizing) should be guided by „duality" principles, while

replacing standard Einstein spaces (whereby within a tensor theory framework one is never told what the spaces really are) by a compact 3-manifold M with minimal surface (e.g. embedded in the Euclidean (Hamiltonian formalism enabling?) 6-space), fulfilling proper conditions, which ensures that the mean curvature is zero (instead of the usual torsions freedom situation of each (semi-) Riemannian geometry). The exterior curvature 2-forms (theta(i,j)) on M are related to the Riemann tensor R(i,j,k,l) by the basis for the 1-forms on M (sigma(j)) in the form 2*theta(i,j)=SUM(R(i,j,k,l)sigma(k)*sigma(l). Potential theory due to W. Hodge might enable such coerciveness conditions, enabling minimal surface properties and unique solutions of the gravitation field equations formulated in a variational theory framework. E. Cartan extended Einstein`s theory by considering torsion and put it into relation with the spin of matter fields. For some research to the latter topic and physical interpretation of torsion in space-time dimensions we refer to [FHe]

- explaining the assymmetry of our world´s time arrow by such a framework.

- ensuring consistency to the Huygens principle by such a framework, which is completely neglected by the current string theory.

The mathematical counterpart of the Huygens principle is the Duhamel integral. The Hodge operator gives the d'Alembertian operator in case of Euclidean spaces, the underlying Green function for the (parabolic) heat resp. the (hyperbolic) wave equation (related to the Duhamel integral) are the Gauss-Weierstrass function resp. the Fourier-inverse of the sinc function; singularities caused by the sinc-function might be manageable by the Hilbert-transform trick from the RH proof.

"The exterior derivative d is a closed densely-defined unbounded operator in an appropriate Hilbert space setting, which is connected to the de Rham complex. Due to the Hodge decomposition and the Poincare inequality the mixed (*), weak formulation of the Hodge-Laplacian is well-posed. Leveraging proofs for well-posed PDE in the framework of the calculus of variations might enable an appropriate definition of a mixed formulation of hyperbolic (differential form) gravitation equations, where e.g. well known concepts modelling shock wave singularities/observations can be applied."

Geometrization of 3-manifolds

Finally we recall the mathematical question of the geometrization of 3-manifolds: R. Hamilton´s introduced the concept of the Ricci flow, which is a PDE model on the space of all metrices on manifolds related to the heat equation. His concept plays the key role in G. Perelman´s proof of W. Thurston's geometrization conjecture and the Poincaré conjecture respectively. In his proof G. Perelmann managed successfully the appearance of singular solutions of the Ricci flow.

The Ricci flow describes the evolution of a time-dependent Riemannian metric g(t) under the nonlinear partial differential equation dg(t)/dt=-2*ric(t), where ric(t) is the Ricci curvature with respect to g(t). Ricci flow is not parabolic itself, but it can be modified by the DeTurck trick into a nonlinear parabolic equation. Short time existence of Ricci flow was proved by Richard Hamilton.

A few (probably stupid) questions

Is there an appropriate alternative model of the current (parabolic) Ricci flow, e.g. analogue to the (hyperbolic) wave equation model (instead of the (parabolic) heat equation producing (only) high regular solutions), which enables a consistent PDE model related to a hyperbolic surface manifold structure (which might be the surface of an appropriate 3-manifold, enriched by certain coerciveness propositions of corresponding curvature 2-forms?)?

Is there a geometrization of a (hyperbolic?) compact 3-manifolds with minimal surface, which allows a Garding type inequality of curvature tensors (resp. the appropriate 2-forms) analogue to Korn`s second inequality for the strain/stress tensors in elasticity problems, enabling subellipticity and coerciveness of underlying PDE operators, which then would enable an analysis in a Hilbert space framework,whereby the differential calculus enables the definition of an appropriate space?

Coerciveness requires a bilinear form, which might be in this case enabled by the duality formula between the Cartan (exterior) derivative and the Co-derivative (with its relation to harmonic forms, the Poincare duality theorem and the Hodge theorem). We note that the Riemannian metric is also called the kinetic-energy metric on M. Can this motivate a less strong condition, i.e. a coerciveness characterization ofan appropriate compact 3-manifold with minimal surface, to ensure unique solutions of the gravitation equations; then this would enable an alternative to Einstein´s gravitation constant (again note, that the tensor theory gives no advive about the space environment its tensor are acting in) by an additional differential term (an additional tensor) added to the Riemann tensor, modelling kinetic AND potential-energy metric solution?

Is Sonoluminescence, where its mechanism is still unsettled, an appropriate phenomenon, giving some evidence how shock wave can create extrem high temperatures?

Sonoluminescence is the emission of short bursts of light from imploding bubbles in a liquid when excited by sound. Sonoluminescence can occur when a sound wave of sufficient intensity induces a gaseous cavity within a liquid to collapse quickly. This cavity may take the form of a pre-existing bubble, or may be generated through a process known as cavitation. Spectral measurements have given bubble temperatures in the range from 2300 K to 5100 K, the exact temperatures depending on experimental conditions including the composition of the liquid and gas. Detection of very high bubble temperatures by spectral methods is limited due to the opacity of liquids to short wavelength light characteristic of very high temperatures.

5. k-Forms and Minimal Surfaces

Smooth manifolds can show singularities after a limit process; the missing compactness leads to difficulties, when proving the existence of minimal surfaces; varifolds are a generalization of the comcept of differentiable manifolds by replacing differentiability requirements with those provided by rectificable sets: they can be loosely described as generalized surfaces (manifolds) endowed with multiplicity. In an Euclidean environment a varifold is a positive Radon measure. Varifolds are "currents" without signed orientation, which can still be integrated with unsigned volume integration.

We recall from G. Alberti [GAl] some definitions/terminologies:

The theory of integral currents provides a class of generalized (oriented) surfaces with well-defined notions of boundary and area (called mass), where the existence of minimizer can be proven by direct methods. This class is large enough to have good compactness properties with respect to the topology that makes the mass a lower semi-continuous functional.

From [FAl] we recall:

varifolds of dimension one or two are curves and surfaces defined in Euclidean space in a measure theory way: integral varifolds provide a mathematical model for all soap film and soap bubbles;

- a k-dim. current in R(3) is a linear functional into R, mapping a differential k-form to the integral of that differential over an oriented k-rectificable subset of A of R(3)

- a k-dim. varifolds in R(3) is a linear functional into R+, mapping a differential form to the integral of that differential k-form into the nonnegative real number space R+.

An exterior algebra of a vector space is the algebra of the wedge (or exterior) product, also called an alternating algebra or an Grassmann algebra. A Grassmann algebra is a associative, anti-commutative algebra with a 1-element. It is a sub algebra of the tensor algebra. The manifolds of differential k-forms builds an exterior algebra.

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Any k-dim. surface in R(n) may be thought of as a k-dim. varifold in R(n). The elements of the weakly topologized space of Radon measures on R(n)xG(n,k), where G(n,k) is the Grassmann manifold of k-dim. linear subspaces of R(n), are called k-dim. varifolds on R(n).

The definition of k-dim. currents closely resembles that of distributions: they are dual of smooth k-forms with compact support. Since every oriented k-dim. surface defines by integration a linear functional on forms, currents can be regarded as generalized oriented surfaces. As every distribution admits a derivative, so every current admits a boundary.

Differentiable extentions of manifolds (varifolds), which allows non-manifolds points (i.e. manifolds, which may have singularities) are the so-called rectificable sets. Integration over differential form over rectificable set gives an unoriented integral. It can also be regarded as real valued function on the space of differential forms, i.e. as linear functional. At the same time this defines signed measures. Those are called rectificable currents. The varifolds, which are obtained by integration differential forms over a rectificable set are called rectificable varifolds.

In general exterior "measures" (monoton, non-negative, sigma-subadditive) on a set Omega are no measures. If a metric is given on Omega there can be defined a metric, exterior "measure". If such an exterior, metric "measure" is restricted to the sigma-algebra of all measurable subsets of Omega, then is becomes a measure (basically restricted to those sets, by which every other set is "splitted additively". This criteria becomes useful, if a sufficiently large number of sets become measurable. This is the case, when every Borel-set in Omega is measurable. In a metric space there is a natural way to define a special class of metric, exterior "measures", the "Hausdorff-measures".

Signed measures (or electric charge distributions) can have negative volumes. This relates to the generalized oriented surface (the k-dim. currents). Haussdorf "measures" (metric, exterior "measures") are used to define the integration of differential forms over manifolds. For example one can ask for the rectificable set having the smallest Hausdorff measure among all sets having a given boundary in some algebraic topological sense.

The notes above shows, that there is a certain degree of freedom, which is still unused in current Geometric Measure Theory, to define a proper measure on varifolds/currents. As a first proposal we suggest to investigate in an alternative underlying metric:

the usual topology on currents is the weak topology based on a Hausdorff measure in an Euclidean vector space environment, i.e. there is still a degree of freedom to define such a weak topology alternatively (see below). We propose to use "Sobolev"- or "Hoelder-continuous" distribution function with scale factor -1/2 with a correspondingly defined exterior "measure" according to the building principle above. The non-integer scale factor indicates that it might be more appropriate to use the concept of integral currents, than that of integral varifolds. This then might enable the application of spectral and distribution theory providing existence, uniqueness and well posted problem formulations in combination with the Ritz-Galerkin procedure to build such appropriate solutions.

From H. Federer, [HFe], we recall the remarkable property of complex integral currents:

an integral curent (of even dimension) in C(n) (or in a Kaehler manifold), which has a complex tangent space almost everywhere is a minimal current. This means, that a piece of a complex curve L in C(2) is absolutely area minimizing when the boundary dL is fixed, even if the surface L has branch points.

A sufficient condition to Banach spaces to ensure sequence compactness with respect to weak convergence is "reflexibility". For the function spaces L(p) this is given for 1<p<infinite; for p=1 is requires the concept of bounded variation functions, which have only jump-type discontinuities. The related function space is a not separable Banach algebra and the Hausdorff measure are related to it. On the other side every Hilbert space is reflexible due to the representation theorem of Riesz. The Fourier transform of the uniform distribution of unit mass over the unit sphere is given e.g. in [BPe].

In the context of a GUT we especially refer to

- the Hildebrand functional to prove the existence of surfaces with given mean curvature H (see below), i.e. $J(u):=a(u,u)+2F(u)$--> Min with $F(u)=(Q(u),u,du)$ and $H=divQ$ resp. the corresponding equivalent variational equation formulation

- the recently published paper [BMo]

- the embedding theorem of Nash (surfaces as m-dim. manifold in a n-dim vector space)

- branch points of minimal surfaces and corresponding singular parametrization in case of $n>3$

- quasi minimal surfaces in pseudo-Riemanian manifolds, [BCh]

- the integral equation method of Theodorsen and Garrick for conformal mapping, [DGa]

References

[GAl] G. Alberti, Geometric Measure Theory, Encyclopedia of mathematical Physics, Vol.2 , pp. 520-527, Elsvier, Oxford, 2006

[WAl] W.K. Allard, On the first variation of a varifold, Ann. of Math. (2) 95 (1972), pp. 417-491

[RAl] R. Almeida, Non-standard Analysis and Differentiable manifolds – foundations, Differential Geometry, Dynamical Systems, Vol. 11 (2009) pp. 1-19

[FAl] F.J. Almgren, Plateau´s Problem, An Invitation to Varifold Geometry, American Mathematical Society

[IAr] I. Aref'eva, I.V. Volovich, Quantization of the Riemann Zeta-Function and Cosmosolgy, internet link, Steklov Mathematical Institue, Moscow, 2007

[EAr] E. Artin, Galois Theory, Notre Dame Mathematical Lectures, Notre Dame University, Indiana, USA, 1948

[JBe] J. Bell, Infinitesimals and the Continuum,

[EBo] E. Bombieri (Ed.), Geometric Measure Theory and Minimal Surfaces, CIME Summer Schools, Varenna, Italy, 1972

[LBr] L. E. J. Brouwer: "Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten". Erster Teil, Allgemeine Mengenlehre. In Koninklijke Akademie van Wetenschappen te Amsterdam, Verhandelingen 1e sectie, deel XII, no 5 (1918), 1:43

[KBr] K. Braun, A spectral analysis argument to prove the Riemann Hypothesis, www.riemann-hypothesis.de

[HBr] H. L. Bray, Black Holes, Minimal Surfaces and Differential eometry, Mathematics Department, MIT, Cambridge, bray@math.mit.edu

[JBu] J.-F. Burnol, Spacetime causality in the study oft he Hankel transforms, urnol@math.univ-lille1.fr

[CCh] C.C. Chang, H.J. Keisler, Model Theory, 3rd enl. Ed. New York, Elsevier, 1990

[BCh] B. Cheng, Black Holes, Marginally Trapped Surfaces and Quasi-Minimal Surfaces, Tamkang Journal of mathematics, Vol. 40, No. 4 (2009) pp. 313-341

[RCo] R. Courant, Dirichlet´s principle, "conformal mapping and minimal surfaces", Dover Publications Inc., Mineola, New York

[MDa] M. Davis: "Applied Nonstandard Analysis", New York 1977

[GDe] In G. Deleuze, Die Falte, Leibniz und der Barock, Keibniz und der Barock, (le pli. Leibniz et le baroque, the fold and the soul), Suhrkamp, 1995

[AEi] A. Einstein, Ueber die formale Beziehung des Riemannschen Krümmungstensors zu den Feldgleichungen der Gravitation, Math. Ann. (1926) pp. 99-103

[JEs] J.-H. Eschenburg, J. Jost, "Differentialgeometrie und Minimalflächen", Springer Verlag, Berlin, Heidelberg, NewYork, 2007

[DHo] D. R. Hofstadter, "I am a Strange Loop", Basic Books (Perseus Books Group), New York, 2007

[HDu] H.-P. Duerr, "Physik & Transzendenz", Scherz Verlag, 1986, with contributions of D. Bohm, N. Bohr, M. Born, A. Einstein, W. Heissenberg, P. Jordan, W. Pauli, M. Planck, E. Schrödinger, C.F. von Weizäcker

[HFe] H. Federer, some theorems on integral currents, presented to the Society, November 28, 1961, under the tile Integral currents in kähler manifolds

[CFe] C. Fefferman, D. Phong, Symplectic Geometry and Positiveness of Pseudo-differential Operators, Proc. Natl. Acad. Sci. USA, Vol. 79, pp. 710-713, 1982

[EFe] E. Fermi, thermodynamics, Dover Publications, Inc. New York, 1936

[DFl] D. Fleisch, A Student's Guide to Maxwell's Equations, Cambridge University Press, 2008

[LFo] L.H. Ford, Quantum Field Theory in Curved Spacetime, Lecture Notes, Institute of cosmology, Tufts University, Massachusetts, July 30, 1997

[KFi] K. Fischer: "Gottfried Wilhelm Leibniz, Leben, Werk und Lehre", marixverlag, www.marixverlag.de

[DGa] D. Gaier, Konstruktive Methoden der konformen Abbildung, Springer Tracts in Natural Philosophy, Vol. 3,1964

[LGa] PL. Garding, Some Points of Analysis and Their History, University Lecture Series, American Mathematical Society, 1997

[VGo] V. Gol`dstein, M. Troyanov, On the naturality oft he exterior differential, C.R. Math. Rep. Sci. Canada Vol. (30) (1) 2008, pp. 1-10

[HHa] H.G. Haefeli, Hyperkomplexe Differentiale, Commentarii Mathematici Helvetici, 20, p. 382-420

[FHe] F.W. Hehl, Found. Phys. 15, 451 (1985) and F. Gronwald, F.W. Hehl, Proc. of the 14th Course of the School of Cosmology and Gravitation on Quantum Gravity, Enrice 1995, World Sci. Singapure, 1996, R.T. Hammond, Contemp. Physics 36, 103 (1995).

[MHe] M. Heidegger, (1950), „Holzwege“, (Forest Paths), Fankfurt: Vittorio Klostermann:

[MHe1] (1935) "the Origin of the Work of Art", transl. A. Hofstadter, in D.F. Krell (ed.) Basic Wirtings, revised edn, San Franscisco: Harper, 1993

[MHe2] (1938) "The Age of the World Picture", in W. Lovitt (trans. and ed.) The Question Concerning Technology and Other Essays, New York: Harper & Row, 1977

[THe] T. Heikonen, Notes on Differential forms and Spacetime, Lecture Notes, (2006)

[DHo] D. R. Hofstadter, "I am a Strange Loop", Basic Books (Perseus Books Group), New York, 2007

[GHo] G. Hoft, Introduction to String Theory, Lecture Notes, Institute for Theoretical Physics, Utrecht University, the Netherlands, 2004

[MKa] M. Kaku, Introduction to Superstrings and M-Theory, Springer-Verlag, New York, 1984

[RKr] R.S. Kraußhar, A characterization of conformal mappings in R(n) by a formal differentiability condition,

[NLa] N. Laskin, Fractional Quantum Mechanics, Lecture notes, University of Toronto, Canada, 2002

[GLe] G.W. von Leibniz: "Neue Abhandlungen über den menschlichen Verstand", Elibron Classics, www.elibron.com, "New essays concerning human understanding", translated by Alfred Gideon Langley, New York, London, 1896

[WLu] W.A.J. Luxemburg, K. Stroyan: "Introduction to the Theory of Infinitesimals", New York, 1976

[PLy] P. Lynds, Zeno's Paradoxes: A Timely solution, New Zealand, peterLynds@xtra.co.nz

[BMa] B. B. Mandelbrot, the Fractal Geometry of Nature, W.H. Freeman and Company, New York, 1977

[WMc] W. McLaughlin, S. L. Miller, An epistemological use of nonstandard analysis to answer Zeno's objections against motion

[BMo] B. Mohammed, Minimal Surfaces of the 3-Dimensional Lorentz-Heisenberg Space, Int. J. of Math. Analysis, Vol. 3, 2009, pp. 473-482

[BMo] B. Morrey, "Multiple Integrals in the Calculus of Variations", Springer Verlag, New York Inc., 1966

[ENe] E. Nelson: "Internal Set Theory: A new approach to Nonstandard Analysis", in Bulletin of the American Mathematical Society, 83, 6, 1977, pp. 1165-1198

[JNe] J. von Neumann, Mathematische Grundlagen der Quantenmechanik, Springer Verlag, 1968

[ROs] R. Ossermann, Ann. Math (2) 91 (1970) pp. 550-569, R. Gulliver, Ann. Math, 97 (1973) pp. 275-305, H.W. Alt, Math. Z. 127 (1972), pp. 333-362, Math. Ann. 201 (1973), pp. 33-55.

[RPe1] R. Penrose: "The Road To Reality", A. Knopf, New York, 2005

[RPe2] R. Penrose: "The Large, the Small and the Human Mind", Cambridge University Press, 1997

[RPe3] R. Penrose: "The structure of space-time", Battelle Rencontres, 1967, Lectures Math. Phys. 121-235 (1968)

[RPe4] R. Penrose: "On the Nature of Quantum Geometry", Magic Without Magic, J. Klauder, Freeman, San franscisco, 1972, pp. 333-354

[BPe] B. E. Petersen, Introduction to the Fourier Transform and Pseudo-Differential Operators, Pitman Advanced Publishing Program

[JPl] J. Plemelj, Potentialtheoretische Untersuchungen, B.G. Teubner, Leipzig, 1911

[PPo] P.V. Polyan, Non-standard analysis of non-classical motion, http://res.krasu.ru/non-standard/eng1.htm

[ARo1] A. Robinson: "Non-standard Analysis", Amsterdam, 1966

[ARo2] A. Robinson: "The metaphysics of the calculus", University of California, LA

[BRu1] B. Russell: "The Philosophy of Leibniz", Routledge Chapman & Hall, 1992

[BRu2] B. Russell: "The Problems of Philosophy", Oxford University Press, 1912

[RSc] R. Schinzinger, P.A.A. Laura, "Conformal Mapping, Methods and Applications", Dover Publications Inc., Mineola, New York

[Esc1] E. Schroedinger: "Space-Time Structure", Cambridge University Press, 1954

[Esc2] E. Schroedinger: "Mind and Matter", Cambridge University Press, 1958

[RSe] R.U. Sexl, H.K. Urbantke, Gravitation und Kosmologie, Spektrum Akademischer Verlag, Heidelberg, Berlin, Oxford, 1995

[ISt] I. Strohmeyer, Quantentheorie und Transzendentalphilosophie, Spektrum Akademischer Verlag Heidelberg, Berlin, Oxford, 1995

[SSt] D.J. Struik, Lectures on Classical Differential Geometry, Dover Publications Inc., New York, 1961

[RTa] R. Taschner, "Der Zahlen gigantische Schatten", dtv, München, 2009

[JTr] J. A. Tropp, Infinitesimals: History & Applications, University of Texas, Texas, Austin

[RWe] R. Weinstock, Calculus of Variations, Dover Publications, New York, 1952

[CWe] C.F. von Weizäcker, "Naturgesetze und Theodizee" about "causality and finality" in the context of G.W. Leibniz`s mathematics and meta-physics

[DWe] D. B. Westra, The Haar Measure on SU(2), 2008

[HWe] H. Weyl, „Raum, Zeit, Materie, Vorlesungen über Allgemeine Relativitätstheorie", Springer-Verlag, Berlin, Heidelberg, New York, 1922