OVERVIEW

The Riemann Hypothesis

The three Millennium problems: RH, NSE, YME

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Albert Einstein, "*we can't solve problems by using the same kind of thinking we used when we created them*",

Wolfgang E. Pauli, "*all things reach the one who knows how to wait*".

This homepage addresses the following three Millennium problems (resp. links to corresponding homepages):

A. The Riemann Hypothesis (RH)
B. The 3D-Navier-Stokes equations (NSE)
C. The Yang-Mills equations (YME)

A helicopter view

From a helicopter point of view there is a common denominator of all solution concepts of this homepage: it is about a common mathematical frame to govern the "infinitesimal small" with respect to truly infinitesimal small "elements" and related "functions" and truly geometrical (i.e. equipped with an inner product) "function spaces", enabling a truly geometrical mathematical modelling framework with corresponding operators (including well defined domains and ranges).

Regarding the RH the "infinitesimal small" is about the challenge to represent the entire Zeta function as a (Mellin-) transform of a self-adjoint operator ([EdH] 10.3). The Hilbert transform is the proposed tool to build a self-adjoint operator enabling the Berry-Keating (Hilbert-Polya) conjecture, because of its property that any Hilbert transformed function has a vanishing constant Fourier term (see also Polya's Bessel function based alternative entire Zeta function representation[EdH], 12.5).

Regarding the NSE and YME, beside the n-dimensional counterpart of the Hilbert transform, the Riesz transforms, the two applied central "objects" are the well-established "differentials" and the distributional "Hilbert scale" concept enabling Pseudo-Differential and Fourier multiplier (weak and strong singular integral) equations (Calderón) and the related (Stieltjes integral like) spectral representation of Hermitian operators.

In the context of the newly proposed "energy-space" $H_{1/2} = H_1 + H_1^{\neg}$ we also refer to the Bose-Einstein Condensation (BEC), where below the critical temperature T_C BEC "normal gas" particles coexist in equilibrium with "condensed" particles. Unlike a liquid droplet in a gas, here the "condensed" particles are not separated in space $(H_{-1/2} = H_0 + H_0^{\neg})$ from normal particles. Instead they are separated in standard momentum space H_1, which is a closed, compactly embedded subspace of the newly proposed "energy-space" $H_{1/2}$. The condensed standard particles all occupy a single quantum state of zero momentum, while normal standard particles all have finite momentum with respect to the $H_1 -$ norm.

The challenge to represent to entire Zeta function as a (Mellin-) transform of a self-adjoint operator is explained in [EdH], 10.3:

Let $\gamma(s)$, $Z(s)$, $z(s)$ denote the Gamma function, the entire Zeta function ([EdH] 1.8 (1)) and the Zeta function ([EdH] 1.4 (3))

$$z(s) = \frac{\gamma(1-s)}{2\pi i}\int_{+\infty}^{+\infty}\frac{(-x)^s}{e^x-1}\frac{dx}{x},$$

which is analytic at all points of the complex s-plane except for a simple pole at $s = 1$. It is equal to the Dirichlet function

$$z(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \quad \text{for} \quad Re((s) > 1 \ .$$

Let further denote $f(x) := e^{-\pi x^2}$ denote the Gaussian function (with mean value $\hat{f}(0) = 1$) and

$$M[f](s) := \int_0^{\infty} f(x)x^s\frac{dx}{x} = \frac{1}{2}\gamma(\frac{s}{2})\pi^{-s/2} \quad , \quad G(x) := \sum_{n=1}^{\infty} f(nx).$$

Then it holds ([EdH] 1.6 (5), 12.5)

$$(*) \qquad \frac{1}{2}\gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}z(s) = \frac{1}{2}\gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}z(1-s)$$

$$(**) \qquad Z(s) = \frac{1}{2}\int_0^{\infty} x^{1-s}\frac{d}{dx}(x^2\frac{d}{dx}G(x))\frac{dx}{x} \ .$$

The function on the left hand side of (*) has poles at $s = 0{,}1$ ([EdH] 1.8), whereby the pole at $s = 0$ is caused by the Gamma function. Therefore, Riemann multiplies it by $s(s-1)$ to define the entire function $Z(s)$; that is an analytical function, which is defined for all values of s , and the functional equation of the Zeta function is equivalent to $Z(s) = Z(1-s)$. The functional equation is applied to derive the Riemann (error) function for the density function ([EdH] 1.13)

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} log z(s)x^s\frac{ds}{s}.$$

The factor $(s-1)$ leads to the Li-function, while the factor $\frac{s}{2}$ is anticipated in the function $\gamma\left(1+\frac{s}{2}\right)$ leading to the famous Riemann error function ([EdH] 1.16). Its convergence property jeopardizes a proof of the RH criterion

$$\pi(x) = Li(x) + O(\sqrt{x}logx) = Li(x) + O(x^{\frac{1}{2}+\varepsilon}) \quad , \quad \varepsilon > 0 \ .$$

Our proposed Kummer function based Zeta function theory is based on the Hilbert transform of the Gaussian function, which is the Dawson function

$$F(z) := e^{-z^2}\int_0^z e^{-t^2}dt$$

and its corresponding Mellin transform

$$M[f_H](s) = \pi^{(1-s)/2}\tan(\frac{\pi}{2}s)\gamma\frac{s}{2}.$$

We note that $F(z)$ satisfies the differential equation $F'(z) + 2F(z) = 1$, leading to the (polynomial) asymptotic $F(\sqrt{x}) = O(x^{-\frac{1}{2}})$, alternatively to the asymptotic $f(\sqrt{x}) = O(e^{-x})$.

With respect to Riemann's "trick" above to build an entire function, this corresponds to an alternative multiplication with the factor $(s-1)\tan(\frac{\pi}{2}s)$ of (*).

We further note the identity $\tan\left(\frac{\pi}{2}s\right) = \cot\left(\frac{\pi}{2}(1-s)\right)$; on the critical line ($s = \frac{1}{2} + it, t \in R$), it holds ($\tan\left(\frac{\pi}{2}s\right) = \cot\left(\frac{\pi}{2}\bar{s}\right)$ (see also ([EdH] 10.2, 10.3, 10.10). From [GrI] 1.317, 1.411, we recall the following identities

$$\frac{\pi}{2}\tan\left(\frac{\pi}{2}x\right) = \frac{\pi}{2}\cdot\frac{1-\cos(\pi x)}{\sin(\pi x)} = \frac{\pi}{2}\cdot\frac{sin(\pi x)}{1+\cos(\pi x)}$$

$$\frac{\pi}{2}\tan\left(\frac{\pi}{2}x\right) = \frac{1}{x}\cdot\sum_{k=1}^{\infty}\frac{2^{2k}-1}{(2k)!}|B_{2k}|\cdot(\pi x)^{2k} \quad , \quad x^2 < 1 \ .$$

With respect to (**) above it formally also holds ([EdH] 10.3)

$$\frac{Z(s)}{s(s-1)} = \frac{1}{2}\int_0^\infty x^{1-s}G(x)\frac{dx}{x}$$

that is, the function is formally the transform of the operator

$$g(s) \to \int_0^\infty g(xs)G(x)dx.$$

But the above operator has no transform at all, as the integral does not converge for any s.

Nevertheless, this statement can be given substance as follows ([EdH] 10.5), which motivated our proposed alternative transform representation below:

A continuous analog of Euler's ludicrous formula

$$\sum_{n=-\infty}^{\infty} x^n = (1 + x + x^2 + \dots) + (x^{-1} + x^{-2} + \cdots) = \frac{1}{1-x} + \frac{1}{x-1} = 0$$

is

$$\int_0^\infty x^{1-s}\frac{dx}{x} = \int_0^1 x^{1-s}\frac{dx}{x} + \int_1^\infty x^{1-s}\frac{dx}{x} = \frac{1}{1-s} + \frac{1}{s-1} = 0 \ .$$

This is, of course, nonsense because the values of s for which the above integrals converge are mutually exclusive – the first one being convergent for $Re(s) < 1$ and the second one being convergent for $Re(s) > 1$, but it does suggest that the formal transform of $g(x) \to \int_0^\infty g(ux)dx$ is zero.

The Hilbert transform is the proposed additional tool to overcome the above challenge to build a self-adjoint operator enabling the Berry-Keating (Hilbert-Polya) conjecture, as any Hilbert transformed function has vanishing constant Fourier term. Let M and H denote the Mellin resp. the Hilbert transformation operators, then it holds

$$M\left[-x\frac{d}{dx}f(x)\right](s) = sM[f](s) = \pi^{-\frac{s}{2}}\gamma(1+\frac{s}{2})$$

$$M\left[-\frac{d}{dx}(x^2\frac{d}{dx}(f(x))\right](s) = (s-1)sM[f](s) = (s-1)\pi^{-\frac{s}{2}}\gamma(1+\frac{s}{2})$$

whereby ([EdH] 1.8 (1))

$$Z(s) = (s-1)\pi^{-\frac{s}{2}}\gamma(1+\frac{s}{2})z(s).$$

In order to anticipate the non-vanishing constant Fourier term of the Gaussian function, Riemann modified the above Mellin transform representation in the form ([EdH] 10.3)

$$Z^{**}(s)=\frac{1}{2}\int_0^\infty x^{1-s}\frac{d}{dx}\left(x^2\frac{d}{dx}(G(x)-1)\right)\frac{dx}{x}$$

which even destroys the (only formally valid) self-adjoint transform representation above (see also ([EdH] 12.5 for Polya's alternative, entire Zeta function definition linked to the Bessel functions).

With respect to the Hilbert-Polya conjecture this leads to a replacement of the above by

$$M\left[-\frac{d}{dx}(xf_H)\right](s)=(s-1)sM[f_H](s)=c_1\frac{tan\frac{\pi}{2}s}{\frac{\pi}{2}s}(s-1)\pi^{-\frac{s}{2}}\gamma(1+\frac{s}{2})\ ,$$

which then defines an alternative entire Zeta function in the form

$$Z^*(s):=c_1\frac{tan\frac{\pi}{2}s}{\frac{\pi}{2}s}Z(s)=c_1\frac{\cot(\frac{\pi}{2}(1-s)}{\frac{\pi}{2}s}Z(s)=c_2\left[-\frac{\pi}{2}(1-s)\cot(\frac{\pi}{2}(1-s))\right]\pi^{-\frac{s}{2}}\gamma(\frac{s}{2})z(s)$$

and its corresponding transform representation given by

$$Z^*(s)=\frac{1}{2}\int_0^\infty x^{1-s}\frac{d}{dx}(xG_H(x)))\frac{dx}{x}\ .$$

The Gaussian function and its Hilbert transform are norm-equivalent with respect to the $L_2=H_0$ −norm, i.e. both are equal in a weak L_2 −sense. The convergence of the transform representations in the critical stripe resp. on the critical line is ensured in a weak $H_{-\alpha}$ −sense ([EdH] 9.7, 9.8).

The Hilbert transform of the Gaussian function is the Dawson function with the asymptotic

$$f_H(\sqrt{x})=O(x^{-\frac{1}{2}})\ ,\ x\to\infty\ .$$

Let $g(x):=e^{-x}$ and $I(x)=\int_{-\infty}^{\infty}\frac{g(t^2)}{x-t}dt$, then this leads to the replacement

$$\gamma(s)=M[g](s)\to M\left[I(\sqrt{x})\right](s)=\pi\cdot\tan(\pi s)\cdot\gamma(s)\ .$$

The expansion of the tan − resp. the $xcot(x)$ −term in series of simple fractions are given by ([GrI] 1.421)

$$\frac{tan\frac{\pi}{2}x}{\frac{\pi}{2}x}=\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2-x^2}\quad,\quad-\frac{\pi}{2}(1-x)\cot\left(\frac{\pi}{2}(1-x)\right)=-1+2\sum_{k=1}^{\infty}\frac{(1-x)^2}{4k^2-(1-x)^2}\quad.$$

With respect to the relationship of the Zeta function and the $\pi xcot(\pi x)$ function we further note the following formulas ([BeB] 5, [TiE] 4.14, [AnG] entry 3, entry 44)

i. $\sum_{n>x}\frac{1}{n^s}=\frac{1}{2\pi x}\int_{x-i\infty}^{x+i\infty}x^{-s}(-\pi zcot(\pi z)dz$

ii. $z(s)=\sum_{n\le x}\frac{1}{n^s}-\frac{x^{1-s}}{1-s}+O(x^{-\sigma})\quad,\ \sigma\ge\sigma_0>0\ ,\ |t|\le\frac{2\pi x}{c}\ ,\ c>1$

iii. $xcoth(x)=1+\frac{x^2}{3\ +}\frac{x^2}{5}+\frac{x^2}{7}\ldots..=1+\frac{x^2}{3}-\frac{x^2}{9}\left[\frac{x^5}{5}+\frac{\frac{4\cdot5}{2\cdot3}x^2}{7}+\frac{\frac{2\cdot3}{4\cdot5}x^2}{9}+\frac{\frac{6\cdot7}{4\cdot5}x^2}{11}+\frac{\frac{4\cdot5}{6\cdot7}x^2}{13}+\cdots\right]$

iv. $(\frac{\pi}{2}s)\cdot coth\left(\frac{\pi}{2}s\right)=1+\frac{s^2}{1\ +}\frac{1^2(s^2+1^2)}{3}+\frac{2^2(s^2+2^2)}{+\quad 5}\frac{3^2(s^2+3^2)}{+\quad 7}\ldots\ldots\ s\in C.$

With respect to the entire Riemann Zeta function Z, the $cot-$ function and the Hermite polynomials h_n (with $\widehat{h_n}(\omega)=(-i)^n h_n$) we further note that $Z\in H_{-1}$, $cot\in H_{-1}^{\#}$, $h_n\in L_2$, i.e. $(Z,logsin)_{-1/2}$, $(Z-cot,logsin)_{-1/2}, (Z,h_n)_{-1/2}$, $(Z-cot,h_n)_{-1/2}$ are defined. In this context we refer to

- the Bagchi reformulation of the Beurling RH criterion, as the $H_{-1/2}$ –Hilbert space is dense in H_{-1} with respect to the $\| \quad \|_{-1}$ norm
- the wavelet section below and to the theory of cardinal series ([WhJ] §11).

We also note that the periodical continuation of the log-Gamma-function $log\gamma(s)$ with domain $(0,1)$ is $\in H_{-1/2}^{\#}(0,1)$, also anticipating the odd sin-Fourier terms in the form $\frac{\log(2\pi n)}{2\pi n}+\frac{\gamma}{2\pi n}$ ([GrI] 6.443). The log-Gamma-function is linked to the log-sin-function by the formula

$$log\gamma(x)+log\gamma(1-x)=\log(2\pi)+log\frac{1}{2\sin(\pi x)}=\log(2\pi)+\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k}$$

$$\frac{\pi}{2}\cot\left(\frac{\pi}{2}x\right)-\frac{\pi}{2}\cot\left(\frac{\pi}{2}(x-1)\right)=\frac{2\pi}{2\sin(\pi x)} .$$

The replacement

$$M\left[-\frac{d}{dx}(x^2\frac{d}{dx}(f(x))\right](s)\rightarrow M\left[-\frac{d}{dx}(xf_H)\right](s)$$

goes along with corresponding reduced regularity requirement. The Berry conjecture is about the Riemann Zeta function as a model for the quantum chaos [BeM]. The combination of both leads to the **Calderón-Zygmund integral-differential operator** with symbol

$$|\omega|=\sum_{k=1}^{n}\omega_k\frac{\omega_k}{|\omega_k|}$$

as alternative "(pseudo) differential operator", which can be defined for any Hilbert scale domain ([BrK2]). It is given by ([EsG] example 3.4, [LiI] example 3.1.4, 3.1.6)

$$L[u]:=\sum_{k=1}^{n}Y_kD_ku=\sum_{k=1}^{n}\frac{\gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\text{p.v.}\int_{-\infty}^{\infty}\frac{x_k-y_k}{|x-y|^{n+1}}\frac{\partial u(y)}{\partial y_k}dy$$

resp

$$\Lambda^{-1}u=\frac{\Gamma(\frac{n-1}{2})}{2\pi^{(n+1)/2}}\int_{-\infty}^{\infty}\frac{u(y)dy}{|x-y|^{n-1}}$$

where Y_k denote the Riesz operators, which are the n-dimensional generalizations of the Hilbert transform. The corresponding **inner product** definition $((\mathbf{du},\mathbf{dv}))$ is provided in [BrK1].

A separable Hilbert scale can be built from the solutions of an eigenvalue equation

$$Kx=\sigma x$$

where K *denotes a symmetric and compact operator:*

Lemma*: for no more than countable values* σ_i *the equation* $Kx=\sigma x$ *possesses non-trivial solutions* x_i *and* $lim_{i\to\infty}\sigma_i=0$.

The **Calderón-Zygmund integral-differential operator** is also proposed as **alternative Schrödinger (-Calderón) momentum operator** with **domain** $\boldsymbol{H_{1/2}} = \boldsymbol{H_0} + \boldsymbol{H_0^{\neg}}$. For space dimension $\mathrm{m} = 1$ it is given by $\mathrm{u(x)} \to \boldsymbol{P}^*[\boldsymbol{u}](\mathbf{x}) \coloneqq -\mathbf{i}\frac{\boldsymbol{d}}{\boldsymbol{dx}}\boldsymbol{H}[\boldsymbol{u}](\mathbf{x})$ ([MeY], 7.1). With respect to the $H_{-1/2}$ −inner product it holds $(P^*[u], v)_{-1/2} \cong (u, v)_0$. In the simplest case (1-D periodical functions) it is given by the convolution **(Stieltjes/Plemelj) integral**

$$\int \cot(\frac{x-y)}{2}\mathrm{d}u_y.$$

In the context of the newly proposed "energy-space" $H_{1/2} = H_1 + H_1^{\neg}$ we further refer to the Bose-Einstein condensation, where below the critical temperature T_C BEC "normal gas" particles coexist in equilibrium with "condensed" particles. Unlike a liquid droplet in a gas, here the "condensed" particles are not separated in space $(H_{-1/2} = H_0 + H_0^{\neg})$ from normal particles. Instead they are separated in momentum space. The condensed particles all occupy a single quantum state of zero momentum, while normal particles all have finite momentum.

In case the domain of such a compact operator is the L_2 Hilbert space the corresponding eigenfunctions build the basis of this Hilbert space. The concept of "wave package" enables also continuous spectra. Therefore, such "wave packages" require a domain extension (e.g. $\mathrm{L}_2 \to \mathrm{H}_{-1/2}$) in order to ensure convergent inner products and related norms.

"Wave packages" are also called "eigen-differentials" (H. Weyl), playing a key role in quantum mechanics in the context of the spectral representation of Hermitian operators (D. Hilbert, J. von Neumann, P. A. M. Dirac).

Regarding the NSE and YME the "infinitesimal small" of "fluids" and "quanta" is and will be all the time out of scope for any human observations. Mathematics is a purely descriptive science with well-established concepts to deal with any kind and "size" of "infinity" (e.g. Cauchy, Dedekind, Bolzano, Weierstrass, Kronecker, Cantor, Gödel, Brouwer). The mathematical tool managing physical "observations" are Partial Differential Equations (PDE), mathematical statistics and approximation theory. Those concepts are also applied in quantum mechanics and quantum field theory. The essential mathematical "objects" are the real numbers (while "nearly all" of those objects are far away from being "real") and the Lebesgue integral building the Lebesgue (Hilbert) space L_2, where all rational numbers build a null set measured by its corresponding norm.

A "helicopter view" regarding “differentials” and “distributional Hilbert scale”

- the real numbers are replaced by differentials with same cardinality as the real number field, but allowing infinitesimal small number objects in the neighborhood of each real number (see the Riesz theorem below for the analogue properties with respect to the "Hilbert scale" objects

- the "measurement/observation/statistical" function space L_2 is replaced by the "much larger" distributional Hilbert space $H_{-1/2}$.

The advantages are the following

1. both conceptual changes do not increase the already existing "infinity" characters of "real" numbers (measured by Cantor's cardinality concept) and the L_2 space, which is a separable Hilbert space, in the same way, as all of the considered Hilbert scale "objects"

2. the "measurement of real numbers is already an approximation by rational numbers, i.e. truly "observations" of irrational number "objects" are not possible; each irrational number is already a full universe, i.e. an approximation of an infinite numbers of rational numbers; extending those number field to ideal numbers is just the same mystery with same cardinality; the key differentiator is related to a measurement of length axiom by given "unit of measure" length

3. the $H_{-1/2}$ provides an alternative model to the (space dimension depending Dirac function regularity)

4. the L_2 is a closed subspace of $H_{-1/2}$, i.e. state of the art statistical analysis is guaranteed

As the L_2 is a closed subspace of $H_{-1/2}$ compactness arguments can be applied, e.g. based on corresponding Garding type inequalities ([AzA], [BrK]).

The theorem of Riesz ensures "quasi-optimal" approximation properties of each "object" of the $H_{-1/2}$ space by an object in the L_2 space:

Theorem (Riesz): *For each ε with $0 < \varepsilon < 1$ there exists a $y \in H_{-1/2}$ with $\|y\|_{-1/2}$ and*

$$inf\{\|x-y\|_{-1/2} | x \in L_2\} \geq \varepsilon$$

Remark*: as 0 is an element of L_2 this means that the inf-term above is at* most *equal 1; therefore the theorem states that this value can be arbitrarily close approximated. This can be interpreted as counterpart of the approximation of an irrational number by rational numbers.*

A "helicopter view" regarding Schrödinger's "purely quantum wave" vision

The Schrödinger (differentiation) operator is not bounded with respect to the norm of L_2, i.e. only on a dense subspace of L_2 a corresponding spectral representation of this operator can be defined. The not vanishing constant Fourier term of the baseline Hermite polynomial (which is the Gaussian function) leads to mathematical challenges with respect to the creation and annihilation operators of the related Hamiltonian operator of the quantum oscillator model. The Hilbert transform of a function f has always vanishing constant Fourier terms. As a consequence, the Hilbert-transformed Schrödinger operator form with extended domain $H_{-1/2}$ is bounded (with respect to the norm of L_2) leads to a bounded Hermitian operator with corresponding spectral form representation.

Basically it is about spectral theory of all considered operators (resp. their inverse operators) with newly defined common Hilbert space domain, while

- the (physical) test space keeps the same, i.e. $L_2 = H_0$
- The current domains of the considered operators are extended to enable a (convergent) energy norm $\|x\|_{1/2}$ and a corresponding weak variation representation of the considered operator equations with respect to **the inner product** $(\boldsymbol{x}, \boldsymbol{y})_{-\mathbf{1/2}}$.

The corresponding notions from variation theory are "energy norm" and "operator norm" with correspondingly defined minimization problems ("energy" resp. "action" minimization problems). The corresponding eigenvalue problem of an operator T is then related to the inner product $(Tx, x)_{-1/2}$.

With respect to the newly proposed Pseudo-differential and Fourier multiplier operators with extended fractional Hilbert scale domain we note the following:

- The Maxwell equations are represented by differential equations or integral equations. Both representations are considered as equivalent.
- The Lagrange ("force") and the Hamiltonian ("energy") formalisms are considered as equivalent. The mathematical proof is based on the Legendre transform, i.e. the equivalence is only valid if the assumptions of the Legendre transform are fulfilled.

In both cases, corresponding (mathematical) regularity assumptions are required to enable those propositions. A restriction of the domain regularity of the considered operators leads to no longer well-defined classical differential equations resp. to no longer valid Lagrange formalism. In other words, the provided consistent model in the distributional framework represents the mathematical/transcendental view of the considered physical world, while the corresponding classical solutions of the several differential equations are mathematical approximations to those physical models. This concept also overcomes the "physical interpretation" challenge of the "Neumann PDE" representation of the pressure p in the NSE model.

As a consequence there is only a "one-energy" (field) concept and corresponding (PDE specific) manifestations/ forms of considered "Nature forces".

The proposed variation Hilbert space frame is built on the space-time frame with dimension $n = m + 1 = 4$. Therefore the Huygens' Principle (which is also valid for the initial value problem of the wave equation) is valid for all considered "wave" PDE, overcoming e.g. the $n > 10$ requirement of the string theory. At the same time, the characteristics roles of a space-time dimension = 4 is also underlined by the specific role of undistorted spherical travelling waves (Courant-Hilbert, "*methods of mathematical physics*", II, VI, §10.3).

Schrödinger's "purely quantum wave" vision is about half-odd integers, rather than integers to be applied to wave-mechanical vibrations which correspond to the motion of particles of a gas resp. the eigenvalues and eigen-functions of the harmonic quantum oscillator still governed by the Heisenberg uncertainty inequality. The alternatively proposed $\mathrm{H}_{1/2}$ energy space enables Schrödinger's vision ([ScE] (7.23) ff):

let ω denotes the angular frequency, h the (h-bar) Planck constant and

$$\mathrm{e} := \frac{\omega \mathrm{h}}{2}$$

Then Schrödinger's "half-odd integer vision" is about the following replacement:

$n = 0$	$E_0 = e$	$\rightarrow$	$E_{1/2} = 1 * e$
$n > 0$	$E_1 = 1 * \omega h$	$\rightarrow$	$E_{3/2} = 2 * e$
	$E_2 = 2 * \omega h$	$\rightarrow$	$E_{5/2} = 3 * e$
...			
	$E_n = n * \omega h$	$\rightarrow$	$E_{(2n+1)/2} = (n+1) * e,\ n = 0,1,2,...$
...			

As a consequence the corresponding eigenvalue and eigenfunction solutions of the number operator (i.e. the product of generation and annihilation operators) start with index $n = 1$, not already with $n = 0$.

Putting $a_n := (2\pi)^{-1/2}(-2)^n n!$, $(n = 0,1,2 \ldots.)$ the corresponding Hermite polynomials are linked to Weber's (Whittaker's) parabolic cylindrical polynomials by ([AbM] (13.1.32) [BuH] p. 215)

$$M_{\left(n+\frac{1}{2}\right)-\frac{1}{4};-\frac{1}{4}}(z) := \sqrt{2}\frac{a_n}{(2n)!} z^{\frac{1}{4}} e^{-\frac{z}{2}} He_{2n}\left(\sqrt{2z}\right) = \frac{n!}{\gamma\left(n+\frac{1}{2}\right)} z^{\frac{1}{2}} z^{-\frac{1}{4}} e^{-\frac{z}{2}} L_n^{-\frac{1}{2}}(z) = z^{\frac{1}{4}} e^{-\frac{z}{2}} M(-n, \frac{1}{2}; z)$$

$$M_{\left(n+\frac{1}{2}\right)+\frac{1}{4};+\frac{1}{4}}(z) := 2\frac{a_n}{(2n+1)!} z^{\frac{1}{4}} e^{-\frac{z}{2}} He_{2n+1}\left(\sqrt{2z}\right) = \frac{n!}{\gamma\left(n+\frac{3}{2}\right)} z^{\frac{1}{2}} z^{\frac{1}{4}} e^{-\frac{z}{2}} L_n^{\frac{1}{2}}(z) = z^{-\frac{1}{4}} e^{-\frac{z}{2}} \left[xM(-n, \frac{3}{2}; z)\right]$$

resp. ([AbM] (13.6) ([GrI] (9.231)

$$M\left(-n, \frac{1}{2}; \frac{z^2}{2}\right) = \frac{n!}{(2n)!}(-\frac{1}{2})^{-n} He_{2n}(z)$$

$$xM\left(-n, \frac{3}{2}; \frac{z^2}{2}\right) = \frac{n!}{(2n+1)!}(-\frac{1}{2})^{-n} He_{2n+1}(z)$$

where

$$M_{\left(n+\frac{1}{2}\right)+\mu;\mu}(z) = \frac{z^{\frac{1}{2}-\mu} e^{\frac{z}{2}}}{(2\mu+1)(2\mu+2)\ldots(2\mu+n)} \frac{d^n}{dz^n}\left(z^{n+2\mu} e^{-z}\right) .$$

The Mellin transform of $h_{\rho;\sigma}(z) := e^{-\frac{z}{2}} M_{\sigma;\rho}(z)$ is given by ([GrI] (7.621)

$$\int_0^\infty x^s\, h_{\rho;\sigma}(s) \frac{dx}{x} = \frac{\gamma(1+2\sigma)\gamma(\rho-s)\gamma(\frac{1}{2}+\sigma+s)}{\gamma(\frac{1}{2}+\sigma+\rho)\gamma(\frac{1}{2}+\sigma-s)} \quad , \quad Re\left(-\frac{1}{2}-\sigma\right) < Re(s) < Re(\rho) \ .$$

The half-odd integers are related to the Fourier coefficients of the convolution (integral) equation

$$\mathrm{Gu} = \mathrm{f}$$

where

$$\mathrm{G[u](y)} := \int_{-\infty}^{\infty} \mathrm{g(y-x)u(x)dx} \ , \ \ \mathrm{g(x)} := \frac{1}{\cosh(\mathrm{x})}$$

with the secans hyperbolicus function ([GrI], 1.232, 1.411)

$$\mathrm{g(x)} := \mathrm{sech(x)} := \frac{1}{\cosh(\mathrm{x})} = \mathrm{cn(x;1)} = \mathrm{dn(x;1)} = 2\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)x}$$

$$= -2\sum_{n=0}^{\infty}(-1)^{n-1}\frac{(n+\frac{1}{2})\pi}{(n+\frac{1}{2})^2\pi^2+x^2} = 1+2\sum_{n=1}^{\infty}\frac{E_{2n}}{(2n)!}x^{2n}$$

defining a distribution function similar to the normal distribution function.

The Mellin, the Hilbert and the Fourier transforms of $\mathrm{g(x)}$ are given by ([GrI], 3.523, 3.981)

$$\mathrm{M}[g](\mathrm{s}) = \int_0^{\infty} g(t)t^s\frac{dt}{t} = 2^{1-s}\gamma(s)\ {}_1F_1(-1;s;\tfrac{1}{2}),$$ (where γ denotes the Gamma function)

$$\mathrm{H}[g](\mathrm{x}) = 2\pi\int_0^{\infty}\frac{\sin(2\pi xy)}{\coth(\frac{\pi}{2}\pi y)}dy = \frac{1}{\pi}\frac{\sinh(4x)}{1+\cosh(4x)} = \frac{1}{\pi}\frac{\cosh(4x)-1}{sinh(4x)}$$

$$= -2\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x}{(n+\frac{1}{2})^2\pi^2+x^2}$$

$$\mathrm{F}[g](\mathrm{x}) := \hat{\mathrm{g}}(\omega) := \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty}\frac{e^{(1-i\omega)t}}{e^{2t}+1}dt = 2\frac{\frac{\pi}{2}}{\cosh(\frac{\pi}{2}\omega)}$$

$$= \pi g(\frac{\pi}{2}\omega) = -2\sum_{n=0}^{\infty}(-1)^{n-1}\frac{(n+\frac{1}{2})}{(n+\frac{1}{2})^2+(\frac{1}{2}\omega)^2}$$

where

$$z_n := -\mathrm{i}\pi(\frac{1}{2}+n) \ , \ n\epsilon\mathrm{Z}$$

are the simple poles of the integrand above with the residuals

$$Res_{z=z_n}\frac{e^{(1-i\omega)z}}{e^{2z}+1} = \frac{e^{(1-i\omega)z_n}}{e^{2z_n}+1} = \frac{i}{2}(-1)^n e^{-\pi\omega(n+\frac{1}{2})}$$

The solution of the above integral equation is then given by

$$\mathrm{u(x)} = \frac{1}{4\pi}\left[\frac{2}{\pi}\cosh(\frac{\pi}{2}\omega)\hat{f}(\omega)\right](-x) \ .$$

The generalized Hermite polynomials satisfy different differential equations for even and odd polynomials enabling corresponding spectral analysis ([KrA]). For the special case of the Schrödinger differential equation the spectrum of its related Schrödinger equation operator L is discrete, consisting of the odd integers. The corresponding eigen-functions form a complete orthogonal set in the weighted-L_2 space [DaD].

The Galois group of the equations $K_n^{(i)}(x) = 0$, $i = 0,1$, is the symmetric group S_n, where

$$K_n^{(0)}(x^2) = H_{2n}(x) \quad , \quad xK_n^{(1)}(x^2) = H_{2n+1}(x)$$

and $H_n(x)$ denote the Hermite polynomials ([ScW]), i.e. the polynomials $K_n^{(i)}(x)$ are non-affine.

Our alternatively proposed Schrödinger (Calderón) equation operator differs from the standard operator L by its combination with the Hilbert-transform operator H (which is an unitary operator with corresponding spectral theorem and a representation

$$H = \cos(A) + i * \sin(A) = e^{iA}$$

with A being a Hermitian operator and a corresponding spectrum on the unit circle) and an extension of the L_2 space. This enables a spectral representation of the alternatively proposed Schrödinger equation operator with *vanishing* (!) constant Fourier term being replaced by a continuous spectrum summand (modelling the "ground state zero" eigen-function/ eigen-differential) "governing" the corresponding complementary space of L_2.

A "helicopter view" regarding a (truly infinitesimal geometry) Hilbert space based quantum gravity theory

The relativistic cosmology is based on the three assumptions, the cosmological principle, the Weyl postulate and the GRT. The GRT is based on the Riemann geometry with its underlying mathematical axiom, that the *Pythagoras theorem* is valid only for the case *when two points are infinitely near*.

We note that the Legendre (contact body) transform, which is applied to prove the equivalent of the Lagrange and Hamiltonian formalism, applied to differentials, is neglecting the $dxdy$ term in

$$d(x+y)=dx+dy \quad , \quad d(xy)=xdy+ydx+dxdy=xdy+ydx$$

which is kind of contradiction to the Pythagoras axiom above.

The framework of the GRT is the affine connected manifolds ([WeH] p. 123). The metric character of a manifold is characterized relatively to a system of reference (= (1) *co-ordinate system* + (2) *calibration*) by two fundamental forms, namely a (1) quadratic differential form and a (2) linear one. They remain invariant during transformations to new co-ordinate systems.

The Weyl curvature tensor is a measure of a pseudo-Riemann manifold. It expresses the "tidal force" that a body feels when moving along a geodesic. It conveys the information how the shape of the body is distorted by this "tidal force". The Ricci curvature tensor, which expresses the trace components of the Riemann curvature tensor, contains the information about how the volumes change in the presence of "tidal forces", so the Weyl tensor is traceless component of the Riemann tensor. It is a tensor that has the same symmetries as the Riemann tensor with the extra condition that it be trace-free. The Weyl curvature is the only component of curvature for Ricci-flat manifolds and always governs the characteristics of the field equations of an Einstein manifold.

[WeH] p. 91: *The transition from Euclidean geometry to that of Riemann is founded in principle on the same idea as that which led from physics based on action at a distance to physics based on infinitely near action The Ohm Law find by the observation, that the current flowing along a conducting wire is proportional to the difference of potential between the ends of the wire. Also the Coulomb Law deals with "actions at a distance". In order to model the physical model in its most general form, one accordingly deduces this law by reducing the measurements obtained to an infinitely small portion of wire. This results in the expression*

$$curl\vec{E}=0 \quad , \quad div\vec{E}=\rho$$

on which Maxwell's theory is founded. Proceeding in the reverse direction, one derives from this differential law by mathematical processes the integral law, which we observed directly, on the supposition that conditions are everywhere similar *(homogeneity). … The fundamental fact of Euclidean geometry is that the square of the distance between two points is a quadratic form of the relative co-ordinates of the two points (Pythagoras Theorem).*

We propose a replacement of the "Pythagoras theorem" in the infinitesimal small by the concept of "rotating differentials", leveraging on the Riesz operators property to be rotation invariant. This goes along with reduced regularity requirements to the corresponding operator domain. It also anticipate Leibniz' living force concept (see below).

At the same time it enables a replacement of the affine connected manifolds concept and its underlying invariant ***exterior*** **(covariant) derivative concept** (of p-forms) (e.g. enabling Hodge's potential theory of closed Riemann manifolds based on differential forms w/o vector fields) by a distributional Hilbert space concept with a corresponding ***inner*** **(differential) derivative product** (defining a corresponding (norm-) metric). The p-forms representations of the Riemann curvature tensor are e.g. given in ([FlH]). The Riemann geometry requires *differentiable* manifold w/o any physical meaning. The alternative (distributional) Hilbert space framework avoids this purely mathematical requirement, enabling also an alternative "orthogonality concept" as being applied in the Weyl postulate, where the world lines of the fluid particles, which act as the source of the gravitational field and which are often taken to model galaxies, should be *hypersurface orthogonal.*

In the context of the newly proposed "energy-space" $H_{1/2} = H_1 + H_1^{\neg}$ (where $H_1^{\neg}$ represents the vacuum energy space) we note that already for the vacuum field equations ($R_{\alpha,\beta} = 0$) there are two solutions, the Minkowski and the Schwarzschild metrics. Therefore those metrics are not compatible with the uniquely defined Hilbert space metric/norm. **Gödel's** example of a new type of **cosmological solutions with non-vanishing density of matter** (and with a cosmological term $\neq 0$) of Einstein's field equations provides **a system with a rotation of matter** relatively to the compass of inertia. This solution, or rather the properties of the four-dimensional space it defines are also provided ([GöK]).

The complementary variational analysis is proposed to characterize the solutions of the quantum gravitational field equations ([ArA]). The method of Noble is based on a system of two operator equations, which is analogue to the Euler differential equations, covering not only (non-linear) partial differential equations, but also integral equations. It is basically about a characterization of the solution as a saddle point of a minimization functional based on a "Hamiltonian" function $W(w', u)$, which is convex with respect to w', and concave with respect to u ([VeW] 6.2.4).

In continuum mechanics, the infinitesimal strain theory is about the deformation of a solid body. The displacement gradient is a 2nd order tensor, where is it possible to perform a geometric linearization of any one of the (infinitely many possible) strain tensors, e.g. the (Lagrange) strain tensor. Considering the linearized strain tensor as the "primary" unknown, instead of the displacement in the pure traction problem of three-dimensional linearized elasticity leads to a well-posed minimization problem, constrained by a weak form of the St Venant compatibility conditions. This approach also provides a new proof of Korn's inequality ([CiP]).

A "space lab" view comment regarding related philosophical concepts

We can think (hear and watch) the Yoda quote "may the FORCE be with us" and mathematics can model this FORCE/POWER/ENERGY in a way that all corresponding physical (law) models are consistent; ... the bad (or good?) news is, that's it and that's all!
From a philosophical perspective we are back to

- Leibniz's ontology of force (e.g. "Fünf Schriften zur Logik und Metaphysik" and "Monadologie")

  "the primitive active and passive forces, the form and matter are in the monadological view understood as features of the perceptions of the monads … in this way the notion of force, … loses its foundational status: primitive force gets folded into the perceptual life of non-extended perceiving things", Garber's monograph: Leibniz: Body, Substance, Monad, 2009)

- Kant's conception of physical matter and the existence of ether, which fills the whole space and time with its moving forces ([WoW])

  "it is the moving forces of the ether that affect us"; …"There exists a matter, distributed in the whole universe as a continuum, uniformly penetrating all bodies, and filling (all spaces) (thus not subject to displacement). Be it called ether, or caloric, or whatever…", … "space is hypostatically", …" Space which can be sensed (the object of the empirical intuition of space) is the complex of moving forces of matter – without which, space would be no object of possible experience", …"Matter does not consist of simple paets, but each part is, in turn, composite…", Each part of matter is a quantum; i.e. matter does not consist of metaphysically simple parts, and Laplace's talk of material points (which were to be regarded as parts of matter) would, understood literally, contain a contradiction." … "Atomism is a false doctrine of nature", …forces "fill a space (both) extensively and intensively", …

- Schrödinger's "*(my) view of the world*" with respect to "*reasons for abandoning the dualism of thought and existence, or mind and matter*"

  "The objective world has only been constructed at the prize of taking the self, that is, mind, out of it remaking it; mind is not part of it; obviously, therefore, it can neither act on it nor be acted on by any of its parts. If this problem of the action of mind on matter cannot be solved within the framework of our scientific representation of the objective world, where and how can it be solved?" ... "No single man can make a distinction between the realm of his perceptions and the realm of things that cause it, since however detailed the knowledge he may have acquired about the whole world, the story is occurring only once and not twice. The duplication is an allegory suggested mainly by communication with other beings."

  A TENTATIVE ANSWER:'"A single experience that is never to repeat itself is biologically irrelevant. Biologic value lies only in learning the suitable reaction to a situation that offers itself again and again, in many cases periodically, and always requires the same response if the organism is to hold its ground." ... But whenever the situation exhibits a relevant differential - let us say the road is up at the place where we used to cross it, so that we have to make a detour - this differential and our response to it intrude into consciousness, from which, however, they soon fade below the threshold, if the differential becomes a constantly repeating feature. .... Now in those fashion differentials, variants of response, bifurcations, etc., are piled up one upon the other in unsurvey able abundance, but only the most recent ones remain in the domain of consciousness, only those with regard to which the living substance is still in the stage of learning or practicing.

  ... I would summarize my general hypothesis thus: consciousness is associated with the learning of living substance; it's knowing how (Können) is unconscious"

- Heidegger's notion of „mathematical" physics ("Holzwege, die Zeit des Weltbildes" (72) ff: die neuzeitliche Physik heisst mathematische, weil sie … eine ganz bestimmte Mathematik anwendet. … Keineswegs wird das Wesen des Mathematischen durch das Zahlenhafte bestimmt. … Durch sie (math. Physik) und für sie wird in einer betonten Weise etwas als das Schon-Bekannte im vorhinein ausgemacht. Der sich geschlossene Bewegungszusammenhang raum-zeitlich bezogener Massenpunkte. … Kein Zeitpunkt hat vor einem anderen einen Vorzug. Jede Kraft bestimmt sich nach dem, … was sie an Bewegung ..in der Zeiteinheit zur Folge hat.

A. The Riemann Hypothesis

All nontrivial zeros of the analytical continuation of the Riemann zeta function have a real part of $1/2$. The Hilbert-Polya conjecture states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self adjoint operator.

We provide a solution for the RH building on a new Kummer function based Zeta function theory, alternatively to the current Gauss-Weierstrass function based Zeta function theory. This primarily enables a proof of the Hilbert-Polya conjecture (but also of other RH criteria like the Bagchi formulation of the Nyman-Beurling criterion or Polya criteria), whereby the imaginary parts of the zeros of the corresponding alternative Zeta function definition corresponds to eigenvalues of a bounded, self adjoint operator with (newly) distributional Hilbert space domain.

Let H and M denote the Hilbert and the Mellin transform operators. For the Gaussian function $f(x)$ it holds

$$M[f](s)=\frac{1}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) \quad , \quad M[-xf'(x)](s)=\frac{s}{2}\pi^{-s/2}\Gamma(\frac{s}{2})=\frac{1}{2}\pi^{-s/2}\Pi(\frac{s}{2})$$

The corresponding entire Zeta function is given by

$$\xi(s):=\frac{s}{2}\Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s)=(1-s)\cdot\zeta(s)M[-xf'(x)](s)=\xi(1-s).$$

The central idea is to replace

$$M[xf'(x)](s)\rightarrow M[f_H(x)](s)$$

by

$$M[f_H(x)](s)=2\pi\cdot M\left[x\cdot{}_1F_1(1,\frac{3}{2},-\pi x^2)\right](s)=\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\tan(\frac{\pi}{2}s)$$

This enables the definition of an alternative entire Zeta function in the form

$$\xi^*(s):=(1-s)\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\tan(\frac{\pi}{2}s)\cdot\zeta(s)$$

with same zeros as $\xi(s)$.

The fractional part function

$$\rho(x):=\{x\}:=x-[x]=\frac{1}{2}-\sum_1^\infty\frac{\sin 2\pi\nu x}{\pi\nu}\in L_2^\#(0,1)$$

is linked to the Zeta function by

$$\zeta(1-s)=(s-1)M[\rho](s-1)=M[-x\rho'(x)](s-1)$$

The Hilbert transform of the fractional part function is given by

$$\rho_H(x)=\sum_1^\infty\frac{\cos 2\pi\nu x}{\pi\nu}=-\frac{1}{\pi}\log 2\sin(\pi x)\in L_2^\#(0,1) \quad , \quad \hat{\rho}_H(0)=0 \quad , \quad \rho'_H\in H_{-1}^\#$$

Applying the idea of above leads to the replacement

$$M[-x\rho'(x)](s-1)=\zeta(1-s) \qquad \rightarrow \qquad M[-\rho_H(x)](s-1)$$

$$M[-\rho_H(x)](s-1)=M\left[\sum_1^\infty\frac{\cos 2\pi\nu x}{\pi\nu})\right](s-1)=2\pi\Gamma(\frac{1-s}{2})\Gamma^{-1}(\frac{s}{2})\cdot\chi(s)\cdot\frac{\zeta(s)}{s}$$

with same zeros as $\zeta(1-s)$.

The proposed framework also provides an answer to Derbyshine's question, ("Prime Obsession")

... "*The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?*"

The answer, in a nutshell:

"identifying "fluids" or "sub-atomic particles" not with real numbers (scalar field, I. Newton), but with hyper-real numbers (G. W. Leibniz) enables a truly infinitesimal (geometric) distributional Hilbert space framework (H. Weyl) which corresponds to the Teichmüller theory, the Bounded Mean Oscillation (BMO) and the Harmonic Analysis theory. The distributional Hilbert scale framework enables the full power of spectral theory, while still keeping the standard L(2)=H(0)-Hilbert space as test space to "measure" particles' locations. At the same time, the Ritz-Galerkin (energy or operator norm minimization) method and its counterpart, the methods of Trefftz/Noble to solve PDE by complementary variation principles (A. M. Arthurs, K. Friedrichs, L. B. Rall, P. D. Robinson, W. Velte) w/o anticipating boundary values) enables an alternative "quantization" method of PDE models (P. Ehrenfest), e.g. being applied to the Wheeler-de-Witt operator.

Regarding the proposed alternative quantization approach we also refer to the Berry-Keating conjecture. This is about an unknown quantization ***H*** of the classical Hamiltonian H=xp, that the Riemann zeros coincide with the spectrum of the operator 1/2+i***H***. This is in contrast to canonical quantization, which leads to the Heisenberg uncertainty principle and the natural numbers as spectrum of the harmonic quantum oscillator. The Hamiltonian needs to be self-adjoint so that the quantization can be a realization of the Hilbert-Polya conjecture.

B. The Navier-Stokes Equations

The Navier-Stokes equations describe the motion of fluids. The *Navier–Stokes existence and smoothness* problem for the three-dimensional NSE, given some initial conditions, is to prove that smooth solutions always exist, or that if they do exist, they have bounded energy per unit mass.

We provide a global unique (weak, generalized Hopf) $H_{-1/2}$-solution of the generalized 3D Navier-Stokes initial value problem. The global boundedness of a generalized energy inequality with respect to the energy Hilbert space $H_{1/2}$ is a consequence of the Sobolevskii estimate of the non-linear term (1959).

The "standard" weak Hopf solution is not well posed due to not appropriately defined domains of the underlying velocity and pressure operators. Therefore, this is also the case for the corresponding classical solution(s).

The proposed solution also overcomes the "Serin gap" issue, as a consequence of the bounded non-linear term with respect to the appropriate energy norm.

The Navier-Stokes Equations (NSE) describes a flow of incompressible, viscous fluid. The three key foundational questions of every PDE is existence, and uniqueness of solutions, as well as whether solutions corresponding to smooth initial data can develop singularities in finite time, and what these might mean. For the NSE satisfactory answers to those questions are available in two dimensions, i.e. 2D-NSE with smooth initial data possesses unique solutions which stay smooth forever. In three dimensions, those questions are still open. Only local existence and uniqueness results are known. Global existence of strong solutions has been proven only, when initial and external forces data are sufficiently smooth. Uniqueness and regularity of non-local Leray-Hopf solutions are still open problems.

Basically the existence of 3D solutions is proven only for “large” Banach spaces. The uniqueness is proven only in “small” Banach spaces. The question of global existence of smooth solutions vs. finite time blow up is one of the Clay Institute millennium problems.

The existence of weak solutions can be provided essentially by the energy inequality. If solutions would be classical ones, it is possible to prove their uniqueness. On the other side for existing weak solutions it is not clear that the derivatives appearing in the inequalities have any meaning.

Basically all existence proofs of weak solutions of the Navier-Stokes equations are given as limit (in the corresponding weak topology) of existing approximation solutions built on finite dimensional approximation spaces. The approximations are basically built by the Galerkin-Ritz method, whereby the approximation spaces are e.g. built on eigen-functions of the Stokes operator or generalized Fourier series approximations.

It has been questioned whether the NSE really describes general flows: The difficulty with ideal fluids, and the source of the d'Alembert paradox, is that in such fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbing on each other, provided they are separated by a streamline. It is clear that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

The question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the NSE blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost. Given a smooth datum at time zero, will the solution of the NSE continue to be smooth and unique for all time?

There is no uniqueness proof for weak solutions except for over small time intervals. The simplest possible model example how a singularity can appear, is the ODE

$$y'(t) = y^2(t) \quad y(0) = y_0$$

with its solution

$$y(t) = \frac{y_0}{1 - t \cdot y_0}$$

which becomes infinite in finite time. For n=3 every positive solution of

$$y'(t) = cy^3(t)$$

blows up, i.e. there is no global estimate by this method.

The global boundedness of our solution is a consequence of the Sobolevskii-estimate of the non-linear term enabling the generalized energy inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le |(Bu,u)_{-1/2}| \le c \cdot \|u\|_{-1/2}\|u\|_1^2$$

Putting

$$y(t) := \|u\|_{-1/2}^2$$

one gets

$$y'(t) \le c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$$

This result into the a priori estimate

$$\|u(t)\|_{-1/2} \le \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \le c\left\{\|u_0\|_{-1/2} + \|u_0\|_0^2\right\}$$

ensuring global boundedness by the a priori energy estimate provided that $u_0 \in H_0$.

C. The Yang-Mills Equations

The YME are concerned with quantum field theory. Its related Millennium problem is about an appropriate mathematical model to govern the current "mass gap" of the YME, which is the difference in energy between the vacuum and the next lowest energy field.

The classical Yang-Mills theory is a generalization of the Maxwell theory of electromagnetism where the *chromo*-electromagnetic field itself carries charges. For given distributions of electric charges and currents the Maxwell equations determine the corresponding electromagnetic field. The laws by which the currents and charges behave are unknown. The energy tensor for electromagnetic fields is unknown for elementary particles. Matter is built by electromagnetic particles, but the field laws by which they are constituted are unknown, as well. The original inertia law (before Einstein's gravity theory) forced to attribute physical-objective properties to the space-time continuum. Analog to the Maxwell equations (in the framework of a short distance theory) Einstein considered the inertia law as a field property of the space-time continuum.

As a classical field theory the Maxwell equations have solutions which travel at the speed of light so that its quantum version should describe massless particles (gluons). However, the postulated phenomenon of color confinement permits only bound states of gluons, forming massive particles. This is the mass gap. Another aspect of confinement is asymptotic freedom which makes it conceivable that the quantum Yang-Mills theory exists without restriction to low energy scales. The problem is to establish rigorously the existence of the quantum Yang-Mills theory and a mass gap. *The conceptual current, to be challenged concept is about the displacement current, which is "just" about a mathematical requirement to enabled consistent mathematical data model.*

Based on an $H_{1/2}$ energy Hilbert space we propose (analog to the NSE solution) a corresponding (weak) variation Maxwell equation representation. Its corresponding generalization (as described above) leads to a modified QED model. In the same manner as the Serrin gap issue has been resolved (as a result of the reduced regularity requirements) the chromo-electromagnetic field /particles can now carry charges. The open "field law" question above and how "particles" are interacting with each other to exchange energy are modeled in same manner as the coherent/incoherent turbulent flows of its NSE counterpart. The corresponding "zero state energy" model is no longer built on the Hermite polynomials but on its related Hilbert transformed Hermite polynomials, which also span the L_2 – Hilbert "test" space.

This provides a truly infinitesimal geometry (H. Weyl), enabling the concept of Riemann that force is a pseudo force only, which results from distortions of the geometrical structure. The baseline is a common Hilbert space framework (for all (nearby action) differential equations)

- providing the mathematical concept of a geometrical structure (while Riemann's manifold concept provides only a metric space and related affine connections)

- replacing "force type" specific gauge fields and its combination model(s) for the electromagnetic, the strong and the weak nuclear power "forces"

- building an integrated (no longer "force" dependent dynamical matter-field interaction laws) universal field model (including the gravity "force")

As a consequence there are no "mass" and therefore no (YME-) "mass gap" anymore, but there is an appropriate vacuum (Hilbert) energy space, which is governed by the Heisenberg uncertainty principle.

Inner products with corresponding norms of a distributional Hilbert scale can be defined based on the eigen-pairs of an appropriately defined operator in the form

$$(x,y)_\alpha := \sum_i^\infty \lambda_i^\alpha (x,\varphi_i)(y,\varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

Additionally, for $t > 0$ there can be an inner product resp. norm defined for an additional governing Hilbert space with an "exponential decay" behavior in the form

$$e^{-\sqrt{\lambda_i} t}$$

given by

$$(x,y)_{(t)}^2 := \sum_{i=1} e^{-\sqrt{\lambda_i} t} (x,\varphi_i)(y,\varphi_i)$$

$$\|x\|_{(t)}^2 := (x,x)_{(t)}^2$$

The distributional $H_{-1/2}$ – Hilbert space is proposed to model quantum states, alternatively to the Hilbert space H_0. A mathematical (wavelet microscopic) analysis of those states is then about an analysis of the "objects"

$$x = x_0 + x_0^\perp \in H_0 \otimes H_0^\perp$$

with

$$\|x_0\|_0 = 1 \qquad \sigma := \|x_0^\perp\|_{-1/2}^2$$

As it holds for any $t, \delta, \alpha > 0$ and $\lambda \geq 1$ the inequality

$$\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{t(\delta^{-1} - \sqrt{\lambda})}$$

the following inequality is valid for any $\in H_0$, governing the approximation "quality" of a quantum state with respect to the norm of H_0:

$$\|x\|_{-1/2}^2 \leq \delta \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2 = \sigma \|x\|_0^2 + e \|x\|_{(\sigma)}^2 = \sigma \|x\|_0^2 + \sum_{i=1}^\infty e^{1-\sqrt{\lambda_i}\sigma} x_i^2$$

It balances the "continuous" view of the overall state with its "discrete" components.

D. The Heisenberg uncertainty inequality

Let $g \in H_0 = L_2(R)$, $\|g\|_0 = 1$, then

$$\mu(g) := t_0\omega_0 := (\int_R (t - t_0)^2 |g(t)|^2)(\int_R (\omega - \omega_0)^2 |\hat{g}(\omega)|^2) \geq \frac{1}{4}$$

whereby t_0 denotes the mean value of the location of the particle and

$$\omega_0 := \int_R \omega |\hat{g}(\omega)|^2 \, d\omega = (-i\frac{d}{dx}\varphi, \varphi)_0$$

the mean value of the momentum of the particle. The localization "uncertainty" $\mu(g)$ of the function g at the phase point (t_0, ω_0) is defined, provided that $-\infty < t_0, \omega_0 < \infty$, i.e. $g \in H_{1/2}$.

The proposed alternative quantum state Hilbert space $H_{-1/2}$ provides an alternative concept to the "Dirac function" calculus. This overcomes current handicaps concerning the regularity of the Dirac function, which depends from the space dimension, i.e. $\delta \in H_{-s}(R^n)$ for $s > n/2$.

The alternatively proposed Hilbert space $H_{-1/2}$ provides a truly "microscopic" mathematical frame (independently from the space dimension), while still supporting the existing physical observation (statistical analysis) subspace. It is also proposed to replace the (continuous & differentiable) manifold concept (and exterior products of differential forms) in Einstein's field theory.

E. The (Hilbert) transform of a differential du (Plemelj) and the corresponding positive definite continued fraction (Stieltjes) transform representation of u

The Stieltjes continued fraction theory provides an integral representation of the continued fraction

$$\frac{1|}{|a_1-z} - \frac{b_1^2|}{|a_2-z} - \frac{b_2^2|}{|a_3-z} - \cdots \quad , \quad a_n \in R, b_n \in R-\{0\}, z \in Z$$

in the form ([HeE], [BrK1])

$$\mathrm{u(z)} = \mathrm{S}[\sigma](z) := \int_{-\infty}^{\infty} \frac{d\sigma(\mu)}{z-\mu} = \frac{1|}{|a_1-z} - \frac{b_1^2|}{|a_2-z} - \frac{b_2^2|}{|a_3-z} - \cdots$$

provided that the series $\sum_{n=1}^{\infty} |p_n(z)|^2$ diverge for at least one non-real $z \in Z-R$ (and therefore for all $z \in Z-R$) whereby the polynomials $p_n(z)$ are defined by the linear homogenous equations

$$(a_1-z)p_1(z) - b_1p_2(z) = 0$$

$$-b_{n-1}p_{n-1}(z) + (a_n-z)p_n(z) - b_np_{n+1}(z) = 0 \quad .$$

The above is about a real bounded J-fraction ([WaH], theorem 27.4). The equivalent function of a positive definite J-fraction can be represented as a Stieltjes transform ([WaH], theorems 65.1 & 66.1).

Just as a side remark we note the result from P. Bundschuh: the number

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}} = \frac{1}{2} + \frac{\pi}{2}\coth(\pi)$$

is transcendental.

The above formula is related to the series ([ChK] VI, §2)

$$\frac{1}{2} - \frac{\pi}{2} z\cot(\pi z) = \frac{1}{2} + \frac{\pi}{2} \mathrm{z}(\frac{d}{dz}\rho_H(\mathrm{z})) = \sum_{n=1}^{\infty} zeta(2n)z^{2n} \quad , |z| < 1 \quad .$$

F. The Boltzmann Entropy

The Boltzmann equation is a (non-linear) integro-differential equation which forms the basis for the kinetic theory of gases. This not only covers classical gases, but also electron /neutron /photon transport in solids & plasmas / in nuclear reactors / in super-fluids and radiative transfer in planetary and stellar atmospheres. The Boltzmann equation is derived from the Liouville equation for a gas of rigid spheres, without the assumption of "molecular chaos"; the basic properties of the Boltzmann equation are then expounded and the idea of model equations introduced. Related equations are e.g. the Boltzmann equations for polyatomic gases, mixtures, neutrons, radiative transfer as well as the Fokker-Planck and Vlasov equations. The treatment of corresponding boundary conditions leads to the discussion of the phenomena of gas-surface interactions and the related role played by proof of the Boltzmann H-theorem.

We provide the relationships of the Boltzmann and Landau equations to the proposed alternative quantum state Hilbert space frame $H_{-1/2}$ resp. its corresponding quantum energy space $H_{1/2}$ and the related NSE global solution, as well as to the proposed alternative Schrödinger (Calderón) momentum operator. At the same point in time we suggest to apply the same frame for a proof of the non-linear Landau damping for the Vlasov equation with only physical relevant mathematical assumptions.

In line with the proposed distributional $H_{-\frac{1}{2}}$ –Hilbert space concept of this paper, we suggest to define "continuous entropy" in a weak $H_{-\frac{1}{2}}$ – frame in the form

$$h(X) := (f, \log\frac{1}{f})_{-1/2} \ ,$$

where X denotes a continuous random variable with density $f(x)$. In this case it can be derived from a Shannon (discrete) entropy in the limit of n, the number of symbols in distribution $P(x)$ of a discrete random variable X ([MaC]):

$$H(X) := \sum_i P(x_i)\log(\frac{1}{P(x_i)}) \ .$$

This distribution $P(x)$ can be derived from a set of axioms. This is not the case, in case of the standard entropy in the form

$$h(X) := (f, \log\frac{1}{f})_0 .$$

G. Wavelets

Wavelets are proposed as appropriate analysis tool for the proposed NMEP, additionally to Fourier analysis technique. There are at least two approaches to wavelet analysis, both are addressing the somehow contradiction by itself, that a function over the one-dimensional space R can be unfolded into a function over the two-dimensional half-plane.

A wavelet transform W_ϑ

$$W_\vartheta[f](a,b) := |a|^{-1/2} \int_R f(t)\vartheta\left(\frac{t-b}{a}\right) dt$$

of a function f with respect to a wavelet function ϑ is an isometric mapping. The admissibility condition is given by

$$0 < c_\vartheta := \int_R \frac{|\hat{\vartheta}(\omega)|^2}{|\omega|} d\omega < \infty$$

The corresponding adjoint operator W_ϑ^* is given by the inverse wavelet transform on its range:

$$W_\vartheta^*[g](a,b) := c_\vartheta^{-1/2} \int_R \int_R |a|^{-\frac{1}{2}} g(a,b) \frac{1}{a} \vartheta\left(\frac{t-b}{a}\right) \frac{da}{a} db$$

The Fourier transform of a wavelet transformed function f is given by

$$\widehat{W_\vartheta[f]}(a,\omega) := (2\pi|a|)^{\frac{1}{2}} c_\vartheta^{-\frac{1}{2}} \hat{\vartheta}(-a\omega)\hat{f}(\omega) \quad .$$

For $\varphi, \vartheta \in L_2(R)$, $f_1, f_2 \in L_2(R)$,

$$0 < |c_{\vartheta\varphi}| := 2\pi \left| \int_R \frac{\hat{\vartheta}(\omega)\bar{\hat{\varphi}}(\omega)}{|\omega|} d\omega \right| < \infty$$

and $|c_{\vartheta\varphi}| \leq c_\vartheta c_\varphi$ one gets the duality relationship ([LoA])

$$(W_\vartheta \, f_1, W_\varphi^* f_2)_{L_2(R^2, \frac{dadb}{a^2})} = c_{\vartheta\varphi}(f_1, f_2)_{L_2}$$

i.e.

$$W_\varphi^* W_\vartheta \, [f] = c_{\vartheta\varphi} f \quad \text{in a } L_2 - \text{sense.}$$

This identity provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets in a (distributional) Hilbert scale framework where the "microscope observations" of two wavelet (optics) functions ϑ, φ can be compared with each other by the above "reproducing" ("duality") formula. The prize to be paid is about additional efforts, when re-building the reconstruction wavelet. The extended admissibility condition above indicates that wavelet "pairs" in the form $(\varphi, \vartheta) \in L_2 x H_{-1}$ would be an appropriate good baseline to start from, when analyzing in the Hilbert space frame $H_{-1/2} = L_2 x L_2^{\neg}$, where $L_2^{\neg}$ denotes the complementary space of L_2 with respect to the $H_{-1/2}$ −norm. The Hilbert transform operator (which is valid for every Hilbert scale) is the "natural" partner to the wavelet-transform operator, as it is skew-symmetric, rotation invariant and each Hilbert transformed "function" has vanishing constant Fourier term. The example in the context above is the Hilbert transform of the Gaussian distribution function, the (odd) Dawson function, with the "polynomial degree" point of zero at +/- infinite.

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