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Cosmology as a Window to Physics beyond Einstein Gravity

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ACADEMIC DISSERTATION

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Seek the company of those who are looking for the truth, but run from those who have found it – Vaclav Havel

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Abstract

This thesis consists of three research papers [1–3], and an introduction which provides some of the relevant background and additional discussions.

The work concentrates, as the title indicates, on the relation between fundamental physics and the spectrum of CMB anisotropies. By fundamental physics we primarily mean string theory. The thesis deals partly with the effect of fundamental physics on the standard treatment of cosmic quantum fluctuations, and partly with the phenomenology of CMB physics in an alternative scenario of cosmology, inspired by string theory, called the pre-big bang scenario.

The thesis contains a self-contained introduction to both the standard theory of inflation and the pre-big bang scenario. The issue of classical and quantum cosmological perturbations is considered in order to explain both the physics of the trans-Planckian problem and the curvaton mechanism.

Acknowledgments

I am eternally grateful to my supervisor Kari Enqvist, who arranged that I could come to Helsinki and for my funding. It has also been a great pleasure to take his (always cheerful) advice in science.

Neither can I thank Fawad Hassan enough, for always being willing to discuss and answer my questions in great detail. I regard him as an additional supervisor and a great friend.

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Back in Denmark there are a few people that I would like to mention on this occasion. At the Niels Bohr Institute I would like to thank my former master thesis supervisor, Paolo Di Vecchia, from whom I learned most of the few things of string theory that I know and about doing research. Poul Olesen and Poul Henrik Damgaard suggested me to go to Helsinki and I will never blame them for that.

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Martin Snoager Sloth
Helsinki, Spring 2003

List of Papers

[1] Kari Enqvist, Martin S. Sloth

ADIABATIC CMB PERTURBATIONS IN PRE - BIG BANG STRING COSMOLOGY.

Nucl.Phys.**B626:**395-409,2002 [hep-ph/0109214]

[2] Martin S. Sloth

SUPERHORIZON CURVATON AMPLITUDE IN INFLATION AND PRE - BIG BANG COSMOLOGY.

Nucl.Phys.**B656:**239-251,2003 [hep-ph/0208241]

[3] S.F. Hassan, Martin S. Sloth

TRANSPLANCKIAN EFFECTS IN INFLATIONARY COSMOLOGY AND THE MODIFIED UNCERTAINTY PRINCIPLE.

hep-th/0204110 (Accepted for publication in Nucl.Phys.**B**)

The contribution of the present author to joint publications

My contribution to publication [1]: Kari Enqvist suggested me to study the isocurvature perturbations. I got the idea of solving the problem of the isocurvature perturbations in the pre-big bang scenario by using a late decaying non-relativistic axion field (curvaton mechanism). I also did the calculations. Kari Enqvist supervised and helped writing the paper.

My contribution to publication [3]: We formed the ideas together and the calculations were done jointly.

In all papers authors are listed alphabetically according to the particle physics convention.

Chapter 1

Preliminaries

Today, in the cosmology community, there are two schools of thought. From a typical particle physicist's point of view the initial conditions of the universe must be determined by symmetries and therefore must be definite, simple, and elegant. On the other hand, an astrophysicist typically thinks in terms of dynamical principles instead of symmetries and believes that the initial condition is some unknown, general, perhaps complicated state. Astrophysicists often speak about how likely a given initial state is. Although the controversy is mostly a philosophical one, it is important for cosmological model building. Today, inflation as a scenario is associated with the astrophysicist's point of view, but it is interesting to reflect upon what the first physicist to suggest inflation thought about the initial state of inflation. In the original paper from 1980 Starobinsky suggested an eternal de Sitter phase (inflationary phase) and "*after that, the Friedmann expansion with the effective equation of state* $p = 0$ *takes place. In terms of the full quantum theory of gravity, the initial most symmetric state can probably be thought of as an eigenstate of the curvature. This scenario of the beginning of the universe is the extreme opposite of Misner's initial chaos.*" [4]. Thus, when de Sitter inflation was first proposed, a part of the motivation was that it is a maximally symmetric space and therefore serves as an appealing initial condition for the universe.

1.1 Introduction

There is continuous interest in what string theory has to say about cosmology. In this thesis we explore two such possibilities. The standard scenario of the early universe is based on slow roll inflation. In the first part of chapter 2 we review some aspects of it, relevant for understanding how string effects (or so called trans-Planckian effects) might influence the quantum fluctuations in the very early universe. In chapter 3 we review the only model so far (Hassan and Sloth [3]), which consistently incorporates the minimum length feature of string theory, and we show how it might solve the problem of uniquely fixing the initial state of the perturbations in the trans-Planckian regime. Also the novel feature called *trans-Planckian damping* is explained.

In second part of chapter 2, we review an alternative scenario of cosmology based on string theory, called the *pre-big bang* scenario. The pre-big bang scenario has been under intensive investigation for over a decade. Until recently, two of the problems with this scenario were the graceful exit and the predicted spectrum of CMB anisotropies. In the last part of section 2, both problems are briefly reviewed. In chapter 4 we then show how the curvaton mechanism can naturally be incorporated into the pre-big bang model, so that it can predict the observed spectrum of CMB anisotropies [1]. This was the first real application of the curvaton mechanism. The curvaton mechanism itself has been known for more than a decade. Later, the curvaton mechanism has been applied in various ways to ordinary inflation. So, while the original application of the curvaton mechanism (Enqvist and Sloth [1]) was motivated by an urgent problem of the pre-big bang model, the later applications to ordinary inflation can instead be considered as alternative scenarios to the single field inflationary model or the usual hybrid model. The first part of chapter 4 contains a general discussion of the curvaton model, based on perhaps the most elegant demonstration of it by Lyth and Wands [5]. In the last part of chapter four we explore the applications to the pre-big bang scenario.

It is impossible to mention all the literature in the field, but there is a recent review on inflationary cosmology by Riotto [6], which has influenced the discussion of standard cosmology in this thesis. There are two extensive reviews of pre-big bang cosmology by Gasperini and Veneziano [7] and by Lidsey *et. al.* [8] which are definitely worth mentioning as they have also been a source of inspiration for the present author.

There have of course also been other developments in the direction of understanding cosmology in terms of fundamental physics and vise versa. Some of the main directions of current research in this field are Brane (anti Brane) inflation [9–13], Brane gasses [14, 15], Holography [16–20], Ekpyrotic [21, 22], and Cyclic models [23] (The references are only examples and should not be considered as complete lists).

Throughout this thesis we will work in natural units $c = \hbar = k_{Boltzmann} = 1$ and we use the metric signature $\eta = diag(-, +, \ldots, +)$. Everywhere we will use the Misner, Thorne, Wheeler sign convention. We denote the Planck-mass by m_p and it is related to the Newton constant G_N by $m_p = 1/\sqrt{G_N} \approx 1.221 \cdot 10^{19}$GeV. The reduced Planck mass is defined as $M_p \equiv 1/\sqrt{8\pi G_N}$.

1.2 Standard Big-Bang model

General relativity combined with the assumption that the universe is homogeneous and isotropic on large scales leads to the standard Big-Bang model (SBB). Using isotropy and homogeneity as an ansatz for the Einstein equation uniquely fixes the metric to be the Friedman-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] , \tag{1.1}$$

where $a(t)$ is the scale-factor of the universe and the parameter k can take values $1, 0, -1$ corresponding respectively to spherical, euclidean and hyperbolic spatial sections which we also refer to as a spatially closed, flat or open universe.

The SBB is a well established model that has been verified in many ways. One controversial (at its time of discovery) and successful prediction of the SBB is the expansion of the universe. The expansion rate $H = \dot{a}/a$ is called the Hubble constant and the experimental value today is $H_0 = 100h$ km sec^{-1}Mpc^{-1}, with $h = 0.71^{+0.04}_{-0.03}$ [24]. The observation of the expansion of the universe by Edwin Hubble lead Einstein to abandon the cosmological constant which he had introduced by hand and fine-tuned to obtain a static universe. However, today we know that much of the energy density of the universe has the form of dark energy which can be mimicked by a cosmological constant, thus the cosmological constant is as important part of cosmology as ever.

The dynamics of the universe is encoded in the Einstein equation, which can be derived from the action $S = S_{EH} + S_M$, where S_{EH} is the Einstein-Hilbert action and S_M is the matter action;

$$S_{EH} = -\frac{1}{2\kappa^2} \int d^D x \sqrt{-g} (R + 2\Lambda) , \qquad S_M = \sum_{fields} \int d^D x \sqrt{-g} \mathcal{L}_{fields} . \tag{1.2}$$

Here Λ is the cosmological constant, $\kappa^2 \equiv 8\pi G_N$ and R is the Ricci scalar. Denoting by $R_{\mu\nu}$ the Ricci tensor and by $R^\lambda_{\mu\nu\sigma}$ the Riemann tensor, with

$$R_{\mu\sigma} = R^\lambda_{\mu\lambda\sigma} , \tag{1.3}$$

the Einstein equation becomes

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \equiv G_{\mu\nu} = 8\pi G_N T_{\mu\nu} + g_{\mu\nu}\Lambda . \tag{1.4}$$

Above the energy-momentum tensor is defined as $T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \delta S_M / \delta g^{\mu\nu}$, however below we will absorb the cosmological constant in $T_{\mu\nu}$ instead of writing it explicitly. Applying the assumptions of homogeneity and isotropy, we can parameterize the energy-momentum tensor in the form $T^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p)$ where ρ is the total energy density and p is the total pressure.

The evolution of the cosmic scale factor is governed by the Friedmann equation

$$H^2 \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G_N \rho}{3} - \frac{k}{a^2} , \tag{1.5}$$

which is obtained from the Einstein equation by inserting the FRW metric (1.1) into equation (1.4). Here ρ is the total energy density of the universe including matter, radiation, vacuum energy, and so on. In a similar fashion one finds from the time component of the Einstein equation

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p) \ . \tag{1.6}$$

The continuity relation $\nabla_\nu T^{\mu\nu} = 0$ can be written in the following form:

$$\dot{\rho} + 3H(\rho + p) = 0 \ . \tag{1.7}$$

This implies that for a given equation of state for the cosmological fluid ($p = w\rho$) one obtains

$$\rho \propto a^{-3(w+1)} \ . \tag{1.8}$$

The Friedmann equation then implies $a \propto t^{2/3(1+w)}$.

It is common to use the Friedmann equation to relate the curvature of the universe to the energy density and the expansion rate

$$\Omega = \frac{\rho}{\rho_c} \ , \qquad \Omega - 1 = \frac{k}{a^2 H^2} \ , \tag{1.9}$$

where $\rho_c = 3H^2/8\pi G_N$ is the critical energy density. If $\Omega < 1$, $\Omega = 1$ or $\Omega > 1$ it implies that the universe is open, flat or closed respectively. Once we know the equation of state we can solve the Friedmann equations for the cosmic scale factor. The "fate of the universe" is determined by the curvature. An open and a flat universe expand forever, while a closed universe re-collapses (this is not strictly true when the role of dark energy is taken into account).

The SBB is not eternal in the past. The open, flat and closed model originate from some finite initial time, usually set to $t = 0$, at which the scale factor vanishes. At the initial time the space-time curvature blows up and it marks a singularity beyond which we can not extend the time coordinate. From general relativity it has been shown that if a space-time satisfy some quite general assumptions, such as the strong energy condition and the absence of closed time-like curves, it always contains an initial singularity[1] [25].

As the cosmic scale factor grows and the universe expands, light becomes red-shifted. The red-shift z is given by the wavelength of the emitted wave λ_e and the wavelength of the received wave λ_r made longer by the expansion ($1 + z = \lambda_r/\lambda_e = a(t_e)/a(t_r)$). It is often convenient to measure time in terms of the redshift. The conversion between t and z can be found from integrating the Friedmann equation (1.5).

Today the contributions of matter and the cosmological constant to the energy density of the universe are comparable. From $\rho \propto a^{-3(w+1)}$ it is seen that the energy density of respectively the cosmological constant, radiation and matter behave as $\rho_\Lambda = const.$, $\rho_r \sim a^{-4}$, $\rho_m \sim a^{-3}$ as the cosmic scale factor goes to zero in the past.

[1]See also section 2.1.5 for more details.

Hence, we conclude that in the past matter must have dominated the universe and before that radiation.

Measurements of the cosmic microwave background radiation (CMBR) tell us that the cosmic photon temperature today is $T_0 = 2.725 \pm 0.002$K [26] and the spectrum is an almost perfect Planck spectrum. This supports the SBB model that tells us that the universe originates from a much hotter and denser state than the present, where interaction between matter and radiation were frequent enough to create a thermal equilibrium. As the universe expands, thermal equilibrium for a given particle species can be maintained, provided the interaction rate Γ is greater than the Hubble parameter, $\Gamma > H$. An important stage in the history of the early universe was when the photons decoupled from the electrons, as the electrons began to recombine with nuclei to form atoms, at $T_{rec} \simeq 3000K$ or $z_{rec} \simeq 1100$ [27]. Since the time of decoupling, the photon-fluid has "free-streamed" to reach us today with little changes, apart from the red-shift due to the expansion of the universe. Thus, the CMBR provides us with a "snapshot" of the very early universe.

One of the greatest triumphs of the SBB-model is its ability to explain primordial nucleosynthesis. Big-Bang nucleosynthesis (BBN) is the only successful theory which explains the origin of the light element isotopes ($D, {}^3He, {}^4He$ and ${}^7Li$). BBN took place when the universe was only 1 to 1000 sec old, at temperatures $T \simeq 10$ MeV to 0.01 MeV. The comparison of the predicted to the observed abundance of light elements provides an important test of the SBB and leads to a baryon density Ω_B constrained by $\Omega_B h^2 = 0.019 \pm 0.002$ [28]. Thus a flat universe ($\Omega = 1$), must consist mostly of non-baryonic matter. From observations, the total energy density of the universe today Ω_0 is constrained to[2] $0.97 \leq \Omega_0 \leq 1.07$ [28], i.e. the universe is flat to a high accuracy.

Throughout this thesis, we will find it convenient to use conformal time τ defined by $ad\tau \equiv dt$ and time independent comoving distances λ_c which are related to the physical distances λ_{ph} by a rescaling $\lambda_{ph} = a\lambda_c$. In a similar way comoving momentum k is related to physical momentum p by $p = k/a$.

1.3 Cosmological Problems

The shortcomings of the SBB model do not indicate any logical inconsistencies, but rather have to do with the underlying assumptions or initial conditions. The flatness, the horizon, and the small-scale inhomogeneity problems indicate that the initial conditions for the SBB are rather special. We would like to understand how or why the universe started in such a special state. In the next chapter we will discuss inflation, which is an attempt to address these questions, by adding to the cosmological evolution an era of accelerated expansion before the Big-Bang. But first we briefly describe the cosmological problems[3].

[2]The new data from WMAP actually suggests a closed universe with $\Omega_0 = 1.02 \pm 0.02$ [24]

[3]A similar discussion can also be found in for instance [6] or [27].

Since the Big-Bang has happened only a finite time ago photons can only have traveled a finite distance. The physical distance traveled by a photon since the Big-Bang is called the particle horizon. Since the photons travel null paths given by $dr = dt/a(t)$, the particle horizon $d_H(t)$ is given by

$$d_H(t) = a(t) \int_0^t \frac{dt'}{a(t')} \sim H^{-1} \qquad a(t) \propto t^\beta \ , \quad \beta < 1 \ . \tag{1.10}$$

Up to a numerical factor the horizon is given by the Hubble radius, H^{-1}. We know that the last scattering was at redshift about $z = 1100$, so we can calculate the length corresponding to the size of the observable universe (the present Hubble radius) at the time of last scattering t_{LS}

$$\lambda_H(t_{LS}) = d_H(t_0) \left(\frac{a_{LS}}{a_0} \right) = d_H(t_0) \left(\frac{T_0}{T_{LS}} \right) \ . \tag{1.11}$$

However, because we can assume that since last scattering the universe has been matter dominated, the Hubble rate has decreased at a different rate $H^2 \propto \rho_M \propto a^{-3} \propto T^3$. So the Hubble radius at last scattering was

$$H_{LS}^{-1} = d_H(t_{LS}) = d_H(t_0) \left(\frac{T_{LS}}{T_0} \right)^{-3/2} \ . \tag{1.12}$$

The crucial observation is that the length corresponding to our present Hubble radius was much larger than the horizon at last scattering. One can for instance compare the volumes corresponding to these two scales:

$$\frac{\lambda_H^3(t_{LS})}{d_H^3(t_{LS})} = \left(\frac{T_0}{T_{LS}} \right)^{-3/2} \approx 10^6 \ . \tag{1.13}$$

This means that there must have been 10^6 causally disconnected regions within the volume that now corresponds to our observable universe. In other words causal processes can not account for the homogeneity of the universe, but rather it must be set as an initial condition of the SBB model.

From equation (1.9) one sees that if the universe is perfectly flat ($\Omega = 1$), it will stay that way at all times. However, if it deviates only slightly from the flat case, Ω will evolve away from unity. Thus we could say that a flat universe is an unstable fixed point. Because the observed value of Ω today is very close to one, the initial value must be fine tuned to one to a very high accuracy. In a radiation dominated era $H^2 \sim \rho_r \sim a^{-4}$, so $\Omega - 1 \sim a^2$, while during a matter dominated period $\rho_m \sim a^{-3}$ and $\Omega - 1 \sim a$. Since the temperature of the universe today is $T_0 \sim 10^{-13}$GeV, the temperature at matter-radiation equality was $T_{eq} \sim 10$eV, and the temperature at the epoch of nucleosynthesis was $T_N \sim 1$MeV, one finds

$$\frac{|\Omega - 1|_{T=T_N}}{|\Omega - 1|_{T=T_0}} \approx \left(\frac{a_N^2}{a_{eq}^2} \right) \left(\frac{a_{eq}}{a_0} \right) \approx \left(\frac{T_{eq}^2}{T_N^2} \right) \left(\frac{T_0}{T_{eq}} \right) \leq 10^{-15} \ . \tag{1.14}$$

If we assume that we can extrapolate the Einstein equations to t_p, where the temperature of the universe was $T_p \sim 10^{19}$GeV one obtains an even more fine-tuned number

$$\frac{|\Omega - 1|_{T=T_p}}{|\Omega - 1|_{T=T_0}} \approx \left(\frac{a_p^2}{a_{eq}^2}\right)\left(\frac{a_{eq}}{a_0}\right) \approx \left(\frac{T_{eq}^2}{T_p^2}\right)\left(\frac{T_0}{T_{eq}}\right) \leq 10^{-60} \ . \tag{1.15}$$

The flatness problem is really the problem of understanding why the (classical) initial conditions correspond to a universe that was so extremely close to flat.

The entropy problem is related to the flatness problem. Let $g_*(T)$ denote the number of degrees of freedom of ultra-relativistic particle species much less massive than the temperature. Then the entropy density in relativistic particles is

$$s = \frac{\rho + p}{T} = \frac{2\pi^2}{45} g_* T^3 \ . \tag{1.16}$$

The entropy per comoving volume is

$$S \propto a^3 s \propto g_* a^3 T^3 \ . \tag{1.17}$$

Most of the entropy of the universe is contained in the radiation fluid which at present consists of photons and neutrinos. The entropy density is proportional to the number density of relativistic particle species and today it is $s = 2905\text{cm}^{-3}$ [6]. During thermal equilibrium the entropy per comoving volume S is constant, which implies that $T \propto a^{-1}$. It also allows us compute the entropy of our universe i.e. the entropy contained in our present Hubble volume

$$S_U = \frac{4\pi}{3} H_0^{-3} s \simeq 10^{90} \ . \tag{1.18}$$

From the Friedman equation (1.5) we know that during a radiation dominated period $H^2 \simeq \rho_r \simeq T^4/m_p^2$. Using this together with equation (1.9) we find

$$|\Omega - 1|_{t=t_p} = \frac{1}{S_U^{2/3}} \approx 10^{-60} \ . \tag{1.19}$$

We note that the universe is so close to flat because the entropy content of our universe is so large. Thus, the flatness problem arises because the entropy in a comoving volume is conserved during adiabatic expansion. It is therefore possible that it could be solved by a short primordial period of non-adiabatic expansion.

The fact that the energy density of the universe is close to the critical energy density leads also to an upper bound on the cosmological constant. Since the energy density contributed by the cosmological constant $\Lambda/8\pi G_N$ is constant as the universe expands, it must be less than the critical density today i.e. $8.45 \cdot 10^{-47} h^2$ GeV^4. There is no known symmetry that forbids a cosmological constant, so we expect it not to vanish. In fact we might have expected $\rho_{vac} \equiv \Lambda/8\pi G_N \sim m_p^4$. The zero point energy of the harmonic oscillator is $\omega/2$ ($\omega = \sqrt{|\mathbf{p}|^2 + m^2}$). The number of momentum states per unit volume is $d^3p/(2\pi)^3$, and so, the vacuum energy of a scalar field due to quantum states with $p < p_{max}$ is

$$\rho_{vac} = \frac{1}{(2\pi)^2}\int_0^{p_{max}} p^3 dp \sim p_{max}^4 \ . \tag{1.20}$$

It is reasonable to take m_p as a cutoff on the theory and therefore we expect $\rho_{vac} \sim m_p^4$ in sharp contrast to the upper observational bound. This is commonly referred to as the cosmological constant problem (see [29] for a recent review).

Super-heavy particles produced in the very early universe can also potentially over-close the universe. In the past when the universe was hot and dense, symmetries existed which are broken today. Heavy particles with weak interactions are likely to be created as a by-product of symmetry breaking and are therefore often called dangerous relics. Typically supersymmetry breaking at the grand unification scale could lead to the creation of magnetic monopoles, gravitinos or moduli fields. Also the creation of cosmic strings and domain walls which could overclose the universe is a potential problem of symmetry breaking.

Even if the universe is very homogeneous and isotropic on large scales, it is not perfectly homogeneous. This is perhaps the most trivial cosmological observation. A less self-evident and very intriguing more recent cosmological observation is the discovery of the anisotropies in the CMB temperature by the COBE satellite in 1992. The source of both structure and CMB anisotropies is believed to be a small primordial density inhomogeneity $\delta\rho/\rho \sim 10^{-5}$. Once the universe becomes matter dominated the small primordial density fluctuations are amplified by gravity and grow into the structure we observe today. The temperature anisotropies are seeded by the same small density contrast, because it causes a difference in the gravitational potential between two points on the last-scattering surface, which again determines the temperature anisotropy on the angular scale. Symbolically we can write[4]

$$\left(\frac{\delta T}{T}\right)_\theta \approx \left(\frac{\delta\rho}{\rho}\right)_\lambda \ , \tag{1.21}$$

here $\lambda \sim 100h^{-1}\mathrm{Mpc}(\theta/\mathrm{deg})$ is the length scale related to the angle θ on the last scattering surface. This is known as the Sachs-Wolfe effect [30]. It is interesting to note that photons which on the last-scattering surface were separated by an angle larger than $\theta_H \approx 1^\circ$ were not in causal contact, although anisotropies of same order of magnitude $\delta T/T \sim 10^{-5}$ are present at larger angles/scales. In other words, the CMB fluctuations seem to be non-causal. Because the large scale density perturbations are fluctuations on scales which were larger than the Hubble radius at last scattering, we say that they were outside the horizon, and as the Hubble radius grew they eventually crossed inside. Since we observe CMBR anisotropies which

[4]Formally the temperature anisotropies are given by the comoving curvature perturbation $\mathcal{R}_k$ on the last scattering surface, which again is related to the density perturbations as we will discuss also in section 2.1.4.

originates from super horizon scales which only recently crossed inside the horizon, causal micro physics alone can not explain the origin of the density perturbations within the SBB model.

Chapter 2

Two Scenarios of Cosmology

From our discussion of the flatness and the entropy problem in the previous chapter, it is clear that in order to solve the flatness problem and produce the large amount of entropy that we observe today, the universe must have gone through an era of non-adiabatic behaviour. In addition the resolution of the causality problem requires that there is a period in the history of the universe where the physical scales λ evolve faster than the horizon scale, such that all of the observable universe could have been in causal contact in the past. This second condition we can express as a constraint for the scale factor. The physical scales redshift as $\lambda \propto a^{-1}$, so requiring that the physical scales evolve faster than the Hubble rate amounts to require

$$\frac{d}{dt}\left(\frac{\lambda}{|H^{-1}|}\right) = \frac{d}{dt}|\dot{a}| > 0 \ . \tag{2.1}$$

Alternatively we can see from the Friedmann equation that $a^2\rho \propto \dot{a}^2 + k$ means that for the cosmological evolution to yield the curvature term negligible, we must have an era where $\partial_t|\dot{a}| > 0$. We conclude that if $\dot{a} > 0$ then we must have $\ddot{a} > 0$ and if $\dot{a} < 0$ then we must have $\ddot{a} < 0$. This is how we will define inflation here, as a period of either accelerated expansion or accelerated contraction.

Inflation is in most models driven by a scalar field (called inflaton), and the needed large entropy production is usually associated with the decay of this field at the end of inflation. This is referred to as reheating and can be viewed as the Big-Bang of the SBB model. Thus, we could say that inflation essentially deals with "what happened before the Big-Bang" [31].

2.1 Slow-Roll Inflation

Slow-roll inflation is perhaps the most popular attempt to find a solution to the cosmological problems. The basic idea has been around more than 20 years and was originally proposed by Allan Guth [32] and Alexei Starobinsky [4, 31].

Assume there is a time interval beginning at time t_i and ending at t_R (the reheating time) during which the universe is to a good approximation exponentially expanding,

$$a(t) \propto e^{Ht} \ , \qquad t_i < t < t_R \tag{2.2}$$

with constant Hubble parameter H (de Sitter phase). Later we will quantify what we mean by "good approximation" in terms of the slow roll parameters. The success of BBN sets an upper bound to the reheating time $t_R << t_N$. Of course during exponential expansion $\ddot{a} > 0$ and from equation (1.6) we conclude that $(\rho + 3p) < 0$, so an accelerating period of expansion is only obtained if the pressure is negative $p < -\rho/3$. This means that during the inflationary era the universe can be neither matter or radiation dominated, but from this perspective vacuum energy is a viable candidate ($p = -\rho$) to have driven inflation. If $p = -\rho$ then from equation (1.6) and (1.7) we know that

$$\rho = constant \ , \qquad H = constant \ , \tag{2.3}$$

and by solving the Friedmann equation (1.5) we find that the universe in this case is an exact de Sitter universe with scale factor as given in equation (2.2). In the rest of this section we will only discuss accelerated expansion, although our definition of inflation also includes accelerated contraction. We postpone the discussion of contracting universes to section 2.2.

2.1.1 Inflation as a Solution to the Cosmological Problems

During an era where the Hubble rate is constant it is clear that physical scales $\lambda \propto a$ will cross outside the horizon as the universe expands. In this way one can explain how scales that entered the horizon during radiation or matter dominated era, has crossed outside the horizon during inflaton and thus originate from inside the horizon. This enables causal micro physics to account for the small observed inhomogeneities in addition to solving the homogeneity and isotropy problem. However to successfully solve these problems the inflationary period must have lasted long enough to make the largest scales we observe today exit the horizon during inflation.

A convenient way to parameterize the length of inflation is in terms of e-foldings N

$$N = \ln \frac{a(t_R)}{a(t_i)} \ . \tag{2.4}$$

In order to solve the horizon and causality problems, the largest observable scale today, which is the present horizon scale H_0^{-1}, must have been inflated from a value $\lambda_{H_0}(t_i)$ smaller than the horizon scale during inflation H_I^{-1}

$$\lambda_{H_0}(t_i) = H_0^{-1} \left(\frac{a(t_R)}{a(t_0)} \right) \left(\frac{a(t_i)}{a(t_R)} \right) = H_0^{-1} \left(\frac{T_0}{T_R} \right) e^{-N} \lesssim H_I^{-1} \ . \tag{2.5}$$

From the above relation we directly obtain a bound on the number of e-foldings:

$$N \gtrsim \ln \left(\frac{T_0}{H_0} \right) - \ln \left(\frac{T_R}{H_I} \right) \simeq 67 - \ln \left(\frac{T_R}{H_I} \right) \ . \tag{2.6}$$

If we ignore the logarithmic term, we need $N \gtrsim 70$ in order to solve the horizon problem. This a very important point which we will return to later in our discussion of the trans-Planckian problem.

Also the solution to the flatness problem gives a bound on the number of e-foldings. Since the Hubble rate is constant during inflation, from equation (1.9) one sees that $\Omega - 1 \propto 1/a^2$, so during inflation

$$\frac{|\Omega - 1|_{t=t_R}}{|\Omega - 1|_{t=t_i}} = \left(\frac{a(t_i)}{a(t_R)}\right)^2 = e^{-2N} \ . \tag{2.7}$$

On the other hand it is clear from the argument that lead us to equation (1.19) that at the beginning of the radiation dominated epoch

$$|\Omega - 1|_{t=t_R} = \frac{m_p^2}{T_R^2}\frac{1}{S_U^{2/3}} \approx \frac{m_p^2}{T_R^2}10^{-60} \ . \tag{2.8}$$

Since the Planck scale marks a reasonable upper bound on the reheat temperature, then even with $|\Omega-1|_{t=t_i}$ of order unity, we see from equation (2.7) that 70 e-foldings is more than enough to solve the flatness problem and yield the small number on the right hand side of equation (2.8).

2.1.2 The Slow-Roll Parameters

In this subsection we will focus on quantum field driven inflation as suggested originally by Guth [32], although there are also other ways to obtain inflation. In Starobinsky's original paper [4], he proposed a model of exponential expansion based on higher derivative curvature terms in the gravitational action.

In quantum field driven inflation, it is the potential energy of a scalar field, the *inflaton* ϕ, that drives the inflation. The action can be written

$$S = \int d^4x\sqrt{-g}\mathcal{L} = \int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right] \tag{2.9}$$

where $V(\phi)$ is the inflaton potential. The equation of motion becomes

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + V_\phi(\phi) = 0 \ , \tag{2.10}$$

with $V_\phi(\phi) \equiv (dV(\phi)/d\phi)$. The energy-momentum tensor becomes $T_{\mu\nu} = \partial_\mu\phi\partial_\mu\phi - g_{\mu\nu}\mathcal{L}$. Now assuming that the scalar field is homogeneous we can ignore the gradient term and from the energy-momentum tensor we find

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi) \ , \qquad p = \frac{\dot{\phi}^2}{2} - V(\phi) \ . \tag{2.11}$$

From the Friedmann equation (1.5) we then obtain the Hubble rate in terms of the scalar field dynamics

$$H^2 = \frac{8\pi G_N}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \ . \tag{2.12}$$

If the velocity of the inflaton is sufficiently small

$$V(\phi) >> \dot{\phi}^2 \tag{2.13}$$

then from equation (2.11) we find the following equation of state

$$p \simeq -\rho \ , \tag{2.14}$$

which leads to an era of approximate de Sitter expansion as discussed in the beginning of this section. From the condition in equation (2.13) it is now obvious why this kind of quantum field driven inflation is also called *slow-roll* inflation. To have enough inflation i.e. to make $\dot{\phi}$ stay small long enough, we might expect that also $\ddot{\phi}$ is negligible. However, this second condition is not strictly required for inflation to happen. One often refers to these condition as the slow-roll conditions.

If the condition in equation (2.13) is satisfied and $\ddot{\phi}$ is negligible, then the Friedmann equation (1.5) and the equation of motion become

$$H^2 \simeq \frac{8\pi G_N}{3}V(\phi) \quad , \qquad 3H\dot{\phi} = -V_\phi(\phi) \tag{2.15}$$

respectively. It is usual to define the slow-roll parameters

$$\epsilon = \frac{1}{16\pi G_N}\left(\frac{V_\phi}{V}\right)^2 \quad , \qquad \eta = \frac{1}{8\pi G_N}\left(\frac{V_{\phi\phi}}{V}\right) \ . \tag{2.16}$$

From equation (2.15) we see that the slow-roll conditions are equivalent to

$$\epsilon << 1 \quad , \qquad |\eta| << 1 \ . \tag{2.17}$$

Using the slow-roll condition we can write the constraint on the number of e-foldings N as

$$N \equiv \ln\frac{a(t_R)}{a(t_i)} = \int_{t_i}^{t_R} H dt \simeq 8\pi G_N \int_{\phi_R}^{\phi_i} \frac{V}{V_\phi} d\phi \gtrsim 70 \ . \tag{2.18}$$

2.1.3 Classical Theory of Cosmological Perturbations

Below we will give a short review of the theory of cosmological perturbations. Detailed treatments and references to the original literature can be found in [33, 34]. Since the fluctuations on scales of the CMB anisotropies were small when the anisotropies were generated, we expect that the fluctuations must have been very small in the early universe and a linearized analysis is justified. In linearized perturbation theory each Fourier modes of the cosmological fluctuations evolves independently, which simplifies the analyzes considerably.

Since the perturbations are small and the background evolution after a short era of inflation can be described by the flat FRW metric

$$ds^2 = a^2(\tau)(-d\tau^2 + d\mathbf{x}^2) \tag{2.19}$$

we will quantize the linear fluctuation about a classical background described by that particular simple form of the metric. Now let us consider the action of gravity plus matter

$$S = \int d^4x\sqrt{-g}R + S_M \ . \tag{2.20}$$

Since we are interested in the evolution of perturbations during inflation, we will assume that the matter action S_M is described by a single minimally coupled scalar field - the inflaton ϕ. We separate the metric and matter into classical background variables $g^{(0)}_{\mu\nu}$, $\phi^{(0)}$ which depends only on time, and fluctuation fields $h_{\mu\nu}$, $\delta\phi$ which depends both on space and time and have vanishing spatial average:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu}(\tau) + h_{\mu\nu}(\tau, \mathbf{x}) \quad , \qquad \phi = \phi^{(0)}(\tau) + \delta\phi(\tau, \mathbf{x}) \ . \tag{2.21}$$

The metric perturbations can be split into three types: scaler, vector and tensor fluctions. However, vector fluctuations decay in expanding cosmological backgrounds and therefore we will only be interesed in scalar and tensor fluctuations.

Let us first consider tensor fluctuations. Tensor fluctuations correspond to gravitational waves and can be viewed as small ripples in the FRW space-time. In the transverse-traceless gauge the perturbed metric has only non-vanishing space-space components $h_{ij}(\tau, \mathbf{x})$ [35],

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right] \ , \tag{2.22}$$

where $|h_{ij}| << 1$. The tensor h_{ij} is symmetric so it has six independent components. The traceless $\delta^{ij}h_{ij} = 0$ and transverse conditions $\partial^i h_{ij} = 0$ add four more contraints and leave only two physical degrees of freedom. This means that we can expand the tensor h_{ij} in terms of two basic traceless and symmetric polarization tensors e^+_{ij} and $e^\times_{ij}$ as

$$h_{ij}(\tau, \mathbf{x}) = h_+ e^+_{ij} + h_\times e^\times_{ij} \ . \tag{2.23}$$

The space and time dependence is contained in the coefficients h_+ and $h_\times$. By expanding the Einstein-Hilbert action around the FRW background, the action for h_+ and $h_\times$ reduces to that of a free, massless, minimally coupled scalar field. In order to obtain the correct normalization, we must multiply with a factor $m_p/\sqrt{2}$, so the canonically normalized scalar fields corresponding to these amplitudes are $\varphi_\lambda \equiv (m_p/\sqrt{2})h_\lambda$, $(\lambda = +, \times)$ [34]. In Fourier space, the action is

$$\delta S_g = \int d\tau \frac{a^2}{2}\left[-\varphi'_{-\mathbf{k}}\varphi'_{\mathbf{k}} + k^2\varphi_{-\mathbf{k}}\varphi_{\mathbf{k}}\right] \ . \tag{2.24}$$

Note that here we have used the definition $' \equiv \partial_\tau$. From the action in equation (2.24) we obtain the following equation of motion

$$\varphi_{\mathbf{k}}'' + 2\frac{a'}{a}\varphi_{\mathbf{k}}' + k^2\varphi_{\mathbf{k}} = 0 \ . \tag{2.25}$$

We can eliminate the Hubble friction term by a change of variables $v_k \equiv a\varphi_k$, yielding

$$v_{\mathbf{k}}'' + \left(k^2 - \frac{a''}{a}\right) v_{\mathbf{k}} = 0 \ . \tag{2.26}$$

This is the equation of motion of an oscillator with a time-dependent frequency, also called a parametric oscillator.

Before we discuss the solution to the equation of motion (2.26) let us first turn to the second type of cosmological perturbations: scalar perturbations. The description of scalar perturbations is generally more complicated, but by working in a fixed gauge it simplifies significantly. Consider the scalar contributions to $h_{\mu\nu}$. There are two possible ways that scalar quantities can enter into h_{ij}. Either we can multiply the spatial part of the background metric g_{ij} with a scalar or we can take covariant derivative of a scalar function with respect to the background metric. In addition we need a scalar function to describe h_{00}, and the three-dimensional covariant derivative of an other scalar function to describe h_{i0}. The line element of the perturbed metric can then be written

$$ds^2 = a^2(\tau)\left[(-1-2A)d\tau^2 + 2\partial_i B d\tau dx^i + ((1-2\psi)\delta_{ij} + 2D_{ij}E)dx^i dx^j\right] \ , \tag{2.27}$$

with $D_{ij} = \nabla_i\nabla_j$. Now if we perform a gauge transformation on the coordinates $\tilde{x}^\mu = x^\mu + \delta x^\mu$, then the metric transforms as $\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \mathcal{L}_{\delta x} g_{\mu\nu}(x)$, ($\mathcal{L}_{\delta x}$ is the Lie derivative along δx). This can be rewritten as variations of the functions A, ψ, B, and E

$$\tilde{A} = A - \xi^{0\prime} - \frac{a'}{a}\xi^0 \ , \qquad \tilde{B} = B + \xi^0 + \xi' \ , \qquad \tilde{\psi} = \psi + \frac{a'}{a}\xi^0 \ , \qquad \tilde{E} = E + \xi \ . \tag{2.28}$$

where we decomposed the vector δx^μ into $\delta x^0 = \xi^0(x^\mu)$ and $\delta x^i = \delta^{ij}\partial_j\xi(x^\mu)$.

Now it is evident that the Bardeen potentials

$$\Phi \equiv -A + \frac{1}{a}\left[(-B + E'/2)a\right]' \ , \qquad \Psi \equiv -\psi - \frac{1}{6}\nabla^2 E + \frac{a'}{a}\left[B - E'/2\right] \ , \tag{2.29}$$

which were first introduced in [36, 37], are gauge invariant.

To simplify matters we will now go to the longitudinal gauge defined by the conditions $B = E = 0$. From equation (2.28) it follows that in the longitudinal gauge the coordinates are totally fixed and there are no complicating residual gauge modes. It is important to note that in this gauge, the gauge invariant potentials Φ and Ψ are just the scalar functions A and ψ. Hence, in the longitudinal gauge the metric takes the form

$$ds^2 = a^2(\tau)\left[-(1+2\Phi)d\tau^2 + (1-2\Psi)\delta_{ij}dx^i dx^j\right] \ . \tag{2.30}$$

Further, if the spatial part of energy-momentum tensor is diagonal then it follows from the Einstein equations that $\Phi = \Psi$. Due to the Einstein equation, the remaining metric fluctuation Ψ is determined by the matter fluctuation. It should then be clear that the physical degrees of freedom of scalar metric fluctuations can be expressed in terms of a single free scalar field χ with a time dependent mass, determined by the time dependent background. It can be shown that this is given by [38]

$$\chi = a\left(\delta\phi + \frac{\phi_0'}{\mathcal{H}}\Psi\right) = z\mathcal{R} \ , \tag{2.31}$$

where $\phi_0(\tau) \equiv \phi^{(0)}$ denotes the background value of the scalar matter field, $\mathcal{H} = a'/a$,

$$z \equiv a\frac{\phi_0'}{\mathcal{H}} \ , \tag{2.32}$$

and $\mathcal{R}$ denotes the gauge invariant comoving curvature perturbation [39]. One can show that the action for the scalar metric fluctuations χ is [33]

$$\delta S_{\mathcal{R}} = \frac{1}{2}\int d\tau \left[-\chi_{\mathbf{k}}'\chi_{-\mathbf{k}}' + k^2\chi_{\mathbf{k}}\chi_{-\mathbf{k}} - \frac{z''}{z}\chi_{\mathbf{k}}\chi_{-\mathbf{k}}\right] \ . \tag{2.33}$$

which leads to the following equation of motion

$$\chi''_{\mathbf{k}} + \left(k^2 - \frac{z''}{z}\right)\chi_{\mathbf{k}} = 0 \ . \tag{2.34}$$

Notice that with the change $a \rightarrow z$ equation (2.34) becomes identical to the equation of motion (2.26) for gravitational waves. In pure de Sitter space-time, ϕ_0' and $\mathcal{H}$ scale with the same power of τ. This implies that z is proportional to a and the evolution of gravitational waves and scalar metric perturbations is identical. During slow-roll inflation the background field value of the inflaton ϕ_0 is nearly frozen and the scalar metric perturbations reduces to the fluctuations in the inflaton field up to a logarithmic time-dependence suppressed by the slow-roll parameter ϵ

$$\chi \simeq a\delta\phi \ . \tag{2.35}$$

Thus, during slow-roll inflation ($\epsilon \rightarrow 0$) we can with good approximation think of the scalar metric fluctuations χ as the canonical normalized inflaton fluctuations, where the linearized action for the inflaton fluctuations is just the covariant action of a free scalar field. In Fourier space the equation of motion of the fluctuations in the inflaton field is then analogous to equation (2.25):

$$\delta\phi_{\mathbf{k}}'' + 2\frac{a'}{a}\delta\phi_{\mathbf{k}}' + k^2\delta\phi_{\mathbf{k}} + V_{\phi\phi}\delta\phi_{\mathbf{k}} = 0 \ . \tag{2.36}$$

One should keep in mind that this equation is approximate and the use of the equation (2.36) has always to be justified. Later we will discuss this aspect more detailed. If the inflaton is coupled to other fields the linearized equation of motion of the fluctuations is of course more complicated.

Let us now discuss the solution of the equation of motion (in equation (2.34) or (2.26)) in more details. In order to first discuss the dynamics qualitatively, it is convenient to rewrite the equation of motion for the inflaton perturbation in terms of the cosmic time t:

$$\ddot{\delta\phi}_{\mathbf{k}} + 3H\dot{\delta\phi}_{\mathbf{k}} + \left(\frac{k}{a}\right)^2 \delta\phi_{\mathbf{k}} + V_{\phi\phi}\delta\phi_{\mathbf{k}} = 0 \; . \tag{2.37}$$

This equation implies that physical wavelengths $\lambda \propto a/k$ of a small perturbation $\delta\phi$ grows in time as $a \propto \exp(Ht)$. Perturbations with $k >> H >> V_{\phi\phi}$ oscillate as $\sin kt$, whereas perturbations with $k << H$, $V_{\phi\phi} << H^2$ do not oscillate, and their magnitude remain almost frozen during inflation due to the large friction term $3H\dot{\delta\phi}$. It is reasonable that fluctuations are frozen on super horizon scales, since the expansion is too fast for causal micro-physics to act on those scales.

As a first approximation we can use the slow-roll approximation $|\eta| << 1$ and neglect the mass term $V_{\phi\phi}$ in equation (2.36). In this case equation (2.36) has same form as equation (2.25). If we also use the approximation $a = \exp(Ht)$, H=constant, then the solution of equation (2.26) and (2.34) is particularly simple:

$$v_{\mathbf{k}} = \chi_{\mathbf{k}} = A(k)\left(1 - \frac{i}{k\tau}\right)e^{-ik\tau} + B(k)\left(1 + \frac{i}{k\tau}\right)e^{+ik\tau} \; . \tag{2.38}$$

More quantitatively let us look at the exact solutions for the equation of motions of the metric fluctuations. We will first discuss the solution of the equation of motion for the gravitational waves (2.26) or (2.25) since this is the simplest case. One common way to proceed, is to write the equation (2.26) in the following generic form

$$v''_{\mathbf{k}} + \left[k^2 - \frac{1}{\tau^2}\left(\nu_T^2 - \frac{1}{4}\right)\right]v_{\mathbf{k}} = 0 \; . \tag{2.39}$$

For real ν_T, it has the general solution [40]

$$v_{\mathbf{k}}(\tau) = \sqrt{-\tau}\left[C_1(k)H^{(1)}_{\nu_T}(-k\tau) + C_2(k)H^{(2)}_{\nu_T}(-k\tau)\right] \; , \tag{2.40}$$

where $H^{(1)}_{\nu_T}$ and $H^{(2)}_{\nu_T}$ are Hankel functions of first and second kind. Using that

$$a = \frac{-1}{\tau H(1-\epsilon)} \; , \tag{2.41}$$

we can write the cosmic acceleration as

$$\frac{a''}{a} \simeq -\frac{1}{\tau^2}(2 - 3\epsilon) \; . \tag{2.42}$$

In this way we find

$$\nu_T \simeq 3/2 - \epsilon \; . \tag{2.43}$$

Now lets turn to the scalar perturbations. One can express z''/z in terms of new slow-roll parameters [41]

$$\frac{z''}{z} = 2a^2H^2\left(1 + \frac{3}{2}\delta + \epsilon_H + \frac{1}{2}\delta^2 + \frac{1}{2}\epsilon_H\delta + \frac{1}{2}\frac{1}{H}\dot{\epsilon}_H + \frac{1}{2}\frac{1}{H}\dot{\delta}\right) \tag{2.44}$$

where the new slow-roll parameters are given by

$$\epsilon_H \equiv \frac{-\dot{H}}{H^2} \quad , \qquad \delta \equiv \frac{\ddot{\phi}_0}{H\dot{\phi}_0} \ . \tag{2.45}$$

To first order in the slow-roll parameters we have $\epsilon \simeq \epsilon_H$ and $\delta \simeq \epsilon - \eta$. Assuming that the new slow-roll parameters (ϵ_H, δ) are constant and using again equation (2.41) we obtain

$$\frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right) \quad , \qquad \nu = \frac{1 + \delta + \epsilon_H}{1 - \epsilon_H} + \frac{1}{2} \ . \tag{2.46}$$

To first order in the slow-roll parameters we thus have $\nu \simeq \frac{3}{2} + 3\epsilon - \eta$. Of course the solution again have the form given in equation (2.40) with $\nu_T \to \nu$

$$\chi_{\mathbf{k}}(\tau) = \sqrt{-\tau}\left[C_1(k)H_\nu^{(1)}(-k\tau) + C_2(k)H_\nu^{(2)}(-k\tau)\right] \ , \tag{2.47}$$

In the limit $\epsilon, \eta \to 0$ the solutions (2.40) and (2.47) reduces to the approximate solution in equation (2.38).

Until now we have discussed the solutions to the classical equations of motion of the field fluctuations. In order to obtain the right normalization we have to quantize the fluctuations and normalize them to a reasonable initial condition which will be the quantum fluctuation of an appropriate chosen vacuum.

2.1.4 Quantum Theory of Cosmological Perturbations

In the previous section we solved the classical equations of motion. This is sufficient to describe the evolution of the perturbations. However, in order to understand the origin of the perturbations and normalize the fluctuations, a quantum treatment is necessary.

It is practical to quantize in the Heisenberg picture where the states are time-independent but the operators evolves. From the action (2.33) it follows that the canonical conjugate to the field χ is $\Pi_{\mathbf{k}} = \chi'_{-\mathbf{k}}$. Following the usual path of canonical quantization we promote χ, Π to canonical conjugate operators $\hat{\chi}$, $\hat{\Pi}$ and impose the following equal time commutation relations

$$[\hat{\chi}(\tau,\mathbf{x}), \hat{\chi}(\tau,\mathbf{x}')] = \left[\hat{\Pi}(\tau,\mathbf{x}), \hat{\Pi}(\tau,\mathbf{x}')\right] = 0 \ , \qquad \left[\hat{\chi}(\tau,\mathbf{x}), \hat{\Pi}(\tau,\mathbf{x}')\right] = i\delta\left(\mathbf{x} - \mathbf{x}'\right) \ . \tag{2.48}$$

From the Heisenberg equation of motion

$$i\partial_\tau \hat{A} = \left[\hat{A}, \hat{H}\right] \ , \tag{2.49}$$

where $\hat{H}$ is the Hamiltonian operator, it follows that the operator $\hat{\chi}$ satisfies the same equation of motion as the classical field χ. Thus, analogous to the quantization of a scalar field in Minkowski space, we expand the operator $\hat{\chi}$ over a complete orthogonal basis of the solution to the classical field equation [42]

$$\hat{\chi}(\tau, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{\sqrt{2k}} \left[\hat{c}_{\mathbf{k}}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{c}^\dagger_{\mathbf{k}}(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \ . \tag{2.50}$$

This is equivalent to

$$\hat{\chi}_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2k}} \left(\hat{c}_{\mathbf{k}}(\tau) + \hat{c}^\dagger_{-\mathbf{k}}(\tau)\right) \tag{2.51}$$

for the Fourier mode. Above we used that the spatial part can be taken to be the basis of plane waves in the spatial flat metric. Of course the commutation relation for the canonical conjugate fields (2.48) translates into the usual commutators for the creation and annihilation operators:

$$\left[\hat{c}_{\mathbf{k}}(\tau), \hat{c}^\dagger_{\mathbf{p}}(\tau)\right] = \delta(\mathbf{k} - \mathbf{p}) \ . \tag{2.52}$$

The vacuum state $|0\rangle$ is defined as the state annihilated by $\hat{c}_{\mathbf{k}}$:

$$\hat{c}_{\mathbf{k}}(\tau) \left|0\right\rangle_\tau = 0 \ . \tag{2.53}$$

Since the background has a non-trivial time-dependence, the vacuum state will in general rotate in time unlike the Minkowski vacuum that is time invariant. This phenomena leads to gravitational particle creation, since the vacuum at one time will be an excited state of the vacuum at a later time. The conventional way to discuss this phenomena in details is by means of a Bogoliubov transformation

$$\begin{aligned} \hat{c}_{\mathbf{k}}(\tau) &= \alpha_k(\tau)\hat{c}_{\mathbf{k}}(\tau_0) + \beta_k(\tau)\hat{c}^\dagger_{-\mathbf{k}}(\tau_0) \ , \\ \hat{c}^\dagger_{\mathbf{k}}(\tau) &= \alpha^*_k(\tau)\hat{c}^\dagger_{\mathbf{k}}(\tau_0) + \beta^*_k(\tau)\hat{c}_{-\mathbf{k}}(\tau_0) \ , \end{aligned} \tag{2.54}$$

where τ_0 is some initial time and α_k, β_k are the Bogoliubov coefficients that must satisfy the normalization condition

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{2.55}$$

in order for the commutation relations to be preserved in time. The solution to the dynamical equations is most easily found by defining

$$f_k(\tau) \equiv \frac{1}{\sqrt{2k}} \left(\alpha_k(\tau) + \beta^*_k(\tau)\right) \ , \tag{2.56}$$

which satisfies the classical equation of motion (2.34) as can be seen from the Heisenberg equation of motion. From equation (2.51) we find

$$\hat{\chi}_{\mathbf{k}}(\tau) = f_k(\tau)\hat{c}_{\mathbf{k}}(\tau_0) + f_k^*(\tau)\hat{c}^\dagger_{-\mathbf{k}}(\tau_0) \ , \tag{2.57}$$

such that the field operator becomes,

$$\hat{\chi}(\tau, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} \left[\hat{c}_{\mathbf{k}}(\tau_0) f_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{c}^\dagger_{\mathbf{k}}(\tau_0) f_k^*(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \ . \tag{2.58}$$

The solution to the classical equation is given in equation (2.47), but we write it again here for f_k:

$$f_k(\tau) = \sqrt{-\tau}\left[C_1 H_\nu^{(1)}(-k\tau) + C_2 H_\nu^{(2)}(-k\tau)\right] \ . \tag{2.59}$$

Finally we are able to normalize the quantum fluctuations by introducing an appropriate initial condition. In this sub-section we will follow the standard (most simple) approach, while in chapter three we shall discuss potential problems of this approach and propose alternative normalization procedures. Following the usual approach we utilize the large argument asymptotic behavior of the Hankel functions for $\tau \to -\infty$

$$H_\nu^{(1)}(-k\tau) \sim \frac{\sqrt{2/\pi}}{\sqrt{-\tau k}} e^{-ik\tau} \quad , \qquad H_\nu^{(2)}(-k\tau) \sim \frac{\sqrt{2/\pi}}{\sqrt{-\tau k}} e^{ik\tau} \ . \tag{2.60}$$

Using the asymptotic expansion (2.60) we can normalize the solution (2.59) to the vacuum state in flat Minkowski space (i.e. the no particle state) in the infinite past by taking

$$C_1 = \sqrt{\pi}/2 \quad , \qquad C_2 = 0 \ . \tag{2.61}$$

Evidently in early times (when $\tau \to -\infty$) the solution becomes the positive frequency mode, which correspond to our usual definition of vacuum in Minkowski space

$$\hat{\chi}_{\mathbf{k}}(\tau) \to \frac{1}{\sqrt{2k}} e^{-ik\tau}\hat{c}_{\mathbf{k}}(\tau_0) + \frac{1}{\sqrt{2k}} e^{+ik\tau}\hat{c}^\dagger_{-\mathbf{k}}(\tau_0) \qquad \text{for} \qquad \tau_0 \to -\infty \ . \tag{2.62}$$

The logic in this choice of normalization is based on the redshift of the physical momentum modes. In order to see this, it is most convenient to first discuss the case with $\epsilon \simeq \eta \simeq 0$. In this case the solution for both scalar metric perturbations and gravitational waves reduces to the solution given in equation (2.38) and using the normalization condition described above we obtain

$$\hat{\chi}_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) e^{-ik\tau}\hat{c}_{\mathbf{k}}(\tau_0) + \frac{1}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right) e^{+ik\tau}\hat{c}^\dagger_{-\mathbf{k}}(\tau_0) \ . \tag{2.63}$$

From this solution the physics is clear. On scales much smaller than the curvature radius i.e. scales far inside the horizon $|k\tau| >> 1$ the universe looks flat and one can ignore the expansion rate H in equation (2.63) which reduces to the oscillating Minskowski vacuum solution. As scales red-shift and eventually crosses outside the horizon the oscillation freezes and their amplitude blows up.

The fluctuations at scales outside the horizon become classical as gravitational particle production takes place on those scales. This can be understood from

$$ {}_{\tau_0}\langle 0|\,\hat{N}_k\,|0\rangle_{\tau_0} = |\beta_k|^2 \ , \tag{2.64} $$

where $\hat{N}_{\mathbf{k}}(\tau) = \hat{c}^\dagger_{\mathbf{k}}(\tau)\hat{c}_{\mathbf{k}}(\tau)$ is the number operator. The Bogoliubov coefficient β_k vanishes on sub-horizon scales, while as the scales cross the Hubble radius it starts to grow since it is proportional to the amplitude of $f_k(\tau)$. This implies that the quantum state becomes highly squeezed and that perturbations becomes effectively classical.

Of course, the observable of a quantum field is the n-point correlators. For a Gaussian free field they can all be expressed in terms of the two-point amplitude. Hence, properties of a the cosmological perturbations can be expressed in terms of the power spectrum $\mathcal{P}(k)$ defined by

$$ {}_{\tau_0}\langle 0|\,\hat{\chi}^\dagger_{\mathbf{k}_1}\hat{\chi}_{\mathbf{k}_2}\,|0\rangle_{\tau_0} \equiv \delta(\mathbf{k}_1-\mathbf{k}_2)\frac{2\pi^2}{k^3}\mathcal{P}_\chi(k) \ , \tag{2.65} $$

From equation (2.57) we find

$$ \mathcal{P}_\chi(k) = \frac{k^3}{2\pi^2}|f_k|^2 \ . \tag{2.66} $$

Above we have everywhere discussed the scalar metric perturbations $\hat{\chi}$, but the discussion is completely general and applies equally well to gravitational waves.

Using the small argument limit of the Hankel functions

$$ H^{(1)}_\nu(-k\tau) \sim \sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{2}}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}(-k\tau)^{-\nu} \ , \tag{2.67} $$

we can calculate the power spectrum of the curvature fluctuation $\mathcal{R} = \chi/z$ on superhorizon scales

$$ \mathcal{P}_\mathcal{R}(k) = \frac{k^3}{2\pi^2 z^2}|f_k|^2 = \frac{2^{2\nu-3}}{(2\pi)^2}\left(\frac{\Gamma(\nu)}{\Gamma(3/2)}\right)^2\left(\frac{H}{a\dot{\phi}_0}\right)^2(-k\tau)^{3-2\nu}(-\tau)^{-2} \ . \tag{2.68} $$

Similarly one can calculate the power spectrum of the tensor perturbations $\varphi = v/a$

$$ \mathcal{P}_\varphi(k) = \frac{k^3}{2\pi^2 a^2}|\tilde{f}_k|^2 = \frac{2^{2\nu_T-3}}{(2\pi)^2 a^2}\left(\frac{\Gamma(\nu_T)}{\Gamma(3/2)}\right)^2(-k\tau)^{3-2\nu_T}(-\tau)^{-2} \ . \tag{2.69} $$

where $\tilde{f}_k$ is now given by the solution (2.26) and can be obtained by taking $\nu \to \nu_T$ in (2.59).

One defines the spectral index n of the scalar fluctuations by

$$n - 1 = \frac{d \ln \mathcal{P}_{\mathcal{R}}}{d \ln k} = 3 - 2\nu \tag{2.70}$$

and the spectral index for the tensor perturbations is defined by

$$n_T = \frac{d \ln \mathcal{P}_{\varphi}}{d \ln k} = 3 - 2\nu_T \ . \tag{2.71}$$

Hence, in the limit $\epsilon = \eta = 0$ the spectrum of both scalar and tensor perturbations is flat $n - 1 = n_T = 0$.

Since the curvature perturbations in the simplest single field inflationary models are constant outside the horizon it is useful to calculate the curvature perturbation at horizon exit[1] $k = aH$

$$\mathcal{P}_{\mathcal{R}}(k) = 2^{2\nu-3}(1-\epsilon)^{2\nu-1} \left(\frac{\Gamma(\nu)}{\Gamma(3/2)}\right)^2 \left(\frac{H}{\dot{\phi}_0}\right)^2 \left(\frac{H}{2\pi}\right)^2 \Bigg|_{k=aH} \ . \tag{2.72}$$

As explained in the first chapter, what we really observe is the temperature fluctuations long after the end of inflation, which are related to the density fluctuations by the relation in equation (1.21). But for modes that enter the horizon when the universe is still radiation dominated, we have

$$\delta_{\mathbf{k}} = \frac{4}{9} \left(\frac{k}{aH}\right)^2 \mathcal{R}_{\mathbf{k}} \ , \tag{2.73}$$

where $\delta\rho_{\mathbf{k}}/\rho \equiv \delta\rho_{\mathbf{k}}$. For modes which enters the horizon during the matter dominated phase one finds

$$\delta_{\mathbf{k}} = \frac{2}{5} \left(\frac{k}{aH}\right)^2 \mathcal{R}_{\mathbf{k}} \ , \tag{2.74}$$

Remarkably, one is able to calculate the final observed temperature fluctuations directly from the initial quantum field fluctuations present at the beginning of the history of the universe.

2.1.5 Open Questions in Inflation

In inflation driven by the potential energy of a quantum field, the end of inflation occurs when the inflaton reaches the minimum of the potential. At this point the vacuum energy thermalizes and inflation is followed by the radiation dominated era. However, since the evolution of the inflaton is influenced by quantum fluctuations, the thermalization does not occur simultaneously everywhere in the universe. The thermalized regions grow in time, but the inflating regions separating them grow even faster, so even if the observable universe is completely thermalized the whole

[1] This is the standard expression that can be found in textbooks such as [34].

universe is newer [43–45]. If inflation is eternal in the future it is natural to ask if inflation can also extend eternally in the past, avoiding the problem of the initial singularity.

The most classical singularity theorem, which is often referred to, is that of Hawking and Penrose [25]. From the Hawking-Penrose theorem follows that a space-time M contains incomplete, inextendible timelike or null geodesics[2] if, in addition to Einstein's equations, the following four conditions hold [25, 35]:

- M contains no closed timelike curves (which violate causality).
- At each event in M and for each unit timelike vector $\mathbf{u}$, the energy-momentum tensor satisfies
$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) u^\mu u^\nu \geq 0 \; . \tag{2.75}$$
This is the strong energy condition.
- Every timelike or null geodesic with unit tangent $\mathbf{u}$ has at least one point where
$$u_{[\alpha}R_{\beta]\gamma\delta[\epsilon}u_{\rho]}u^\gamma u^\delta \neq 0 \; . \tag{2.76}$$
- The space-time contains a trapped surface, meaning that light rays emitted perpendicular to the surface converge toward each other as they propagate (the singularity sucks the photons into it).

For an energy-momentum tensor of the form $T^\mu{}_\nu = (-\rho, p, p, p)$, the strong energy condition is equivalent to requiring

$$\rho + 3p \geq 0 \; . \tag{2.77}$$

While this condition is always satisfied by ordinary matter and radiation, it is violated during inflation where the universe is dominated by vacuum energy with negative pressure $\rho = -p$. Thus, the Hawking-Penrose theorem allows in principle for past eternal inflation.

In the early nineties Borde and Vilenkin argued for a new singularity theorem [46]. They showed that if inflation is future eternal and the weak energy condition is satisfied[3], then space-time can not be past null geodesic complete i.e. it has an initial singularity. The weak energy condition is the requirement that for every timelike vector $\mathbf{u}$, the energy-momentum tensor satisfies

$$T_{\mu\nu}u^\mu u^\nu \geq 0 \; . \tag{2.78}$$

[2]If a space-time contains timelike or null geodesics that terminates after finite proper time/affine parameter and the space-time can not be extended beyond the termination point (e.g. due to infinite curvature), then it is said to be singular. Although the usual definition of singular space-times is more slightly more general [35].

[3]Actually they also had two more mathematical assumptions. They assumed that the universe is past causally simple and open. Past causally simple just means that all points q in the past light cone of a point p, are connected to the point p by a future-directed timelike curve.

It essentially means that the energy density is positive for every observer. With $T^\mu{}_\nu = (-\rho, p, p, p)$ it corresponds to

$$\rho \geq 0 \quad , \qquad \rho + p \geq 0 \; . \tag{2.79}$$

Naively, it seems that this condition is satisfied by inflation and thus inflation can not be eternal. But in the late nineties several groups argued that even the weak energy condition might be too strong. For instance when quantum fluctuations of the inflaton are important and takes the inflaton up and down the potential it will make the Hubble rate decrease in some domains and increase in others, violating the weak energy condition [47].

In the beginning of the new millennium even that last way out for inflation to be past eternal was closed. Borde, Guth, and Vilenkin showed that even if the weak energy condition is violated, inflation can not be eternal in the past [48]. The argument is based on requiring that the Hubble constant is always positive on average along a past directed null or timelike geodesic (a condition which is violated in pre-big bang cosmology). We shall repeat the simple version of the argument below.

Consider the flat FRW metric

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 \; . \tag{2.80}$$

For simplicity we will only consider null geodesics. From the geodesic equation, one finds that a null geodesic, with affine parameter λ, satisfies

$$d\lambda \propto a(t)dt \; . \tag{2.81}$$

One can normalize the affine parameter by choosing $d\lambda = [a(t)/a(t_f)]dt$, so $d\lambda/dt = 1$ when $t = t_f$. By multiplying equation (2.81) by the Hubble parameter, H, and integrating from some initial time t_i to t_f, we find

$$\int_{\lambda(t_i)}^{\lambda(t_f)} H(\lambda)d\lambda = \int_{a(t_i)}^{a(t_f)} \frac{da}{a(t_f)} \leq 1 \; . \tag{2.82}$$

Defining H_{av} to be an average over the affine parameter,

$$H_{av} = \frac{1}{\lambda(t_f) - \lambda(t_i)} \int_{\lambda(t_i)}^{\lambda(t_f)} H(\lambda)d\lambda \leq \frac{1}{\lambda(t_f) - \lambda(t_i)} \; , \tag{2.83}$$

we see that any backward-going null geodesic with $H_{av} > 0$ must have a finite affine length. In the original paper it was shown that this conclusion actually holds for also timelike geodesics, that is shown to terminate at finite proper time, and in any space-time, not only FRW which we assumed here.

Note that even if de Sitter space is geodesic complete, the inflationary coordinates covers only half of it (see fig. (2.1). This means that there are geodesics which are

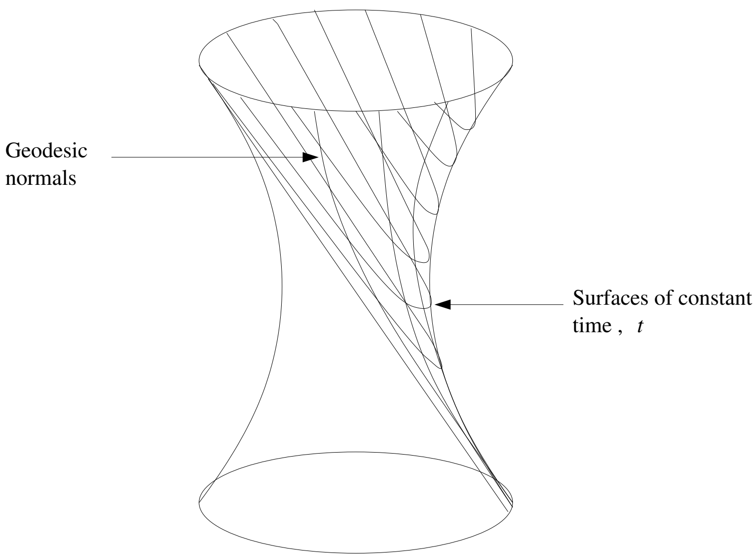

Figure 2.1: The de Sitter space is the full hyperboloid, but inflationary coordinates only cover one half of the de Sitter space-time.

complete in the full de Sitter space, but incomplete in the patch covered by the inflationary coordinates. In the Penrose diagram in fig. (2.2), the causal structure of de Sitter inflation is shown. The upper triangle is the one covered by the inflationary coordinates, while the lower triangle is the other part. It is clear from the Penrose diagram, that inflation can be eternal in the past only in the proper time of comoving observer, while another observer will see the universe as having existed only a finite proper time. This is analogous to the black hole example, where the observer at rest away from the horizon, never sees the observer falling toward the black hole actually crossing the horizon.

Of course the discussion above is based purely on Einstein gravity. One might speculate that the regulating effect of string theory also extends to cosmic singularities. However, string theory on time-dependent backgrounds, such as de Sitter space, is not fully understood so it is not yet known what exactly string theory has to say about cosmic singularities [49].

Since at present it seems that inflation can not be eternal in the past[4], we might ask - what is required to get inflation started and whether the initial conditions needed to start inflation are acceptable? Of course we know that the gradient of the scalar field driving inflation has to be negligible compared to the potential. Or framed otherwise, the inflation has to be homogeneous in a large enough region. Indeed it has been shown that if the weak energy condition is satisfied, inflation requires initial homogeneity on super-Hubble scales [51, 52]. In this case inflation does not solve the homogeneity problem but merely improves the situation.

[4]See [50] for a loophole.

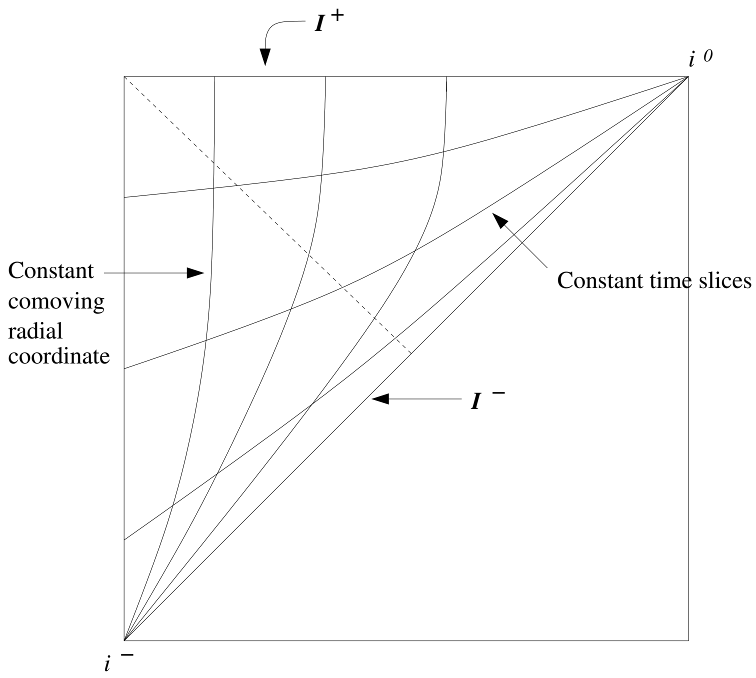

Figure 2.2: The Penrose diagram of de Sitter space, with the inflationary coordinates shown. I^- is the null surface of $t = -\infty$ and I^+ is the space-like surface of $t = +\infty$. The dashed line is the horizon for the observer on the left vertical boundary of the diagram.

In the chaotic version of inflation [53], a small Planck size universe emerges from the space-time foam or from nothing with initial density of order the Planck density $\rho \sim M_p^4$. The argument is that, only after this moment when $\rho \lesssim M_p^4$ it can be described classically. Thus, at this initial moment the sum of the kinetic energy density, gradient energy density, and the potential energy density of the inflaton is of order Planck density

$$\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi) \simeq M_p^4 \ . \tag{2.84}$$

The constraint on the inflaton field is $\phi \lesssim \phi_p$ where $V(\phi_p) = M_p^4$. Now if by any chance $\phi \simeq \phi_p$ (due to some quantum fluctuation which generally violates the weak energy condition), the potential term will dominate and the mini (Planck size) universe will start inflating. Such a quantum probabilistic solution for providing the necessary initial homogeneity involves the anthropic principle to some extend. One inescapable ends up discussing the probability for our universe to be created by such a random process, but it is very difficult to make sense of probability arguments when you only observe one universe. Either way, the basic principle behind inflation is that it greatly enhances the probability of creating the observed universe as compared to the SBB model.

Some of the problems mentioned above are of course of a more or less philosophical nature and could be called meta-cosmology because it does not relate directly to observational predictions. But there are also a few problems with inflation that does relate to its predictive power. The cosmological constant problem is perhaps the most severe, but it is not intrinsically linked to inflation. On the other hand the trans-Planckian problem is a problem of theoretical consistency and perhaps observational significance. Actually, one could argue, it is misleading to call it a problem. More adequately it could be called a promise of observing effects of physics beyond Einstein gravity, using inflation as a microscope.

From our discussion of the horizon problem in the beginning of this chapter, it became clear that we really need more than about $60-70$ e-foldings to solve the causality problem. But about 70 e-foldings before the end of inflation, scales corresponding to the size of our entire observable universe fitted inside a Planck-volume. It does not make any sense to apply Einstein gravity on such scales, hence determining the initial state for the quantum fluctuation becomes non-trivial. In this way it seems that inhomogeneities that we observe today originates from a trans-Planckian regime and we expect that the CMB physics should be sensitive to fundamental physics. Only if we have a specific model for the trans-Planckian physics at hand, it is possible to uniquely determine the initial condition for the cosmic perturbations. Such a model was suggested by Hassan and Sloth [3], motivated by the minimum length property of string theory. In chapter 3 we will investigate this in more details.

The worst problem of inflation remains the sheer multitude of models. Inflation must be viewed as a paradigm that can be incorporated in a vast number of models often with different inflaton potentials. This is a motivation for those who want to derive an inflationary-like scenario from first principle with definite, sound initial conditions. The hope being that one after all can find an observational signature that distinguish that particular model from the generic inflationary scenario. In the next section we shall introduce the pre-big bang scenario which is perhaps the most ambitious such attempt.

2.2 Pre-Big Bang Inflation

The initial singularity in the SBB naturally leads to the question, "what came before the Big-Bang". The standard inflationary scenario does not really address this question in full details, although it seems as it was one of the main motivations for one of the pioneers of inflation. The following quote is from the original paper of Starobinsky: *"It is well known that many solutions of the classical Einstein equations, in particular, the Friedmann-Robertson-Walker isotropic homogeneous cosmological model, contain singularities and can not be analytically continued beyond them. So a fundamental cosmological problem arises: what was there before the Big-Bang, i.e. before the stage of classical expansion of the universe?"* [4].

In this section we will introduce the pre-big bang scenario invented by Gasperini and Veneziano[5] [54–57]. We will discuss how the pre-big bang scenario solves the cosmological problems. An important point is that the problem of the initial state is well separated from the issue of the Big-Bang singularity, unlike in slow-roll inflation as discussed in the previous section. Pre-big bang string cosmology is the first scenario where string theory is attempted to be incorporated in order to regulate the cosmic singularity. In this section we shall also address the phenomenology of the scenario. Especially we will try to emphasize observational predictions of the pre-big bang scenario that differ from the predictions of the standard slow-roll scenario. This issue will also be discussed in more details in chapter 4.

2.2.1 Toroidal Compactification of the String Effective Action

Since the pre-big bang scenario is based on the four dimensional low energy effective action of string theory, it is natural to first discuss the compactification of the 10-dimensional action. Thus, in this subsection we shall discuss the toroidal compactification of the NS-NS sector of the type II and heterotic effective action

$$\hat{S} = \int d^{D+d}\sqrt{-\hat{g}}e^{-\hat{\phi}}\left[\hat{R} + \partial_\alpha\hat{\phi}\partial^\alpha\hat{\phi} - \frac{1}{12}\hat{H}_{\alpha\beta\gamma}\hat{H}^{\alpha\beta\gamma}\right] \tag{2.85}$$

All fields in $D+d$ dimensions are written with hat and indices with Greek letters in the beginning of the alphabet $\alpha = 0, \ldots, D+d-1$. Latin indices will denote compact dimensions $a = D, \ldots, d-1$, and Greek letters in the end of the alphabet will denote the external dimension $\mu = 0, \ldots, D-1$. Thus if M denotes the external and K the internal space-time, then the full space-time can be viewed as the product $M \times K$. The coordinates of M is then denoted by x^μ and the coordinates of K is denoted by y^a. We assume that all the fields are independent of y.

Notice that, even if we for simplicity restrict our considerations here to toroidal compactification (assuming K is a d-Torus T^d), in many cases the discussion in this section will also apply to more general settings whenever the only modulus field that is dynamically important represents the volume of the internal space.

We can parameterize the complete $(D+d)$-dimensional metric as

$$ds^2 = \hat{g}_{\alpha\beta}dx^\alpha dx^\beta = g_{\mu\nu}dx^\mu dx^\nu + h_{ab}(dy^a + A^a_\mu dx^\mu)(dy^b + A^b_\nu dx^\nu)\ . \tag{2.86}$$

This is the most general form of the $D+d$ dimensional metric which is invariant under translations in y^a. The determinant is given by $\hat{g} = gh$.

The $(D+d)$-dimensional Ricci curvature scalar is related to the Ricci scalar of the lower dimensional manifold M by

$$\hat{R}_{D+d}(\hat{g}) = R_D(g) + \frac{1}{4}\nabla_\mu h^{ab}\nabla^\mu h_{ab} + \nabla_\mu(\ln\sqrt{h})\nabla^\mu(\ln\sqrt{h}) - \frac{2}{\sqrt{h}}\Box\sqrt{h} - \frac{1}{4}h_{ab}F^a_{\mu\nu}F^{\mu\nu b} \tag{2.87}$$

[5] There are two extensive reviews, [7] and [8].

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ is the field strength of A^a_μ. Inserting equation (2.87) into equation (2.85) implies that the dilaton-graviton sector of the dimensionally reduced action is given by [58]

$$S_D = \int d^D x \sqrt{-g} e^{-\phi} \left[R_D + \partial_\mu \phi^\mu \phi + \frac{1}{4} \partial_\mu h^{ab} \partial^\mu h_{ab} - \frac{1}{4} h_{ab} F^a_{\mu\nu} F^{\mu\nu b} \right] . \quad (2.88)$$

Above, the D-dimensional effective string coupling is given by the shifted dilaton field

$$\phi = \hat{\phi} - \frac{1}{2} \ln h . \quad (2.89)$$

Now let us consider how the field strengths of some generic form fields behave under toroidal compactification. The field strength of a $(n-1)$-form $\hat{B}_{\alpha_1 \ldots \alpha_{n-1}}$ is

$$\hat{H}_{\alpha_1 \ldots \alpha_n} = n \hat{\partial}_{[\alpha_1} \hat{B}_{\alpha_2 \ldots \alpha_n]} . \quad (2.90)$$

Under the compactification the $(n-1)$-form separates into two components, a $(n-1)$-form $B^{(n-1)}_{\mu_1 \ldots \mu_{n-1}} = \hat{B}_{\mu_1 \ldots \mu_{n-1}}$ and a $(n-2)$-form $B^{(n-2)}_{\mu_1 \ldots \mu_{n-2}} = \hat{B}_{\mu_1 \ldots \mu_{n-2} y}$. These have field strengths

$$H^{(n)}_{\mu_1 \ldots \mu_n} = n \partial_{[\mu_1} B^{(n-1)}_{\mu_2 \ldots \mu_n]} = \hat{H}_{\mu_1 \ldots \mu_n} \quad (2.91)$$

and

$$H^{(n-1)}_{\mu_1 \ldots \mu_{n-1}} = (n-1) \partial_{[\mu_1} B^{(n-2)}_{\mu_2 \ldots \mu_{n-1}]} = \hat{H}_{\mu_1 \ldots \mu_{n-1} y} . \quad (2.92)$$

It is now clear that the compactificaton of the NS-NS 3-form field strength in the action in equation (2.85) on the d-torus produces field strengths for all n-form potentials with $n = 0, 1, 2$. The full calculation leads to the following lowest order dimensionally reduced action [58]

$$S_H = - \int d^D \sqrt{-g} e^{-\phi} \left[\frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \frac{1}{4} H_{\mu\nu a} H^{\mu\nu a} + \frac{1}{4} H_{\mu ab} H^{\mu ab} + \frac{1}{12} H_{abc} H^{abc} \right] . \quad (2.93)$$

Here $H_{abc} = 0$, since $\hat{B}_{ab} = B_{ab}$ is y independent. Also, $H_{\mu ab} = \partial_\mu B_{ab}$. Similarly,

$$H_{\mu\nu a} = \tilde{F}_{\mu\nu a} - B_{ab} F^b_{\mu\nu} , \quad (2.94)$$

where $\tilde{F}_{\mu\nu a} = \partial_\mu \tilde{A}_{\nu a} - \partial_\nu \tilde{A}_{\mu a}$ and

$$\tilde{A}_{\nu a} = \hat{B}_{\mu a} + B_{ab} A^b_\mu . \quad (2.95)$$

Finally $H_{\mu\nu\rho}$ is given by

$$H_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]} - \frac{3}{2} \left(A^a_{[\mu} \tilde{F}_{\nu\rho]a} + F^a_{[\nu\rho} \tilde{A}_{\mu]a} \right) , \quad (2.96)$$

where

$$B_{\mu\nu} = \hat{B}_{\mu\nu} + \frac{1}{2} A^a_\mu \tilde{A}_{\nu a} - \frac{1}{2} A^a_\nu \tilde{A}_{\mu a} - a^a_\mu B_{ab} A^\beta_\nu . \quad (2.97)$$

To summarize, the lowest order dimensionally reduced action $S = S_D + S_H$ can be written in the form

$$S = \int d^D x \sqrt{-g} e^{-\phi} \mathcal{L} \; . \tag{2.98}$$

The Lagrangian $\mathcal{L}$ is given by $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$, where

$$\begin{aligned} \mathcal{L}_1 &= R_D + \partial_\mu \phi \partial^\mu \phi \\ \mathcal{L}_2 &= \frac{1}{4}\left(\partial_\mu h_{ab} \partial^\mu h^{ab} - h^{ac} h^{bd} \partial_\mu B_{ab} \partial^\mu B_{cd}\right) \\ \mathcal{L}_3 &= -\frac{1}{4}\left(h_{ab} F^a_{\mu\nu} F^{\mu\nu b} + h^{ab} H_{\mu\nu a} H^{\mu\nu}_b\right) \\ \mathcal{L}_4 &= -\frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \; . \end{aligned} \tag{2.99}$$

Now let us discuss some assumptions which are typically invoked to simplify the action in equation (2.98). If we assume that $\hat{B}_{\alpha\beta}$ and the metric $\hat{g}_{\alpha\beta}$ has block diagonal form ($\hat{B}_{\mu a} = 0$, $A_{\mu a} = 0$), then $H_{\mu\nu a} = F^a_{\mu\nu} = 0$ and the action is given by

$$\begin{aligned} S &= \int d^D x \sqrt{-g} e^{-\phi} \left[R_D + \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right. \\ &\quad \left. + \frac{1}{4}\left(\partial_\mu h_{ab} \partial^\mu h^{ab} - h^{ac} h^{bd} \partial_\mu B_{ab} \partial^\mu B_{cd}\right) \right] \; . \end{aligned} \tag{2.100}$$

For the discussion of the pre-big bang scenario, we will obviously be most interested in the special case $D = 4$. In 4-dimensions the Poincaré dual of a 3-form is a one-form ${}^*H_\sigma = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma} H^{\mu\nu\rho}$. Due to the Bianchi identity the field equation for $H_{\mu\nu\rho}$ is automatically satisfied if

$${}^*H_\mu = e^\phi \bar{H}_\mu \; , \tag{2.101}$$

where $\bar{H}$ is itself an antisymmetric one-form field strength derived from a scalar potential; $\bar{H}_\mu = \partial_\mu \sigma$. Hence, we can write down a dual action to the effective action in equation (2.100) in terms of the pseudo-scaler axion field, σ

$$H^{\mu\nu\rho} = \epsilon^{\mu\nu\rho\kappa} e^\phi \partial_\kappa \sigma \; . \tag{2.102}$$

The dual action then takes the form

$$\begin{aligned} S &= \int d^4 x \sqrt{-g} e^{-\phi} \left[R_4 + \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{2\phi} \partial_\mu \sigma \partial^\mu \sigma \right. \\ &\quad \left. + \frac{1}{4}\left(\partial_\mu h_{ab} \partial^\mu h^{ab} - h^{ac} h^{bd} \partial_\mu B_{ab} \partial^\mu B_{cd}\right) \right] \; . \end{aligned} \tag{2.103}$$

The moduli fields arising from the internal degrees of freedom h_{ab} and B_{ab} are a set of massless scalar fields. Considering only four dimensional homogeneous and isotropic cosmologies, where all fields are homogeneous and have same dynamics, the dynamics of the external space-time can be determined by a single modulus field, β.

This is equivalent to do a compactification of the $4+d$-dimensional effective NS-NS action on an isotropic d-torus, where the components of the two-form potential on the internal space are assumed to be trivial. The moduli field β is called the *breathing mode* and determines the volume of the internal dimensions. We can then assume that the $(4+d)$-dimensional metric simplifies to

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j + e^{2\beta}\delta_{ab}dy^a dy^b \ . \tag{2.104}$$

With this choice of metric, the effective four dimensional dilaton is

$$\phi = \hat{\phi} - 6\beta \tag{2.105}$$

and substituting $h_{ab} = \exp(\sqrt{\frac{2}{d}}\beta)\delta_{ab}$ into the action in equation (2.103) yields

$$S = \int d^4x\sqrt{-g}e^{-\phi}\left[R_4 + \partial_\mu\phi\partial^\mu\phi - 3\partial_\mu\beta\partial^\mu\beta - \frac{1}{2}e^{2\phi}\partial_\mu\sigma\partial^\mu\sigma\right] \ . \tag{2.106}$$

The action in equation (2.106) is an obvious starting point for the exploration of string cosmology because it contains the general terms common for most more general dimensionally reduced string actions. However, it should be noted that important work has also been done in order to understand the relevance of R-R fields in string cosmology.

2.2.2 Scale Factor Duality

In this subsection we will discuss the symmetries of the equations of motion derived from the action in equation (2.106). When the axion field is trivial, $\dot{\sigma} = 0$, the equations derived from the action (2.106) are invariant under duality transformations [7],

$$a \to \tilde{a} = a^{-1} \ , \qquad \phi \to \tilde{\phi} = \phi - 6\ln a \ . \tag{2.107}$$

This invariance, called *scale factor duality* (SFD) and was first discussed in [54]. When $a \to a^{-1}$ the Hubble parameter H goes to into $\tilde{H} = -H$. However, the cosmological equations are also invariant under time reversal $t \to -t$, which takes $H \to -H$. Thus, in string cosmology a solution has in general four branches: $a(t), a(-t), a^{-1}(t), a^{-1}(-t)$. Two of them describe expansion ($H > 0$) and the other two describe contraction ($H < 0$) (see fig.(2.3)).

SFD represents one of the key motivations behind the pre-big bang scenario. This is because to any given decelerated, expanding solution, $H(t) > 0$, with decreasing curvature, $\dot{H}(t) < 0$, is always associated an inflationary dual partner describing accelerated expansion, $\tilde{H}(-t) > 0$, and growing curvature, $\dot{\tilde{H}}(-t) > 0$. This duality of solution has no analogue in Einstein gravity, where there is no dilaton, and suggest that if string theory regularizes the singularity it will lead to a completion of standard cosmology, in which the universe smoothly evolves from the inflationary pre-big bang branch $\tilde{H}(-t)$ to the post-big bang branch $H(t)$ (see the dashed line in fig.(2.3)).

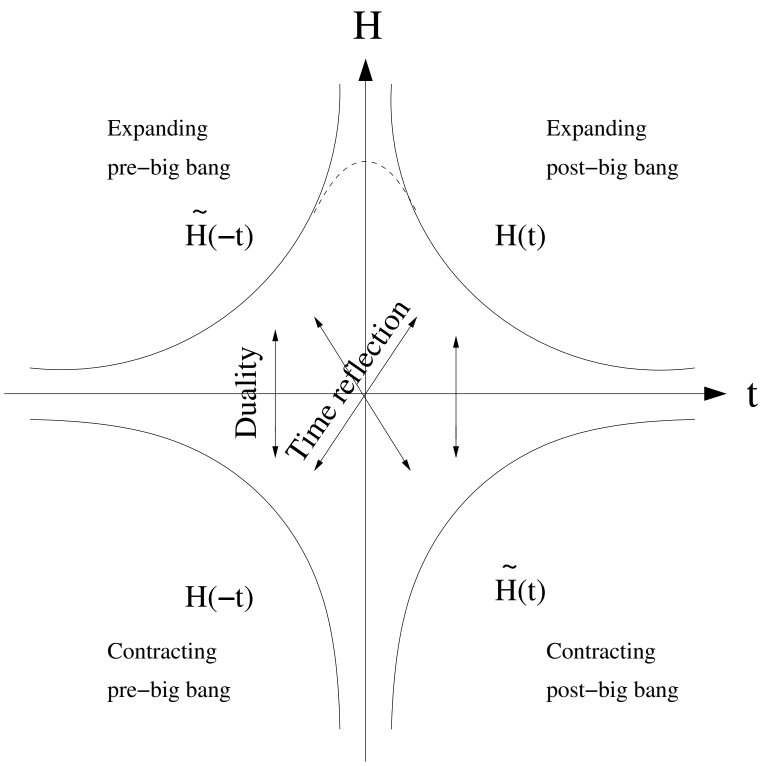

Figure 2.3: The four different branches of the string cosmology backgrounds [7]. The regularized solution, due to higher order string effects, is indicated by the dashed line.

Now consider the addition of a matter action S_m to the action in equation (2.106). Let us further assume for simplicity that the internal moduli is frozen and that the matter takes the form of an isotropic fluid with diagonal energy-momentum tensor,

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \ , \qquad T_\mu^\nu = diag.(-\rho, p, \ldots, p) \ , \qquad p/\rho = w = const. \qquad (2.108)$$

If one introduce the "shifted" dilaton, density, and pressure

$$\bar{\phi} = \phi - 3 \ln a \ , \qquad \bar{\rho} = \rho a^3 \ , \qquad \bar{p} = p a^3 \ , \qquad (2.109)$$

then it turns out that the equations of motion, derived from the action in equation (2.106) with S_m added, are invariant under time reflection and the duality transformation [7]

$$a \to a^{-1} \ , \bar{\phi} \to \bar{\phi} \ , \qquad \bar{\rho} \to \bar{\rho} \ , \qquad \bar{p} \to \bar{p} \ . \qquad (2.110)$$

It is interesting to note that this transformation reflects the barotropic equation of state $w \to -w$.

Let us follow [7], and consider a solution to the equations of motion with the following ansatz

$$a \sim t^\alpha \ , \qquad \bar{\phi} \sim -\gamma \ln t \ , \qquad p = w\rho \ . \qquad (2.111)$$

For $w \neq 0$ the time evolution of a and $\bar{\phi}$ is given by

$$a \sim t^{\frac{2w}{1+3w^2}} \ , \qquad \bar{\phi} = -\frac{2}{1+3w^2} \ln t \ , \tag{2.112}$$

and for ρ, ϕ the solutions are

$$\rho = \bar{\rho} a^{-3} = \rho_0 a^{-3(1+w)} \ , \qquad \phi = \bar{\phi} + 3 \ln a = \frac{2(3w-1)}{1+3w^2} \ln t \ . \tag{2.113}$$

The four solutions related by time reversal and duality transformation is characterized by the scale factors

$$a_{\pm}(\pm t) \sim (\pm t)^{\pm 2w/(1+3w^2)} \ . \tag{2.114}$$

Interestingly, for $w = 1/3$ and $t > 0$ one obtains the standard solution for a radiation dominated universe with constant dilaton

$$a \sim t^{1/2} \ , \qquad \rho = 3p \sim a^{-4} \ , \qquad \phi = constant \ , \qquad t \to \infty \ , \tag{2.115}$$

describing decelerated expansion and decreasing curvature

$$\dot{a} > 0 \ , \qquad \ddot{a} < 0 \ , \qquad \dot{H} < 0 \ , \qquad \dot{\phi} = 0 \ . \tag{2.116}$$

This solution is related by a duality and time reversal transformation to the pre-big bang solution [7],

$$a \sim (-t)^{-1/2} \ , \qquad \phi \sim -3 \ln(-t) \ , \qquad \rho = -3p \sim a^{-2} \ , \qquad t \to -\infty \ , \tag{2.117}$$

which describes accelerated expansion (inflation), with growing dilaton and curvature

$$\dot{a} > 0 \ , \qquad \ddot{a} > 0 \ , \qquad \dot{H} > 0 \ , \qquad \dot{\phi} > 0 \ . \tag{2.118}$$

This shows, that if we use SFD to postulate a phase prior to the big-bang which is dual to the phase of a radiation dominated universe, then via string theory we are automatically lead to a phase of growing curvature *and* growing dilaton.

Scale factor duality is a subgroup of a more general $O(d,d)$ covariance of the action in equation (2.100). In general the groups of duality transformations, that relates equivalent string backgrounds, corresponds to discrete subgroups of non-compact, global symmetries of the effective supergravity actions. The global symmetries are broken to the discrete subgroups by quantum effects. Since the discrete subgroup $O(d,d;Z)$ of the group $O(d,d)$ is the T-duality group of the toroidally compactified string, SFD is in this way related to the T-duality of superstring theory.

2.2.3 The Dynamics of Pre-Big Bang Cosmology

In this subsection we will discuss the dynamics of the pre-big bang scenario. In subsection 2.2.1, we showed that the four-dimensional low energy effective tree-level action of string theory can be written in the following generic form,

$$S_{eff} = \frac{1}{2\alpha'} \int d^4x \sqrt{-g} e^{-\phi} \left[\mathcal{R} + \partial_\mu \phi \partial^\mu \phi - 3 \partial_\mu \beta \partial^\mu \beta + \mathcal{L}_{matter} \right] \ , \tag{2.119}$$

where $\sqrt{\alpha'} = \sqrt{8\pi}/M_s$. The matter Lagrangian, $\mathcal{L}_{matter}$, is composed of gauge fields and axions which we assume to be trivial constant fields that do not contribute to the background. Hence, only the quantum fluctuations of the matter fields will be important, as we will discuss later.

The solution to the equations of motion for the background fields can in the spatially flat case be parameterized as [7, 8, 54, 56, 57]

$$a = a_s \left| \frac{\tau}{\tau_s} \right|^{\frac{\delta}{1-\delta}} \quad , \quad e^{\phi} = e^{\phi_s} \left| \frac{\tau}{\tau_s} \right|^{\frac{3\delta-1}{1-\delta}} \quad , \quad e^{\beta} = e^{\beta_s} \left| \frac{\tau}{\tau_s} \right|^{\frac{\zeta}{1-\delta}} \ , \tag{2.120}$$

where δ and ζ satisfy the Kasner constraint,

$$3\delta^2 + 6\zeta^2 = 1 \ . \tag{2.121}$$

In proper time, the solution becomes

$$a = a_s \left| \frac{t}{t_s} \right|^{\delta} \quad , \quad e^{\phi} = e^{\phi_s} \left| \frac{t}{t_s} \right|^{3\delta-1} \quad , \quad e^{\beta} = e^{\beta_s} \left| \frac{t}{t_s} \right|^{\zeta} \ , \tag{2.122}$$

also subject to the constraint in equation (2.121).

Obviously, in order to reproduce an expanding universe with $H > 0$ at late times, we must choose a solution with $\delta > 0$ for $\tau > 0$ whose dual solution has $\delta < 0$, $H > 0$ for $\tau < 0$.

The solution for the shifted dilaton satisfies

$$\dot{\bar{\phi}} = -\frac{1}{\delta} H \ , \tag{2.123}$$

which shows that $\dot{\bar{\phi}}$ and H have opposite sign at late times and same sign at early times, so we say that the pre-big bang era is on the "+" branch and the post-big bang solution is on the "-" branch. In between the two branches, which are related by SFD, the scalar curvature H blows up and both branches reach a singularity at $\tau = 0$. Of course, it is thought that this is only an artifact of our effective theory. Remember that SFD is a symmetry only of the low-energy effective theory related to the T-duality of the full string theory. As H approaches the string scale $(1/\sqrt{\alpha'})$ and/or the dilaton becomes order one, higher order terms in α' and/or the string coupling $\exp(\phi)$ becomes non-perturbative. It is believed that the regularizing effect of string theory will also extend to this problem and smoothen the singularity out into a continuous regular transition. Note that the vacuum solution for the dilaton implies

$$\dot{\phi} = H\left(3 - \frac{1}{\delta}\right) , \tag{2.124}$$

so on the "+" branch it will always be growing. If a stringy mechanism can bridge the "+" and the "-" branch in a smooth fashion, we are given a cosmological scenario which naturally incorporates inflation on the "+" branch and where singularities are absent due to the properties of small scale physics.

The solution (2.120) suggests that the universe emerged an infinite time ago in a state of low curvature ($H \to 0$) and weak coupling ($\phi \to -\infty$). This is actually the major conceptual difference from the standard lore. Reciting Veneziano [59]:

The Universe started its evolution from the most simple initial state conceivable in string theory, its perturbative vacuum. This correspond to an (almost)

Empty, Cold, Flat, Free

Universe as opposed to the standard

Dense, Hot, Highly-curved

initial state of conventional cosmology.

Let us now consider how the horizon problem is solved in the pre-big bang inflation setup. Let us denote by r, the scaling of the size of the proper horizon scale with respect to the proper size of a homogeneous region, which is given by

$$r(t) \sim \frac{H^{-1}(t)}{a(t)} , \tag{2.125}$$

such that for a generic power-law evolution $a \sim t^{\gamma}$, the ratio scales linearly with respect to the conformal time,

$$r \sim t^{1-\gamma} \sim \int a^{-1} dt \sim \tau . \tag{2.126}$$

In the pre-big bang setup, it is reasonable to assume that the standard radiation dominated era can be extended back in time down to the Planck scale, where it is immediately preceded by a phase of accelerated expansion. Thus, at the beginning of the radiation dominated era the horizon size is naturally set by the Planck length l_p, while the size of our universe today, set by the present Hubble radius, red-shifted back to the beginning of the radiation dominated era is $\sim 10^{30} l_p$ i.e. the homogeneous region is much larger than the horizon at that time. To explain this large mismatch, one concludes that during inflation the ratio r must have increased at least by a factor 10^{30}, or more precisely, as shown in equation (2.6), inflation must have lasted more than 70 e-foldings. From equation (2.126), we see that this means that we must require

$$|\tau_f|/|\tau_i| \lesssim 10^{-30} \tag{2.127}$$

for the duration of the pre-big bang inflationary phase (τ_i and τ_f marks respectively the initial and the final conformal time for the inflationary era).

For simplicity we assume that the dilaton freezes out at the end of the inflationary epoch with $\exp(\phi_f) \sim 1$, such that the string scale and the Planck scale are of the same order. During the pre-big bang era the scale factor evolves as (see (2.122))

$$a \sim |t|^{\delta} \sim |\tau|^{\frac{\delta}{1-\delta}} \ . \tag{2.128}$$

It then follows from equation (2.127), that scale factor is at most reduced by $a_i/a_f \sim 10^{-30/(1-\delta)}$. This means that even at the beginning of inflation the homogeneous region is still very large. In the slow-roll scenario (see previous section) the scale factor grows like $a \sim |\tau|^{-1}$ and at the initial stage of inflation a homogeneous region is of order Planck size. In the pre-big bang case the increase in the size of the homogeneous region is reduced by a factor of at least

$$10^{30\delta/(1+\delta)} \tag{2.129}$$

relative to that of the slow-roll inflationary scenario. In the case of static internal dimension $\delta = 1/\sqrt{3}$, it means that the initial homogeneous region should be of order 10^{19} in string units [54, 60, 61], while in chaotic inflation it is of order l_p.

It is interesting to view this also in terms of the string coupling $g_s = \exp(\phi/2)$ or the related Planck length l_p. During the pre-big bang era the string coupling must increase at least by

$$\frac{g_s(\tau_i)}{g_s(\tau_f)} = \frac{l_p(\tau_i)}{l_p(\tau_f)} = \left(\frac{\tau_f}{\tau_i}\right)^{\frac{1}{2}\frac{3\delta-1}{1-\delta}} \lesssim 10^{-15\frac{3\delta-1}{1-\delta}} \ . \tag{2.130}$$

Then, for $|\delta| = 1/\sqrt{3}$, one finds that the size of the initial homogeneous region in Planck units must be extremely large $\sim 10^{45} l_p$ [7, 61].

Naively, we might think that we could just as well have postulated the universe to be homogeneous at the beginning of the radiation dominated era where the homogeneous patch is anyway $10^{30} l_p$ or greater. However, that would amount to postulate the universe to be homogeneous on super-horizon scales, while in the pre-big bang the initial homogeneous patch might be large, but it is still sub-horizon. This is because, as we go backward in time, the horizon increases by a factor $H_i^{-1}/H_f^{-1} = (\tau_i/\tau_f)^{1/(1-\delta)}$, so at the beginning of inflation, $H_i^{-1} \sim 10^{30/(1-\delta)} l_p$.

It is really a question of which is the natural scale for the initial homogeneous patch: the Planck scale or the horizon scale. When the universe is initially in a high curvature, quantum gravity regime, the Planck scale is arguable the natural scale to use. However, in the pre-big bang scenario the initial condition is set in a weak-coupling, low-curvature, classical regime, where physics is insensitive to the Planck scale and thus the horizon scale H^{-1} might be a more reasonable scale to determine the size of the initial homogeneous patch [7].

2.2.4 The Graceful Exit

In our discussion of the dynamics of the pre-big bang scenario, we discussed the "+" and the "-" branch and how they are related by SFD. But we did not discuss the transition from the one branch to the other. In this subsection we will briefly discuss some of the possibilities of how such a transition might work and some of the problems involved from a theoretical point of view.

There exists no-go theorems, which states that neither a dilaton potential nor matter sources in the form of a perfect fluid can accommodate the transition [62, 63]. On this basis, it is argued that the transition can not occur while the curvature is below the string scale and the string coupling is weak. Higher order corrections in the string coupling g_s, controlling the string-loop corrections, or in α', controlling the finite-string-size correction, must be important for the transition, leading to an intermediate string phase of high curvature or strong coupling [64].

If the initial value of the string coupling is small enough, it is possible that the universe will reach a high curvature regime while the string coupling is still weak. One can then consider the effect of including only the first order correction in α'

$$S_{\alpha'} = \frac{1}{8}\int d^4x\sqrt{-g}e^{-\phi}\alpha' R^{\sigma}_{\mu\nu\rho}R^{\mu\nu\rho}_{\sigma} \tag{2.131}$$

while neglecting terms beyond the lowest order in g_s. The action (2.131) contains ghosts due to the R^2 term [65], while string theory is ghost free, and it was argued in [66] that additional R^2 and $R^{\mu\nu}R_{\mu\nu}$ terms must be added to remove the higher derivatives and yield a theory without ghosts. This is commonly done by replacing the square of the Riemann tensor with the Gauss-Bonnet invariant [66]

$$R^2_{GB} \equiv R^{\sigma}_{\mu\nu\rho}R^{\mu\nu\rho}_{\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2 \ . \tag{2.132}$$

This comes at the price of introducing dilaton dependent α' corrections. One can make a field redefinition [8, 67]

$$\begin{aligned} \tilde{g}_{\mu\nu} &= g_{\mu\nu} + 2\alpha'\left[R_{\mu\nu} - \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\partial_\sigma\phi\partial^\sigma\phi\right] \\ \tilde{\phi} &= \phi - \frac{1}{2}\alpha'\left[R + 3\partial_\sigma\phi\partial^\sigma\phi\right] \ , \end{aligned} \tag{2.133}$$

such that, dropping the tilde over the redefined fields, the α' correction can be written as

$$S_{\alpha'} = \frac{1}{8}\int d^4x\sqrt{-g}e^{-\phi}\alpha'\left(R^2_{GB} - (\partial\phi)^4\right) \ . \tag{2.134}$$

It is interesting that the pre-big bang equations of motion, with this α' correction added, possesses fixed point solutions for constant H and $\dot{\phi}$ and that numerical integrations have shown that the solutions with pre-big bang like initial conditions are attracted toward the fixed point [67, 68]. In the string phase where H and $\dot{\phi}$ are constant, they can be parameterized as [69],

$$a(\tau) = \frac{-1}{H_s\tau} \ , \qquad \phi(\tau) = \phi_s - 2\beta\ln|\tau/\tau_s| \ , \qquad \tau_s < \tau < 0 \ . \tag{2.135}$$

Although this solution does not exhibit a full transition to the post-big bang FRW universe, it does show an indication of how the finite-string-size effects regularize the theory and yield a non-singular solution.

Also a full transition to the post-big bang era has been found [70], but in general it requires some temporary violation of the null energy condition due to quantum corrections to the string effective action in order to escape the fixed point of constant curvature and linearly growing dilaton. However, the precise form of quantum loop corrections is not known, and the full transition has been demonstrated only in principle. An other aspect is, that the dilaton is still growing linearly in the intermediate string phase, and we are thus likely to encounter a strongly coupled regime where the light modes of the effective action are different and related to the original ones by S-dualty. This signals the entering to a D-brane dominated phase[6] [68, 72].

However, even if the quantum correction turn out to have the right form to provide an exit from the string phase to a phase of decelerated expansion, the stabilization of the dilaton remains problematic, even if the generalized second law seems to suggest that a solution must exist [73–76].

2.2.5 String Frame vs. Einstein Frame

Until now we have used the string frame in our discussion of the pre-big bang scenario, which is the most natural from a string theory point of view. However, in the string frame the dilaton is not minimally coupled to the metric, and in order to compare with other scenarios it is convenient to go to the Einstein frame, where the dilaton is minimally coupled. The theory formulated in the string frame and the Einstein frame does not correspond to two different models, but are simply different kinematic representations of the same scenario in two different frames related by a conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \;, \qquad \Omega^2 \equiv \exp\left[\frac{-2}{d-2}\phi\right] \tag{2.136}$$

on the d-dimensional metric, together with the field redefinition

$$\tilde{\phi} \equiv \sqrt{\frac{2}{d-2}}\phi \;. \tag{2.137}$$

In this way the action in equation (2.100) transforms into an "Einstein-Hilbert" form [8]

$$\begin{aligned} S \;=\; & \int d^D x \sqrt{-\tilde{g}} \left[\tilde{R}_D + \tilde{\partial}_\mu \tilde{\phi} \tilde{\partial}^\mu \tilde{\phi} - \frac{1}{12} e^{-\sqrt{8/(d-2)}\tilde{\phi}} \tilde{H}_{\mu\nu\rho} \tilde{H}^{\mu\nu\rho} \right. \\ & \left. + \frac{1}{4}\left(\tilde{\partial}_\mu h_{ab} \tilde{\partial}^\mu h^{ab} - h^{ac} h^{bd} \tilde{\partial}_\mu B_{ab} \tilde{\partial}^\mu B_{cd} \right) \right] \;. \end{aligned} \tag{2.138}$$

[6] In fact, by putting the pre-big bang universe on the brane, the author of [71] claims to obtain a graceful exit at low curvature and weak coupling, without violating the Null Energy Condition.

in the Einstein frame.

From the conformal transformation in equation (2.136), we see that in the Einstein frame, the scale factor is given by

$$\tilde{a} = ae^{-\phi/(d-2)} \tag{2.139}$$

and the cosmic time is given by

$$d\tilde{t} = e^{-\phi/(d-2)}dt \ . \tag{2.140}$$

Then in the Einstein frame, the solution given in equation (2.122), becomes

$$\tilde{a} \sim (-\tilde{t})^{1/3} \ , \qquad e^{\tilde{\phi}} \sim (-\tilde{t})^{-2\frac{3\delta-1}{3\delta+3}} \ , \qquad \tilde{t} < 0 \ , \qquad \tilde{t} \to 0_- \ . \tag{2.141}$$

It is evident that in this frame, the solution describes accelerated contraction with growing dilaton and growing curvature:

$$\frac{d\tilde{a}}{d\tilde{t}} < 0 \ , \qquad \frac{d^2\tilde{a}}{d\tilde{t}^2} < 0 \ , \qquad \frac{d\tilde{H}}{d\tilde{t}} < 0 \ , \qquad \frac{d\tilde{\phi}}{d\tilde{t}} > 0 \ . \tag{2.142}$$

It is important to note that both in the string frame and the Einstein frame, the event horizon is given approximately by H^{-1}, just like in the quasi de Sitter case.

To summarize, let us consider a power-law evolution of the scale factor in cosmic time. As we showed already in the introduction to this chapter, to solve the flatness problem, we need a sufficiently long era where $\dot{a}$ and $\ddot{a}$ has same sign. Following Gasperini [77], the two classes of backgrounds are

- Class I: $a \sim t^\beta$, $\beta > 1$, $t \to +\infty$. *This class of backgrounds corresponds to what is conventionally called "power inflation", describing a phase of accelerated expansion and decreasing curvature scale,* $\dot{a} > 0$, $\ddot{a} > 0$, $\dot{H} < 0$. *This class contains, as a limiting case, the standard de Sitter inflation,* $\beta \to \infty$, $a \sim e^{Ht}$, $\dot{H} = 0$, *i.e. accelerated exponential expansion at constant curvature.*

- Class II: $a \sim (-t)^\beta$, $\beta < 1$, $t \to 0_-$. *This is the class of backgrounds corresponding to the string cosmology scenario. There are two possible subclasses:*

  - IIa: $\beta < 0$, *describing superinflation, i.e. accelerated expansion with growing curvature scale,* $\dot{a} > 0$, $\ddot{a} > 0$, $\dot{H} > 0$.

  - IIb: $0 < \beta < 1$, *describing accelerated contraction and growing curvature scale,* $\dot{a} < 0$, $\ddot{a} < 0$, $\dot{H} < 0$.

The string frame formulation of the pre-big bang scenario belongs to class IIa, while the equivalent Einstein frame description belong to class IIb.

In the contracting frame, the flatness and horizon problems are solved because the energy density of the dilaton becomes dominant over the curvature as $\tilde{a} \to 0$ and the horizon decreases much faster than the scale factor.

2.2.6 Phenomenology

In the previous section, we already treated the theory of quantum and classical cosmological perturbations in the context of slow-roll inflation. In Pre-big bang cosmology the CMB fluctuations and the large scale structure are created in a similar fashion by quantum field fluctuations which are pushed outside the horizon and become classical. Also in the pre-big bang scenario we can normalize the modes to the Bunch-Davies vacuum in the infinite past, because the universe starts out in the infinite past, in an empty state with vanishing curvature. We shall therefore not repeat the details of the formalism, but rather restrict ourself to highlight the differences in a qualitative manner. In chapter four, we will deal with some of the phenomenological aspects in a more quantitative way, as we will describe how the observed spectrum of CMB perturbations is created by the late decay of a stringy axion field.

In the Einstein frame we can parameterize the tensor and scalar perturbations of the metric just like in equation (2.22) and (2.27) respectively. Let us first consider the simpler tensor perturbations. The perturbation equation for the polarization modes h_k, is the same as in equation (2.25) [8]:

$$h_k'' + 2\frac{\tilde{a}'}{\tilde{a}}h_k' + k^2 h_k = 0 \ . \tag{2.143}$$

From the background solution in the Einstein frame (2.141), it is easy so see that $\tilde{a}(\tau) \sim (-\tau)^{1/2}$ and the solution is of the form

$$h_k = \left[C_1 H_0^{(1)}(-k\tau) + C_2 H_0^{(2)}(-k\tau) \right] \ . \tag{2.144}$$

Now comparing with equation (2.40), we see that in the pre-big bang scenario $\nu_T = 0$ and from equation (2.71), we find that the spectral index for the tensor fluctuations is

$$n_T = 3 \ . \tag{2.145}$$

This is in contrast to the conventional inflation models which requires $n_T < 0$, and it can be considered to be a robust and unique prediction of the pre-big bang scenario. There is a possibility that such a spectrum could be observed by future gravitational wave detectors [78–81].

Now let us consider the scalar perturbations. In the longitudinal gauge the growing mode blows up in such a way that the linear perturbation theory breaks down. This is however a gauge artifact and instead one should work in another gauge. It is common to choose the off-diagonal gauge (also called "uniform curvature" gauge). It is defined by taking $\psi = 0 = E$ in equation (2.27). Comparing with equation (2.28), we see that this corresponds to the gauge transformation

$$\xi^0 = -\frac{\tilde{a}\psi}{\tilde{a}'} \ , \qquad \xi = E \ . \tag{2.146}$$

The non-vanishing gauge-invariant scalar quantities are then from (2.28),

$$\tilde{A} \equiv A + \psi + \left(\frac{\tilde{a}\psi}{\tilde{a}'}\right)' , \qquad \tilde{B} \equiv B - E' - \frac{\tilde{a}\psi}{\tilde{a}'} . \tag{2.147}$$

In Fourier space, the perturbed Einstein equation yields the following evolution equation for the Fourier mode $\tilde{A}_k$ [8]:

$$\tilde{A}_k'' + 2\frac{\tilde{a}'}{\tilde{a}}\tilde{A}_k' + k^2\tilde{A}_k , \tag{2.148}$$

and the constraint

$$\tilde{A}_k = -\left(\tilde{B}_k' + 2\frac{\tilde{a}'}{\tilde{a}}\tilde{B}_k\right) \tag{2.149}$$

for the Fourier mode of $\tilde{B}$. The equation for $\tilde{A}_k$ is identical to the equation for the gravitational waves, and by using the background solution $\tilde{a}(\tau) \sim (-\tau)^{1/2}$, we find

$$\tilde{A}_k = \left[A_1 H_0^{(1)}(-k\tau) + A_2 H_0^{(2)}(-k\tau)\right] . \tag{2.150}$$

Expressing $\tilde{A}$ in terms of the Bardeen potentials, it can be seen that $\tilde{A}$ is related to the curvature perturbation on uniform energy density hypersurfaces ζ. In fact, using $\tilde{a}(\tau) \sim (-\tau)^{1/2}$ one can show that [8]

$$\zeta = \frac{\tilde{A}}{3} . \tag{2.151}$$

The curvature perturbation on slices of uniform energy density ζ is the same as the comoving curvature perturbation $\mathcal{R}$ on large scales. It can also be shown that $\tilde{A}$ is related to the perturbations of the dilaton ϕ, and the moduli, β, as [8]

$$\tilde{A} = \frac{\phi'}{4}\frac{\tilde{a}}{\tilde{a}'}\delta\phi + \frac{\beta'}{4}\frac{\tilde{a}}{\tilde{a}'}\delta\beta , \tag{2.152}$$

and by normalizing $\delta\phi$ and $\delta\beta$ to the Bunch-Davies vacuum in the infinite past, one finds[7] [8]

$$\mathcal{P}_\zeta = \frac{8}{\pi^2} l_p^2 \tilde{H}^2 (-k\tau)^3 \left[\ln(-k\tau)\right]^2 \tag{2.153}$$

at late times ($k\tau \to 0$). One can then calculate the spectral index of the curvature perturbations

$$n \equiv 1 + \frac{d\ln\mathcal{P}_\zeta}{d\ln k} = 4 . \tag{2.154}$$

This is ruled out by observations which requires $n \simeq 1$ [24].

[7] From equation (2.152) one sees that the spectral index of the scalar perturbations is basically given by the spectral index of the dilaton and the moduli perturbations.

However, in a multi field inflationary scenario, the density perturbations at late times are not only related to ζ and there is as well the possibility that a change in the equation of state on super horizon scales can induce a significant curvature perturbation after the end of inflation (see chapter 4). Thus, in the presence of axion fields the situation is not so simple and in fact we will show in chapter 4 that the axion fields can give rise to the observed spectrum of CMB perturbations.

Chapter 3

The Trans-Planckian Problem of Inflation

In this section we shall address the trans-Planckian problem of the standard inflationary scenario. In the first and second chapter, we emphasized that in order to solve the horizon and flatness problem, inflation must have lasted more than $65-70$ e-foldings. If this is true, then if we redshift the scale corresponding to the size of our universe today back to the onset of inflation, it must have been smaller than the Planck length.

To see this, we consider the following argument. The reheat temperature, T_{RH}, is related to the Hubble parameter at the time of reheating by [27]

$$H^2 \simeq \frac{8\pi^3 g_*}{90 m_p^2} T_{RH}^4 \ . \tag{3.1}$$

If the conversion of the inflaton potential energy is perfectly efficient and the Hubble rate during inflation is $H_I \sim 10^{-5} m_p$, it will lead to a reheat temperature of about $T_{RH} \sim 10^{16}$ GeV. Since the temperature of the photon fluid T_r is inversely proportional to the scale-factor $T_r \sim 1/a$ and the universe today has temperature $T_0 \sim 0.1$ meV, the scale-factor has increased by

$$\frac{a_0}{a_{RH}} \sim 10^{29} \tag{3.2}$$

since the reheating time. If the scale factor has increased by 70 e-foldings during the inflationary period prior to the reheating, it means that from the beginning of inflation until now the scale factor has increased by

$$\frac{a_0}{a_i} \sim 10^{55} \ . \tag{3.3}$$

The largest observable scale in the CMB anisotropies corresponds to the size of the Hubble scale at decoupling, at $z \simeq 1100$. Today the Hubble parameter is $H_0 \sim 10^{-42}$GeV with the scale factor dependence $\sim a^{-3/2}$ during the matter dominated era, so the largest observable scale is $\lambda_{l=1} \sim 10^{37}$GeV. The observable scales in the CMB anisotropies span 3 orders of magnitude, so the smallest observable scale in the CMB anisotropies corresponds to $\lambda_{l=1000} \sim 10^{33}\text{GeV}^{-1}$. Thus the smallest observable scale in the CMB anisotropies is $\lambda_{l=1000} \sim 10^{52} l_p$, which at the beginning of inflation would have been 3 orders of magnitude less than the Planck length. Thus if inflation lasts more than 70 e-foldings the observable scales in the CMB spectrum originate from sub-Planckian scales.

However, Einstein gravity is believed to break down at small scales near the Planck length, so the standard treatment of perturbations seems to be incomplete at the initial stages of inflation [82, 83].

Only if the Hubble parameter during inflation is much smaller than[1] $10^{-5} m_p$, so that the reheat temperature is very small and inflation does not last much longer than 70 e-foldings, the trans-Planckian problem can be avoided. One could view this as merely a theoretical constraint on a given inflationary scenario, but on the other hand there exist popular inflationary scenarios, such as chaotic inflation [53], which do not satisfy such a theoretical constraint. Hence, it is interesting to explore the effect of short scale physics on the spectrum of the CMB anisotropies [3, 17, 85–115].

Naively, one might think that one could specify the initial state for the perturbation modes just when they emerge from the trans-Planckian regime. But in the next subsection we will show that such a choice is not unique. Further, the standard Bunch-Davies vacuum might in fact be an unnatural choice for the vacuum of the modes at the moment when they become larger than the Planck length [94, 109, 115]. Then, in subsections 3.2 and 3.3, we will show how a minimum length scale can be incorporated into quantum field theory in curved space, essentially protecting the modes from the trans-Planckian regime and selecting a natural vacuum for the modes [3]. This approach is inspired by the non local nature of string theory and quantum gravity, which shows that in nature, most likely, a minimum length scale exists. Finally in subsection 3.4, we shall discuss a novel feature, the trans-Planckian damping, and how it might help in fixing a unique initial state for the quantum fluctuations.

3.1 Ambiguity in the Choice of Initial State

In the section 2.1.4 on cosmological quantum perturbations, we discussed how the spectrum of quantum perturbations depends on the normalization to the vacuum at the initial time. In this section we shall discuss what happens if we take the initial time to be the time when a given mode emerges from the trans-Planckian regime and normalize it to the vacuum at that point.

The theory of metric perturbations basically reduces to a free scalar field theory in curved space, as can be seen from equations (2.24) and (2.33). We can expand

[1] In [84], a scenario was proposed where the inflationary potential can be as low as at the GeV scale.

the canonically normalized quantum field $\hat{\chi}$, describing either tensor or scalar metric fluctuations, as in equation (2.120),

$$\hat{\chi}_{\mathbf{k}}(\tau) = f_k(\tau)\hat{c}_{\mathbf{k}}(\tau_0) + f_k^*(\tau)\hat{c}^\dagger_{-\mathbf{k}}(\tau_0) \ , \tag{3.4}$$

where the mode function f_k satisfies the same equation of motion as the classical field χ, which in pure de Sitter space is given by

$$\chi''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right)\chi_{\mathbf{k}} = 0 \ . \tag{3.5}$$

The canonical conjugate field $\hat{\pi}$ can be expanded in a similar fashion [116],

$$\hat{\pi}_{\mathbf{k}}(\tau) = -i(g_k(\tau)\hat{c}_{\mathbf{k}}(\tau_0) - g_k^*(\tau)\hat{c}^\dagger_{-\mathbf{k}}(\tau_0)) \ . \tag{3.6}$$

In terms of the Bogoliubov coefficients defined in equation (2.54)

$$\begin{aligned} f_k(\tau) &= \frac{1}{\sqrt{2k}}\left(\alpha_k(\tau) + \beta^*(\tau)\right) \ , \\ g_k(\tau) &= \sqrt{\frac{k}{2}}\left(\alpha_k(\tau) - \beta^*(\tau)\right) \ . \end{aligned} \tag{3.7}$$

As can be seen from equation (2.59) with $\nu = 3/2$, the general solution to the equation of motion in de Sitter space, yields

$$f_k = \frac{\tilde{C}_1}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)e^{-ik\tau} + \frac{\tilde{C}_2}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right)e^{ik\tau} \tag{3.8}$$

and one can also verify that

$$g_k = \tilde{C}_1\sqrt{\frac{k}{2}}e^{-ik\tau} - \tilde{C}_2\sqrt{\frac{k}{2}}e^{ik\tau} \ . \tag{3.9}$$

In section 2.1.4, we showed how the standard normalization to the Bunch-Davies vacuum in the infinite past corresponds to setting $\tilde{C}_1 = 1$ and $\tilde{C}_2 = 0$. This is a natural choice in the infinite past, when the term proportional to $1/k\tau$ vanishes and the $\tilde{C}_1$-term just becomes the positive frequency mode.

Let us follow the argument of [109] and investigate the consequence of normalizing the modes to the vacuum at a finite time τ_0, when $1/k\tau_0$ is not vanishing. By comparing equation (3.7) with equation (3.8) and (3.9), we find the solution for the Bogoliubov coefficients

$$\begin{aligned} \alpha_k &= \frac{1}{2}\left[\tilde{C}_1\left(2 - \frac{i}{k\tau}\right)e^{-ik\tau} + \tilde{C}_2\frac{i}{k\tau}e^{ik\tau}\right] \\ \beta_k^* &= \frac{1}{2}\left[\tilde{C}_2\left(2 + \frac{i}{k\tau}\right)e^{ik\tau} - \tilde{C}_1\frac{i}{k\tau}e^{-ik\tau}\right] \end{aligned} \tag{3.10}$$

where the normalization constraint in equation (2.55), leads to

$$|\tilde{C}_1|^2 - |\tilde{C}_2|^2 = 1 \ . \tag{3.11}$$

Defining the mode to be in the vacuum at time τ_0 amounts to requiring $\beta_k^*(\tau_0) = 0$, such that the vacuum expectation value of the number operator vanishes at that point. One then finds

$$\tilde{C}_2(k) = \frac{ie^{-2ik\tau_0}}{2k\tau_0 + i}\tilde{C}_1(k) \ , \tag{3.12}$$

which implies

$$|\tilde{C}_1(k)|^2 = \frac{1}{1 - |A_k|^2} \ , \tag{3.13}$$

where

$$A_k = \frac{i}{2k\tau_0 + i} \ . \tag{3.14}$$

Using the definition of the power-spectrum in equation (2.66), one obtains [109]

$$\begin{aligned} \mathcal{P}_\phi(k) &= \frac{1}{a^2}\mathcal{P}_\chi(k) \sim \frac{1}{4\pi^2\tau^2}\left(|\tilde{C}_1(k)|^2 + |\tilde{C}_2(k)|^2 - \tilde{C}_1^*(k)\tilde{C}_2(k) - \tilde{C}_1(k)\tilde{C}_2^*(k)\right) \\ &= \left(\frac{H}{2\pi}\right)^2 \left(1 + |A_k|^2 - A_k e^{-2ik\tau_0} - A_k^* e^{2ik\tau_0}\right) \frac{1}{1 - |A_k|^2} \ , \end{aligned} \tag{3.15}$$

where $\phi = \chi/a$ is the inflaton field fluctuations. Note that in the limit $\tau_0 \to -\infty$ on obtains $A_k = 0$ and $\mathcal{P}_\phi(k) = (H/2\pi)^2$ in agreement with the results in section 2.1.4 for the pure de Sitter case with $\nu = 3/2$. However, following the line of reasoning in [109], for a given k we choose a finite τ_0 such that the physical momentum p corresponding to the mode k, has the fixed value Λ at τ_0,

$$p(\tau_0) = k/a(\tau_0) = \Lambda \ . \tag{3.16}$$

Here Λ corresponds to the scale of new physics. Also,

$$\tau_0 = -\frac{\Lambda}{Hk} \tag{3.17}$$

so τ_0 depends on k. It is natural to assume that $\Lambda/H >> 1$, and in this limit one finds as in [109]:

$$\mathcal{P}_\phi(k) = \left(\frac{H}{2\pi}\right)^2 \left(1 - \frac{H}{\Lambda}\sin\left(\frac{2\Lambda}{H}\right)\right) \ . \tag{3.18}$$

A generalization of this result to general backgrounds has been given in [115]. It is interesting to note the periodic modulation of the amplitude which is a genuine (but not unique) signature of normalizing the modes to the vacuum as they exit from the trans-Planckian region. It has been argued that if the scale of fundamental physics, Λ, is only 2 orders of magnitude larger than the Hubble parameter during inflation, then this effect will in principle be observable in the CMB spectrum [114].

It was first suggested in [94] that the modes should be in a state of minimum energy density as they cross the new physics boundary. It was also noted that the adiabatic vacuum for each mode, at this point, has a higher energy density.

One should keep in mind that the procedure above is not unique. In [115] it was shown that the procedure above corresponds to defining an initial state that minimize the following Hamiltonian as the mode exits from the trans-Planckian regime:

$$H^{(1)}(\tau) = \int d^3x \frac{1}{2}\left[\frac{\Pi^2}{a^2} + a^2(\partial_i\phi)^2\right] , \tag{3.19}$$

where $\phi = \chi/a$ and $\Pi = a\pi$. In [115] it was also shown that this is not the same state that minimize other Hamiltonians obtained by suitable canonical transformations of the original canonical fields and Hamiltonian. Though the different Hamiltonians describe the same dynamics, the initial state crucially depends on which Hamiltonian is minimized on the new physics hyper-surface. In fact, it was shown that defining the initial state by using the canonical related Hamiltonians

$$\begin{aligned} H^{(2)}(\tau) &= \int d^3x \frac{1}{2}\left[\pi^2 + 2\frac{a'}{a}\chi\pi + (\partial_i\chi)^2\right] , \\ H^{(3)}(\tau) &= \int d^3x \frac{1}{2}\left[\tilde{\pi}^2 - \frac{a''}{a}\chi^2 + (\partial_i\chi)^2\right] , \end{aligned} \tag{3.20}$$

where $\tilde{\pi} = \chi'$, both yields much smaller trans-Planckian effects.

The approach described above is of course a reasonable approach when one does not have an understanding of the trans-Planckian physics, but in addition to the ambiguities discussed, it is not even clear that the modes should be in the vacuum when they exit from the trans-Planckian regime. If the trans-Planckian dynamics is sufficient non-adiabatic they could exit in an excited state. To understand this, one needs an understanding of the trans-Planckian regime. In the next section, we will make an attempt in this direction.

3.2 The Minimum Length Uncertainty Principle

The Planck scale physics is not very well understood and various approaches has been suggested in the literature in order to take it into account.

Inspired by black-hole physics [117–119], Brandenberger and Martin used an *ad hoc* modified dispersion relation in order to check whether inflation is robust to changes in the quantum field theory at Planck scales [85, 87]. The *Unruh* and the *Corley-Jacobsen* dispersion relations, which were first suggested in the context of Hawking evaporation of black-holes, were adopted to the trans-Planckian problem of inflation:

$$\omega_{phys} = M_\Lambda \tanh^{1/\gamma}\left[\left(\frac{p}{M_\Lambda}\right)^\gamma\right] , \qquad \text{Unruh}$$

$$\omega_{phys}^2 = p^2 \left[1 + b_m \left(\frac{p}{M_\Lambda} \right)^{2m} \right] , \qquad \text{Corley} - \text{Jacobsen} . \tag{3.21}$$

Here p is the physical momentum and ω_{phys} is the physical frequency of a given mode. The scale of new physics is given by M_Λ and γ, m, b_m are adjustable parameters. The Corly-Jacobson dispersion relation has two quantitatively different behaviors depending on the sign of b_m, see also figure (3.1). Of these dispersion relations, only the Corley-Jacobsen with $b_m < 0$ turned out to give an observable signature in the CMB spectrum [108].

Although interesting, it is not clear how exactly these *ad hoc* dispersion relations should be related to some new physics. The most promising theory of physics at Planck scales is string theory. However, string theory is not well understood in de Sitter space-time and in Minskowski space it is Lorentz invariant unlike the theory obtained from the modified dispersion relations.

But even if a direct derivation of the trans-Planckian effects from a fundamental theory is not yet possible, there seems to be at least one robust characteristic feature, which is crucial for the understanding of the trans-Planckian problem. This is the existence of a minimum length scale[2].

That perturbative string theory behaves like there is a minimal length scale is a well known feature [121, 122] (see also [123, 124]). As one tries to probe distances shorter than the Planck length with increasing energies, the string gets excited and stretches [125]. The minimum length appears as if the usual uncertainty principle has to be replaced by a *minimum length uncertainty principle*:

$$\Delta x \geq \frac{1}{2} \left(\frac{1}{\Delta p} + l_s^2 \Delta p + \ldots \right) . \tag{3.22}$$

The minimum length uncertainty principle can be related to a modification of the standard space-momentum commutation relations [126, 127].

In this connection it is also interesting to note that Atick and Witten argued that "*outside distances of order $\sqrt{\alpha'}$, string theory is properly described in terms of strings; at shorter distances some more novel description is needed. This novel description involves a system with far fewer degrees of freedom per unit volume than one would ever have in relativistic field theory.*" [128]. Atick and Witten compare this with the ultraviolet divergences of classical physics, where the solution was a reduction of the degrees of freedom by postulating that the continuum of classical phase-space does not exist, and must be replaced by a system which on average has one degree of freedom per $2\pi\hbar$ of area. Of course that did not lead people to think that phase-space is a lattice. The actual cutoff in classical physics is more subtle, involving Heisenberg's $\mathbf{x}, \mathbf{p}$ commutation relation. It is then suggested that: "*A new version of Heisenberg's principle - some non-commutativity where it does not usually arise - may be the key to the thinning of the degrees of freedom that is needed to describe string theory correctly*" [128].

[2]For a review of how a minimum length appears in quantum gravity and string theory, see [120]

In [3], we followed some of these lines of thinking. As we will discuss below, we used the minimum length uncertainty principle as a guiding principle to incorporate a minimum length into quantum field theory, using a modification of Heisenberg's $\mathbf{x}, \mathbf{p}$ commutators. It does not only lead to a resolution of the trans-Planckian problem, but also to a reduction of the degrees of freedom similar to what was found by Atick and Witten [128].

The minimum length uncertainty principle in string theory is a dynamical effect rather than a fundamental feature. It arises due to the dynamics of the high-energy theory, for instance in the high-energy string scattering amplitudes which effectively makes it impossible to probe distances much shorter than $\sqrt{\alpha'}$ [122]. This dynamical effect can then be encoded in the structure of the effective field theory, by using the modified uncertainty principle (i.e. modified commutators) as a trick to obtain the effective field theory, consistent with the minimum length.

Suppose one could really derive an effective field theory description of some high-energy process from a fundamental theory, for example, by summing up to all orders the relevant terms of a perturbative expansion in powers of energy. Such an effective field theory would automatically incorporate the minimum length feature of (3.22) in its structure. But instead of adding up the high-energy corrections, that encode (3.22), we can absorb them into a modification of the $\mathbf{x}, \mathbf{p}$ commutators. This will leave the apparent structure of the field theory unchanged while changing instead the relation between space and momenta. In this way, the modified commutators are only a trick, without fundamental significance, to guess the form of the high energy corrections to a field theory consistent with the minimal length principle.

Now let us consider a general class of modified commutation relations, consistent with rotational invariance, that lead to a minimum length uncertainty[3] [126, 127],

$$[\mathbf{x}^i\,,\,\mathbf{p}^j] = i\,\delta^{ij} f(\mathbf{p}) + i\,g(\mathbf{p})\,\mathbf{p}^i\mathbf{p}^j\,, \qquad [\mathbf{x}^i\,,\,\mathbf{x}^j] = 0\,, \qquad [\mathbf{p}^i\,,\,\mathbf{p}^j] = 0\,. \tag{3.23}$$

The information about the modification is entirely contained in the function $f(p)$. The term with $g(p)$ is required by the Jacobi identities and is fully determined in terms of $f(p)$ (see, for example, [126]). The obvious restriction on $f(p)$ is that for small enough p it should reduce to 1, leading back to the standard commutators. It is understood that p is measured in units of some high-energy scale, say, the Planck mass, l_s^{-1}.

To understand how the modified commutator implies a minimum length, we use a useful construction in Minkowski space introduced in [126]: Introduce auxiliary variables ρ^i such that on functions $\phi(\rho)$,

$$\mathbf{x}^i\phi(\rho) = i\,\frac{\partial}{\partial\rho^i}\,\phi(\rho)\,. \tag{3.24}$$

[3] Below we follow the convention in [3] and let bold-face letters $\mathbf{x}^i, \mathbf{p}^i, \boldsymbol{\rho}^i, \cdots$ denote quantum mechanical operators corresponding to $x^i, p^i, \rho^i, \cdots$, while $p, \rho, \cdots$ denote the magnitudes of vectors p^i, ρ^i, *etc.*

The ρ^i are given in terms of the momenta p^i as,

$$\rho^i = \frac{p^i}{f(p)} \,. \tag{3.25}$$

Now, using a formal power series expansion for $f(p)$, it is easy to verify that (3.23) and (3.24) are equivalent. Further, let us assume that $f(p)$ is such that as p varies from 0 to ∞, ρ stays bounded between 0 and some maximum value ρ_{max}. Here, p and ρ denote the magnitudes of p^i and ρ^i, respectively.

One can now understand the essential consequence of (3.23) without getting into the details of the construction of the relevant function spaces. Equation (3.24) implies the commutator

$$[\,\mathbf{x}^i, \boldsymbol{\rho}^j\,] = i\,\delta^{ij}\,, \tag{3.26}$$

and hence, the associated uncertainty relation $\Delta x^i \geq 1/(2\Delta\rho^i)$. Since we assume ρ to be bounded by ρ_{max}, the maximum uncertainty in ρ^i is $2\rho_{max}$. Thus, there is an associated minimum length uncertainty,

$$(\Delta x)_{min} \sim \frac{1}{\rho_{max}} \,. \tag{3.27}$$

This is required to be of order one, in units of the l_s appearing in (3.22). The conditions on $f(p)$ are summarized below:

- As $p \to 0$, $f(p) \to 1$ and $\rho \to p$.
- $\rho = p/f(p)$ is bounded by 1 in units of l_s^{-1}, which we take to be the Planck mass.
- The most natural functions f satisfying the last condition are those for which ρ increases monotonically with p, approaching its maximum value ρ_{max} as $p \to \infty$. An example is
$$\rho = \tanh^{1/\gamma}(p^\gamma)\,. \tag{3.28}$$
which has the form as the Unruh dispersion relation in equation (3.21).

In formulating a theory based on the modified commutator, it may be tempting to speculate on identifying ρ as the physical momentum with a cut-off. However, a momentum cut-off is not natural from the point of view of fundamental physics. For example, in string theory loop diagrams are naturally regulated as a consequence of modular invariance, and not by a momentum cut-off. Moreover, if the modified commutator (3.23) is obtained by analyzing physical processes in some fundamental theory, then p will manifestly correspond to the physical momentum.

The construction above can easily be generalized to a flat RW space with metric

$$ds^2 = a^2(\tau)\left(-d\tau^2 + dy^i dy^j \delta_{ij}\right) \,. \tag{3.29}$$

As usual we denote the comoving momentum (wave number) conjugate to y^i by k_i. Physical distances at time τ are simply given by $a(\tau)\, y^i$. Also, the physical momentum undergoes a red-shift with time and, in the metric above, is given by $k_i/a(\tau)$. The minimum length uncertainty should be time independent and apply to the physical lengths. At the same time, the modified commutator should be consistent with the residual symmetries of the metric (3.29), in terms of comoving coordinates.

Only a subset of general coordinate transformations keeps the form of the RW metric in comoving coordinates unchanged. Besides rigid spatial rotations and translations, which are in common with flat space, this subset also contains constant rescalings of the coordinates that amount to rescaling a, keeping the form of the metric unchanged. Then, the analogue of the modified commutator (3.23) in the flat RW space, consistent with the residual covariance of the metric $\mathrm{g}_{\mu\nu}$ in the comoving coordinates, is

$$[\, \mathbf{y}^i \,,\, \mathbf{k}^j \,] = i\mathrm{g}^{ij}\, f(\mathbf{k}) + ig(\mathbf{k})\, \mathbf{k}^i\, \mathbf{k}^j \,, \tag{3.30}$$

where, $k^i = \mathrm{g}^{ij} k_j = a^{-2}\delta^{ij} k_j$. This should not lead to a fixed minimum length uncertainty in the comoving coordinates y^i, which would result in a rather large uncertainty in the proper distance ay^i at the present epoch. It is easy to see that it is actually the physical distance that has a fixed minimum uncertainty: In order to compare with the flat space case, let us introduce $n_i = k_i$, $n^i = \delta^{ij} n_j = a^2 k^i$, so that $k^2 = n^2/a^2$. In terms of this, the commutation relation takes the form

$$[a\mathbf{y}^i \,,\, \frac{\mathbf{n}^j}{a}] = i\delta^{ij}\, f(\frac{\mathbf{n}}{a}) + ig(\frac{\mathbf{n}}{a})\, \frac{\mathbf{n}^i}{a}\, \frac{\mathbf{n}^j}{a} \,. \tag{3.31}$$

This has the same structure as (3.23) and hence implies a minimum position uncertainty in the physical or proper distance ay^i as desired, with n^i/a as the physical momentum. More precisely, in analogy with the flat space case, one can introduce the "physical" auxiliary variables ρ_i as well as "comoving" ones, $\hat{\rho}_i$, given by

$$\rho_i = \frac{n_i/a}{f(n/a)} \,, \qquad \hat{\rho}_i = a\, \rho_i \,. \tag{3.32}$$

The commutator (3.31) now becomes

$$[a\mathbf{y}^i \,,\, \boldsymbol{\rho}^j] = i\, \delta^{ij} \,. \tag{3.33}$$

This leads to a minimum length in ay^i as $(a\Delta y)_{min} \sim \rho_{max}^{-1}$. Since ρ is a function of a single variable, this value is τ-independent. In terms of the "comoving" version of the auxiliary variable $\hat{\rho}_i = a\rho_i$, the commutator is $[\mathbf{y}^i \,,\, \hat{\boldsymbol{\rho}}_j] = i\, \delta^i_j$. Note that unlike the flat space case above, f and ρ_i have now become time dependent through the red-shift of the physical momentum. However, they still satisfy the restrictions described below equation (3.27).

A field theory based on the modified commutator relations will contain new effects that become important at high energies, which as explained above, are expected to reproduce an effective field theory that provides a description of high energy interactions in a fundamental theory. Especially we find a modification of the dispersion relation leading to a minimum bound on the wavelengths, and an increase in the phase volume occupied by a quantum state at high momenta leading to *trans-Planckian damping*. The effect of modified dispersion relations on the predictions of inflationary cosmology is, as already mentioned, well studied, but the trans-Planckian damping is a novel feature of the minimum length formalism [3]. But while the dispersion relations has previously been modified *ad hoc* in the literature, the minimum length principle provides a deeper understanding of the physics of the modifications and even some constraints.

3.3 Scalar Field with Modified Commutator

Let us consider the standard free scalar field theory in Robertson-Walker space-time,

$$S = -\frac{1}{2}\int d^4x\sqrt{-\mathrm{g}}\,\mathrm{g}^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = -\frac{1}{2}\int d\tau\, d^3y\, a^2\Big(\partial_\tau\phi\,\partial_\tau\phi - \sum_{i=1}^{3}\partial_{y^i}\phi\,\partial_{y^i}\phi\Big)\,. \quad (3.34)$$

The equation of motion in terms of $\chi(\tau, y) = a(\tau)\,\phi(\tau, y)$ is

$$\Big(\partial_\tau^2 - \sum_{i=1}^{3}\partial_{y^i}\partial_{y^i} - \frac{\partial_\tau^2 a}{a}\Big)\chi = 0\,. \quad (3.35)$$

The quantum mechanical commutators becomes relevant to the structure of free field dispersion relations. Since $\mathbf{y}^i$ and $\hat{\boldsymbol{\rho}}_i$ satisfy the standard commutation relation, we can write the Fourier transform

$$\phi(\tau, y) = a^3\int_{-l_s^{-1}}^{l_s^{-1}} d^3\rho\,\tilde{\phi}(\tau,\hat{\rho})\,e^{iy^i a\rho_i}\,. \quad (3.36)$$

One has to be careful since the Fourier transform is now time-dependent. Keeping in mind that this formalism is only a trick to incorporate a minimum length into the field theory and requiring that the theory should be consistent with the homogeneity of the background, it can be argued that the extra terms arising from the time-dependence of the Fourier transform should be dropped[4] [3]. Then in terms of the Fourier modes, and after expressing $\hat{\rho}$ in terms of n from (3.32), one obtains the action

$$S = -\frac{1}{2}\int d\tau\, d^3n\, J(\frac{n}{a})\, a^2\Big(\partial_\tau\tilde{\phi}^*\,\partial_\tau\tilde{\phi} + \frac{n^2}{f^2(n/a)}\,\tilde{\phi}^*\,\tilde{\phi}\Big)\,. \quad (3.37)$$

[4]There is nothing that tells us that the action in equation (3.34) is really the right starting point. In fact, the extra terms that we ignore, are a sign that equation (3.34) is not exactly right at high energies, and the formulation of the theory in Fourier space in equation (3.37) is more correct. See [3] for a more detailed discussion of this aspect.

We use the same notation $\tilde{\phi}$ for the field as a function of ρ and n since the difference is clear form the context. Here, $n^2 = \sum_{i=1}^{3} n_i n_i$ and $J(n/a)$ is the Jacobian for the change of variables determined by (3.32),

$$J(v) = \frac{\rho^2}{v^2}\frac{\partial \rho}{\partial v}\,, \qquad \left(v = \frac{n}{a}\right)\,. \tag{3.38}$$

The restrictions on ρ fix the asymptotic behaviour of J such that $J \to 1$ for $v \ll 1$, and $J \to 0$ for large physical momentum v. The equation of motion for $\tilde{\phi}(\tau, n)$ can be evaluated from (3.37) and is given by,

$$\left(\partial_\tau^2 \,+\, a^2\,\rho^2(\frac{n}{a}) \,-\, \frac{\partial_\tau^2(\sqrt{J}\,a)}{\sqrt{J}\,a}\right)\tilde{\chi}(\tau, n) = 0\,, \tag{3.39}$$

where $\tilde{\chi}(\tau, n) = a\,\sqrt{J}\,\tilde{\phi}(\tau, n)$.

The standard equation of motion in equation (3.5) corresponds to the small momentum limit of (3.39), where $a\rho \sim n$ and $J \sim 1$. The effects of Planck scale physics, as encoded in the modified commutation relations, result in the modification of the dispersion relation and in the appearance of J in the equation of motion. The J dependence is a novel feature which is responsible for the trans-Planckian damping which we will discuss in more details in the next section.

In the equation of motion (3.39), the free field dispersion relation $\omega_{phys}^{free} = n/a$, is modified to a non-linear, time dependent one,

$$\omega_{phys} = \rho \equiv \frac{n/a}{f(n/a)}\,. \tag{3.40}$$

This is a generalization of the modified dispersion relations considered in [85–87, 90–92, 94, 100, 101, 106, 108, 129–131]. However, the modification is no longer *ad hoc*, but is determined by the nature of the uncertainty principle (3.30), through the function f. As summarized in the paragraph following equation (3.27), this function satisfies certain restrictions. In terms of ω_{phys}^2, the restrictions translate to the following:

- The linear dispersion relation $\omega_{phys} = n/a$ emerges at small physical momenta n/a.
- $\omega_{phys}^2 \geq 0$ and ω_{phys} is always real.
- ω_{phys} is bounded by ρ_{max} which is fixed by the minimum length scale as $1/l_s$.

Thus, while there is no cut-off on the physical momentum n_i/a, there is always a cut-off on the effective frequency ω_{phys}. These restrictions exclude some of the dispersion relations that have appeared in the literature as being inconsistent with the minimum length uncertainty. Some frequently used dispersion relations are shown in figure (3.1).

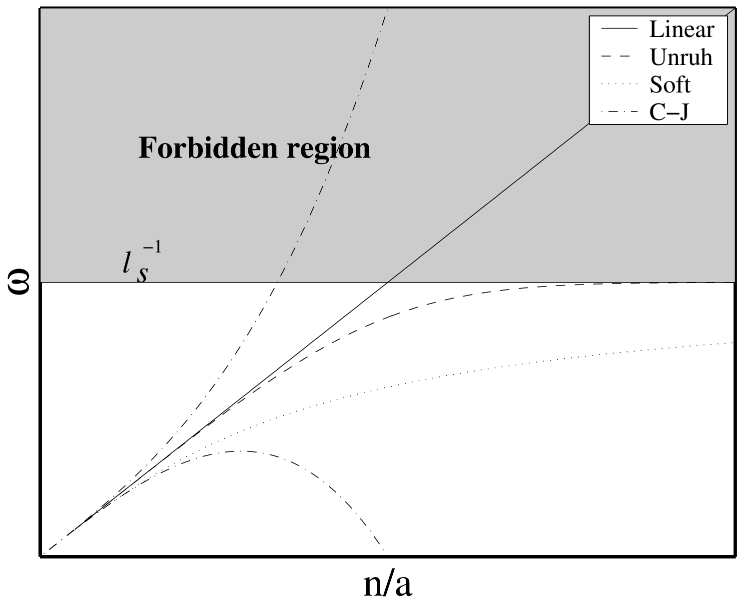

Figure 3.1: Different dispersion relations.

The scalar field theory with the modified dispersion relation can now be second quantized in terms of $\tilde{\chi}(\tau, n)$ in the usual fashion. The modification of the commutation relation in the first quantized theory does not affect the field commutators of the second quantized theory.

The most natural class of solutions to the restrictions on f consists of ρ increasing monotonically with n/a. This avoids the problem of associating multiple momenta with the same frequency. For example, let us consider the class of solutions in (3.28), leading to

$$\omega_{phys} = \tanh^{1/\gamma} \left(\frac{n}{a}\right)^{\gamma} . \tag{3.41}$$

For these dispersion relations, the equation of motion in equation (3.39) has been analyzed before in the special case of $J = 1$ [85]. The conclusion was that this class of trans-Planckian modifications does not modify the spectrum of cosmological perturbations calculated with the linear dispersion relation. We will discuss the case of $J \neq 1$ in the next section.

Another form of f appearing in the literature is [127]

$$f(n/a) = \frac{2(n^2/a^2)}{\sqrt{1 + 4(n^2/a^2)} - 1} . \tag{3.42}$$

One can numerically verify that the dispersion relation corresponding to this function has the same qualitative behaviour as the Unruh form (3.41), with an appropriate value of γ. In figure (3.1), this curve is labeled as "soft". Hence one can infer that for $J = 1$, it leaves the spectrum of cosmological perturbations unchanged. This conclusion is in disagreement with the result of [88] (and the follow-up analysis in [93, 98]) which uses the same form of f, but implements the modified commutator in a different manner. In their implementation the metric has off diagonal elements mixing time and space components, making their definition of comoving momentum problematic[5].

Next, we consider the generalized Corley-Jacobson type of dispersion relations [85, 118] (see also equation (3.21),

$$\rho^2 = \frac{n^2}{a^2}\left(1 + \sum_{q=1}^{m} b_q \,(\frac{n^2}{a^2})^q\right) . \tag{3.43}$$

Since ρ is not bounded, such dispersion relations cannot be associated with a modified uncertainty principle with a minimum length scale. However, one could regard these as low-momentum expansions of expressions that are bounded. Then ρ will remain bounded as long as the expansion is valid. Contrary to the previous two cases, the Corley-Jacobson dispersion relations need not be monotonic functions of n. For $J = 1$ the effect of these on the cosmological perturbation spectrum has been studied in [85, 87, 90, 92, 100, 101, 108] where it was found, as mentioned also in the beginning of the section, that they could give rise to deviations from the predictions of linear dispersion relation when $b_m < 0$.

Until now we have seen how trans-Planckian physics, encoded in the modified commutators, implies a bound on the free field dispersion relation. As discussed in the beginning of this chapter, in models where inflation lasts for a sufficiently long period of time, the CMBR anisotropies at the present epoch seem to have their origin in fluctuations whose wavelengths, in the beginning of inflation, were smaller than a Planck length. This assumes the validity the of standard red-shift formula for the physical wavelength, $\lambda_{phys} = 2\pi\, a(\tau)/n$, down to the Planck length and beyond. One may try to understand, at least heuristically, the time evolution of wavelengths, based on the modified commutators. The modified commutators lead to a wavelength that cannot decrease below the Planck length. In an adiabatic approximation, the wavelength of a plane wave is given by,

$$\lambda = \frac{2\pi}{\rho} = 2\pi\,\frac{a}{n}\, f(\frac{n}{a})\, . \tag{3.44}$$

We take ρ to be a monotonically increasing function. At late times (large a), the function f tends to identity and one recovers the usual red-shift formula. As we go backwards in time (decreasing a), λ tends to a constant 2π (in units of l_s). Near $a = 0$, a small wavelength band corresponds to a large momentum interval. This behaviour is shown in figure (3.2), for an f associated with the Unruh dispersion relation (3.41) with $\gamma = 1$. Thus equation (3.44) insures that at the end of the inflationary era one still finds fluctuations of all possible wavelengths within the

[5] Consider for example equation (4) in [88].

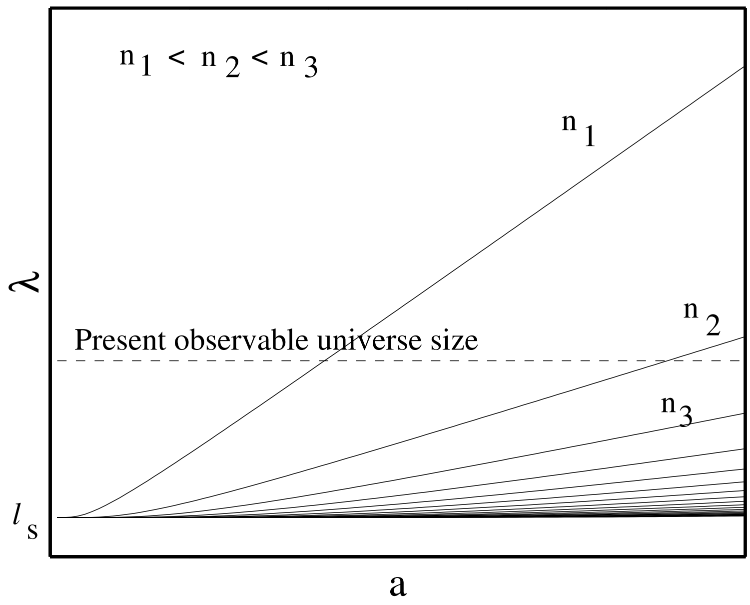

Figure 3.2: Time evolution of wavelength λ associated with momentum mode n.

observable universe without requiring sub-Planckian wavelengths in the early epochs. One also does not need to continuously generate Planck scale modes at later epochs. This argument, although reasonable, still remains heuristic since a plane wave with the above wavelength is not really a solution of the equation of motion (3.39), due to the time dependence of a. However, since the theory has an inbuilt minimum length uncertainty, this picture does capture the qualitative behaviour of length scales associated with the fluctuations.

3.4 Trans-Planckian Damping

In this section we will summarize some of the consequences of the the trans-Planckian damping due the appearance of J in the equation of motion (3.39). We shall focus on two interesting aspects. The trans-Planckian damping helps fixing the initial condition of the inflationary quantum fluctuations in a natural manner and it provides a reduction of the degrees of freedom at high energies.

Note that, when the dispersion relation is modified by hand, one misses the Jacobian J in the equations. This corresponds to $J = 1$ for which equation (3.39) reduces to

$$\left(\partial_\tau^2 \, + \, a^2 \, \rho^2(v) \, - \, \frac{\partial_\tau^2 a}{a} \right) \tilde{\chi}_{MB}(\tau, n) = 0 \, . \tag{3.45}$$

This is the equation that has been used before the appearance of [3], in order to study the cosmological implications of modified dispersion relations. It was analyzed in detail by Martin and Brandenberger [85] for the Unruh and Corley-Jacobson type dispersion relations. But now we turn to discuss the consequences of the appearance of J in the last term of equation (3.39). For simplicity we assume that the background metric is that of de Sitter space-time, such that the Hubble rate is constant.

In order to see the effect of J, we write the equation of motion (3.39) in the following more convenient form,

$$\left(\partial_\tau^2 + \omega_{total}^2\right) \tilde{\chi}(\tau, n) = 0\,, \qquad \omega_{total}^2 = a^2\,\rho^2(v) - \frac{\partial_\tau^2(a\sqrt{J})}{a\sqrt{J}}\,. \tag{3.46}$$

Here, $v = n/a$ is the magnitude of the physical momentum and $\tilde{\chi}(\tau, n)$ is the canonically normalized field given by

$$\tilde{\chi}(\tau, n) = a\sqrt{J}\,\tilde{\phi}(\tau, n)\,. \tag{3.47}$$

The second term in ω_{total}^2 is

$$\frac{\partial_\tau^2(a\sqrt{J})}{a\sqrt{J}} = \frac{v^2}{2}\left[\frac{\partial_v^2 J}{J} - \frac{1}{2}\left(\frac{\partial_v J}{J}\right)^2\right]\left(\frac{\partial_\tau a}{a}\right)^2 + \left[1 - \frac{v}{2}\frac{\partial_v J}{J}\right]\left(\frac{\partial_\tau^2 a}{a}\right)\,. \tag{3.48}$$

The behaviour of this term depends on the choice of the dispersion relation which determines J through (3.38). During de Sitter expansion with Hubble parameter H one has

$$\left(\frac{\partial_\eta a}{a}\right)^2 = H^2\,a^2\,, \qquad \left(\frac{\partial_\eta^2 a}{a}\right) = 2H^2\,a^2\,. \tag{3.49}$$

Therefore the J-dependent term is suppressed by the small number H^2.

The momentum range can be divided into four regions according to the behaviour of the solution:

- At late times, when $v << H$ (region III), the dispersion relation is linear and $\rho^2 = v^2$ (and the effect of J) can be neglected as compared to $\partial_\tau^2 a/a$. The relevant solution is $\tilde{\chi}^{(\mathrm{III})} = C_n^{(\mathrm{III})}\,a$.

- When $H << v << 1$ (region II), the dispersion relation is still linear, but now $\partial_\tau^2(a\sqrt{J})/(a\sqrt{J})$ can be neglected and one gets an oscillatory solution $\tilde{\chi}^{(\mathrm{II})} = C_{1n}^{(\mathrm{II})}\,e^{in\tau} + C_{2n}^{(\mathrm{II})}\,e^{-in\tau}$. If the dispersion relation had not been modified, then region II would have extended beyond the Planck scale, $v = 1$, all the way up to the beginning of inflation. This is the standard scenario that consequently suffers from the trans-Planckian problem.

- For non-linear dispersion relations, when $v > 1$ one enters region I. Then, for the Unruh dispersion relation, $\rho^2 \simeq 1$. Also, $J \simeq 1$ is still effectively constant and can still be ignored. In the de Sitter phase, $a^2 = (H\tau)^{-2}$ and the equation has an exact solution, $\tilde{\chi}^{(\mathrm{I})} = C_{1n}^{(\mathrm{I})}|\tau|^{\delta_1} + C_{2n}^{(\mathrm{I})}|\tau|^{\delta_2}$. The exponents are determined in terms of H, $\delta_{1,2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{9 - 4H^{-2}}$, and contain imaginary components. Therefore, the solution in region I has an oscillatory nature. In the case of Corley-Jacobson dispersion with $b_m < 0$, at early enough times, ρ^2 becomes negative and the solution is damped.

- For dispersion relations with non-vanishing J, when v is sufficiently large, one enters region 0. This happens as $\partial_v J$ becomes large. It appears as a damping term analogous to the Hubble damping term, yielding the effective frequency imaginary $\omega^2_{total} < 0$. For the Unruh dispersion relation, the solution can typically be given on the form $\tilde{\chi}^{(0)} = C^{(0)}_{1n} e^{-v^\gamma} + C^{(0)}_{2n} e^{v^\gamma}$.

The power spectrum of scalar fluctuations is given by $\mathcal{P} = n^3 |C^{(\mathrm{III})}_n|^2 / 2\pi^2$. For dispersion relations for which $J \simeq 1$ for all momenta, such that the trans-Planckian damping is absent, region 0 does not exist and the coefficient $C^{(\mathrm{III})}_n$ is determined in terms of $C^{(\mathrm{II})}_{1n,2n}$ and $C^{(\mathrm{I})}_{1n,2n}$ by matching the solution and its first derivative across the boundaries of regions III, II and I. The coefficients $C^{(\mathrm{I})}_{1n,2n}$ are in turn determined by the initial conditions at some (arbitrary) time τ_i in region I, which extends to the beginning of the inflationary era. Depending on the dispersion relation, the power spectrum may or may not acquire a non-trivial dependence on the initial state in the form of a modified dependence on n. As such, there is no natural choice for the initial state and, for oscillatory solutions, the best one can do is to pick up the local vacuum state at time τ_i. This is based on the implicit assumption that the modes are created in their ground state and that nothing drastic happens from the beginning of inflation until time τ_i. This of course also applies to the situations where the dispersion relation is modified by hand and the J factor does not appear at all. However, if $J \neq 1$ the initial state problem can be addressed.

As an example we consider the Unruh dispersion relation (3.41) for $\gamma = 3$. The variations of $\rho^2(v)$ and $a^{-2}\partial^2_\tau(a\sqrt{J})/(a\sqrt{J})$ with v are shown in figure (3.3). We have taken $H = 10^{-5}$ in Planck units. Below the Planck scale, $v < 1$, (regions III and II) the modifications due to the modified uncertainty relation are absent. So we will concentrate on the trans-Planckian regime. As v increases above the Planck scale in region I ($1 < v < 32$), ρ^2 soon approaches 1 and J (not shown in the figure) decreases rapidly. However, the J-dependent term grows large in spite of the suppression by H^2. At the boundary between regions I and 0, the two terms are equal and $\omega^2_{total} = 0$. Beyond this, we enter region 0 where the J-dependent term dominates and $\omega^2_{total} < 0$. The appearance of imaginary frequencies in this model is a consequence of the rapid expansion during which the presence of J and the red-shift of physical momenta induce the trans-Planckian damping term. In other words, in Planck scale processes at the present epoch, the frequencies are real and are given by $\rho^2 > 0$ alone. In this sense, the appearance of imaginary frequencies is not an unphysical feature of the theory. In particular, one avoids the multiple valuedness of momenta for a given energy which is a feature of non-monotonic dispersion relations

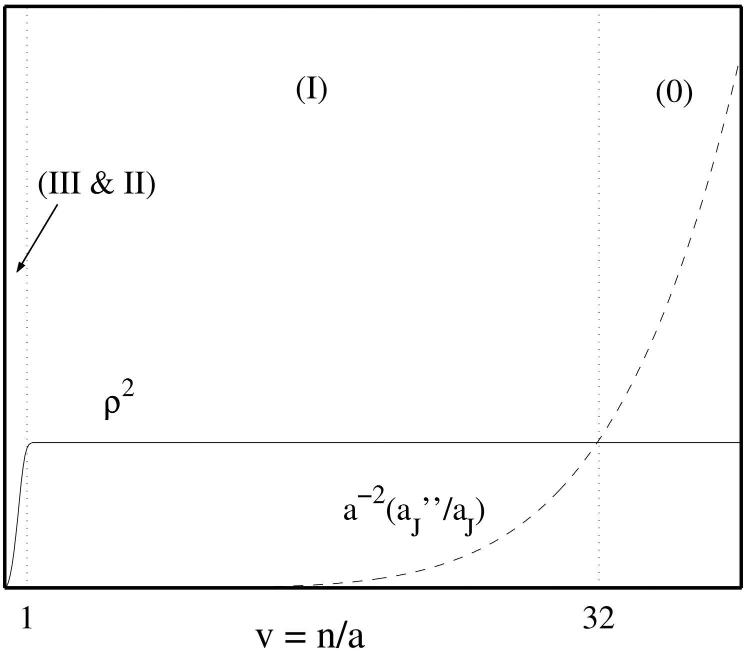

Figure 3.3: The evolution of ρ^2 and $a^{-2}\partial_\tau^2 a_J/a_J$ ($a_J \equiv a\sqrt{J}$) for the Unruh dispersion relation with $\gamma = 3$, as a function of momentum v.

considered in the literature[6]. Region 0 is a new region that has not appeared in earlier models.

Deep in region 0, we can ignore ρ^2 in (3.46). The equation then has an obvious solution,

$$\tilde{\chi}_1^{(0)}(\tau, n) \approx a\sqrt{J}\, C_{1n}^{(0)} \sim a\, e^{-v^3}\, C_{1n}^{(0)} \ , \tag{3.50}$$

where $C_{1n}^{(0)}$ is an n-dependent constant. This is similar to the solution in region III. But now the modes are frozen because, due to the presence of $\sqrt{J}$, they perceive the universe as expanding much faster than it really does. The other solution is

$$\tilde{\chi}_2^{(0)}(\tau, n) = a\sqrt{J}\int^\tau \frac{d\tau}{a^2 J} \approx \frac{a}{\sqrt{J}}\, C_{2n}^{(0)} \sim a\, e^{v^3}\, C_{2n}^{(0)} \ . \tag{3.51}$$

One can show explicitly that the dependence on n is given by $C_{1,2n}^{(0)} = n^{-3/2} C_{1,2}^{(0)}$, which by successively matching to the solution in region I, II and III yields a flat spectrum [3]. Comparing with (3.47) one finds that $\tilde{\phi}_1^{(0)} = C_{1n}^{(0)}$ and $\tilde{\phi}_2^{(0)} = C_{2n}^{(0)}/J$. The energy density in terms of the field $\tilde{\phi}$ is given by

$$\varepsilon(v)\, d^3v \sim \frac{1}{2}\, v\, J\left((\partial_\tau\tilde{\phi})^2 + a^2\rho^2\tilde{\phi}^2\right) d^3v \ . \tag{3.52}$$

[6] To compare with the earlier models, note that although we started with the Unruh dispersion relation, ω_{total} is reminiscent of the Corley-Jacobson case with $b_m < 0$.

As we go backward in time, the energy density for the first solution vanishes as $J(\equiv 1-\rho^6)$, while the one corresponding to the second solution blows up as $1/J$. Therefore, it is reasonable to impose $C_{2n}^{(0)}=0$ as a boundary condition. Thus, deep in the trans-Planckian region, the field $\tilde{\chi}_1^{(0)}$ starts from an extremely small value and increases with time as the momentum is red-shifted. At the boundary between regions 0 and I, ω_{total} turns real and the solution starts oscillating. Since $\tilde{\chi}^{(0)}$ is real and has only one branch, $\tilde{\chi}^{(\mathrm{I})}$ is also real and oscillates as a cosine. Note that the damping effect of the J-dependent term has fixed the "initial state" in region I in terms of the solution in region 0. For the Unruh dispersion relation, this is no longer the adiabatic vacuum used in the literature.

Since the solution is strongly damped in region 0 and goes rapidly to zero, it is possible to effectively fix the initial state at $\tau_i=-\infty$ even if inflation is not really eternal in the past. This removes any initial time dependence of the final power spectrum and thereby the arbitrariness in the initial state.

The behavior of ω_{total} with v could change appreciably from the one depicted in figure (3.3) for other dispersion relations. For the Unruh dispersion relations (3.41), the length of region I decreases with increasing γ. Another interesting case is the dispersion relation corresponding to (3.42), depicted as "soft" in figure (3.1). In this case, the J dependent term in ω_{total} always remains very small and as a result in region I, well above the Planck scale, $\omega_{total}^2\approx\rho^2\approx 1$, and there will be no region 0. Thus, the problem is very similar to the case of Unruh dispersion relation analyzed in [85].

Finally let us see how the J-factor reduces the degrees of freedom. For simplicity we restrict our considerations below to Minkowski space. In Minkowski space we can write the action in Fourier space simply as

$$S=\frac{1}{2}\int dtd^3pJ(p)\left(\partial_t\tilde{\phi}^*\partial_t\tilde{\phi}-\rho^2(p)\tilde{\phi}^*\tilde{\phi}\right)\ . \tag{3.53}$$

Thus, in flat space-time, the only effect of J is to modify the effective phase-space volume as it will not effect the equation of motion, simply because J is now time-independent. Reintroducing $\hbar$, $\int d^3pJ(p)/\hbar$ can be regarded as the continuum limit of a sum over momentum states, which indicates that now a momentum mode p occupies phase-space volume $\sim\hbar^3/J$ as opposed to $\sim\hbar^3$ in ordinary quantum mechanics. For a $\rho(p)$ consistent with the minimum length uncertainty, J decreases for large p and hence the phase-space volume occupied by a momentum mode p increases. Equivalently, the total number of degrees of freedom at high energy decreases. In string theory there is a similar reduction of degrees of freedom at high energies which manifest itself by a change in the free energy near the Hagedorn temperature T_{Hag} [128]. To compare, we will now consider the effect of J on the free energy at high temperatures. For simplicity, consider the Maxwell-Boltzmann statistics for which the free energy is given by[7]

$$-\frac{F(T)}{T} = \int \frac{d^3q d^3p}{(2\pi\hbar)^3} J(p) e^{-E(p)/T} \ . \tag{3.54}$$

We can now conveniently use a model independent feature of ρ. We know that ρ is only compatible with a minimum length if it is bounded. This implies

$$-\frac{F(T)}{T} = \frac{V}{(2\pi\hbar)^3} \int_0^{\Lambda} d\rho \rho^2 e^{-\rho/T} \ . \tag{3.55}$$

where we also used $E^2 \simeq \rho^2$ for a non interacting particle. By the usual method we substitute $x \equiv E/T = \rho/T$,

$$-\frac{F(T)}{T} = \frac{V}{(2\pi\hbar)^3} T^3 \int_0^{\Lambda/T} dx x^2 e^{-x} = \frac{-V}{(2\pi\hbar)^3} T^3 \left[e^{-\Lambda/T} \left(\frac{\Lambda^2}{T^2} + 2\frac{\Lambda}{T} + 2 \right) - 2 \right] \ . \tag{3.56}$$

Obviously when $T << \Lambda$ one recovers the usual result $-F(T) \sim VT^{d+1}$, where $d+1$ is the space-time dimension. But for $T >> \Lambda$ one finds

$$-\frac{F(T)}{T} = \frac{V}{3(2\pi\hbar)^3} \Lambda^3 \ . \tag{3.57}$$

This relation is somewhat similar to one obtained in a lattice field theory, although here we are just incorporating dynamical effects into the field theory without really thinking that we live on a discrete space. More interestingly this behaviour can be compared to that of an open string above the Hagedorn temperature T_{Hag}. It has been suggested long ago by Atick and Witten that above the Hagedorn temperature, $T > T_{Hag}$, the free energy of an open string behaves like $-F(T)/VT \sim const$ like in equation (3.57) [128].

[7] There are some interesting related discussions in [132, 133].

Chapter 4

Rescuing Pre-Big Bang Phenomenology

In section 2.2.6 we showed that the adiabatic density perturbations generated in the pre-big bang model has a spectral index $n = 4$ which is not comparable with observational constraints. However, in multifield models it is possible that the curvature perturbation on superhorizon scales is not constant unlike in the single field inflationary model discussed in section 2.1.4 (and 2.2.6). This has been known for some time [8, 134–139], and in particular it has been known that a late decaying non-relativistic field could convert isocurvature perturbations into adiabatic curvature perturbations[1] [134–136]. The isocurvature perturbation is typically given by a field that does not give any contribution to the total energy density perturbation or which only contributes to second order, such that the perturbations in the field does not contribute to the perturbations of the metric (isocurvature perturbations). However, if the energy density of the field later starts to dominate the total energy density, the field will at this point determine the metric perturbations. When it decays into radiation, it effectively becomes the source of all the perturbations in the cosmic fluid. This is similar to what happens in the ordinary reheating process in the single field inflationary model. Thus, if the initial curvature perturbation is small, the final adiabatic curvature perturbation will come only from the decaying field[2]. The first to apply this mechanism to explain the observed CMB anisotropies were Enqvist and Sloth [1], in order to show that the pre-big bang scenario in this way can explain the observed adiabatic density perturbations with spectral index $n \simeq 1$ [1, 2, 143, 144]. Lyth and Wands later named this mechanism[3], *the curvaton mechanism* [5].

[1]Pure cold dark matter and baryon isocurvature models are ruled out by data [140]. However, a correlated mixture of adiabatic and isocurvature perturbations are allowed to some level [141] (See also [142]).

[2]If it does not decay or does not decay completely, the perturbations in the field will be isothermal or entropy perturbations.

[3]The curvaton mechanism has lately been the source of a considerable amount of research. An incomplete list of work on the curvaton mechanism is [1, 2, 5, 143–174].

4.1 The Curvaton Mechanism

The most elegant exposition of the curvaton mechanism is perhaps given by Lyth and Wands [5] (see also [137, 145]). Here it is shown, without explicit solving the gravitational field equations, that a change in the curvature perturbation on uniform-density hypersurfaces is due to the presence of a non-adiabatic pressure perturbation.

More explicitly, let us consider the slicing (choice of constant time hypersurfaces) of uniform energy density, which is defined to be the slicing where there is no perturbation in energy density, $\delta\rho = 0$. Using the notation introduced in section 2.1.3, the perturbation of a scalar quantity f, transforms as $\tilde{\delta f} = \delta f - f'\xi^0$ under a gauge transformation. This means that $\delta\rho \to \delta\rho - \rho'\delta\tau$ for a transformation on constant time hypersurfaces $t \to t + \delta\tau$. In this way we deduce from

$$\delta\rho \to \delta\rho_{unif} = \delta\rho - \rho'\delta\tau = 0 \tag{4.1}$$

that

$$\delta\tau = \frac{\delta\rho}{\rho'} \tag{4.2}$$

is the time-displacement needed to go from a generic slicing to the slicing of uniform energy density where $\delta\rho_{unif} = 0$. From equation (2.28), we find that the curvature perturbation ψ transforms into

$$\psi \to \psi_{unif} = \psi + \mathcal{H}\delta\tau = \psi + \mathcal{H}\frac{\delta\rho}{\rho'} \ , \tag{4.3}$$

where $\mathcal{H} = a'/a$. Hence, the curvature perturbation on slices of uniform energy density is given by

$$\zeta \equiv \psi_{unif} = \psi + \mathcal{H}\frac{\delta\rho}{\rho'} = \psi + H\frac{\delta\rho}{\dot{\rho}} \ . \tag{4.4}$$

This quantity is gauge invariant by construction. Using the equation of energy conservation

$$\dot{\rho} + 3H(\rho + p) = 0 \ , \tag{4.5}$$

it can also be written as

$$\zeta = \psi - \frac{\delta\rho}{3(\rho + p)} \ . \tag{4.6}$$

The density perturbations can in general be split in two components: the adiabatic and the isocurvature component. If we let the cosmic fluid be composed of two components i, j, then the gauge invariant definition of the entropy fluctuation is

$$S_{ij} = 3H\left(\frac{\delta\rho_i}{\dot{\rho}_i} - \frac{\delta\rho_j}{\dot{\rho}_j}\right) = 3(\zeta_i - \zeta_j) \ . \tag{4.7}$$

Adiabatic fluctuations are characterized by having $S_{ij} = 0$, which means that all components fluctuate in the same way. Isocurvature fluctuations correspond instead to relative fluctuations between components.

If inflation is driven by only one field, the entropy perturbation vanishes on superhorizon scales and the perturbations are adiabatic, but if more fields are present during inflation then isocurvature perturbations will in general be generated.

Now, the pressure perturbation can be split into an adiabatic and an isothermal (non-adiabatic) part

$$\delta p = c_s^2 \delta\rho + \dot{p}\Gamma \tag{4.8}$$

where $c_s^2 \equiv \dot{p}/\dot{\rho}$. The non-adiabatic pressure perturbation is $\delta p_{nad} \equiv \dot{p}\Gamma$ and

$$\Gamma \equiv \frac{\delta p}{\dot{p}} - \frac{\delta\rho}{\dot{\rho}} \ . \tag{4.9}$$

Sometimes one also talks of the total entropy perturbation S, defined by

$$S \equiv -H\left(\frac{\delta p}{\dot{p}} - \frac{\delta\rho}{\dot{\rho}}\right) \ . \tag{4.10}$$

The entropy perturbation Γ (or S) is gauge invariant and represents the displacement between hypersurfaces of uniform pressure and energy density.

In [137], it was demonstrated that on large scales[4] the curvature perturbation, ζ, is constant if there is no non-adiabatic pressure perturbation. It can be shown that on sufficiently large scales the change in the curvature perturbation is in fact given by

$$\dot{\zeta} = -\frac{H}{\rho + p}\delta p_{nad}. \tag{4.11}$$

Generally it is reasonable to expect that during the inflationary epoch, there will be scalar fields present which gives a sub-dominant contribution to the total energy density. We will assume that one of these fields, *the curvaton* σ, is displaced from the minimum of its effectively quadratic potential at the end of inflation. As the Hubble rate drops below the mass of the field σ, it will start to oscillate in its potential, and its energy density will scale like the energy density of non-relativistic matter $\rho_\sigma \sim a^{-3}$. After the end of inflation, when the universe is radiation dominated, the energy density of the universe scales like $\rho_r \sim a^{-4}$, and it is possible that the curvaton field at some point will start to dominate the total energy density, before it decays into radiation.

To see how the curvaton field, σ, converts the isocurvature perturbation into a curvature perturbation, it is convenient to consider the curvature perturbation ζ_i corresponding to the different components of the energy density. They are defined on slices of uniform ρ_i, corresponding to the gauge-invariant definition (see equation (4.4))

$$\zeta_i \equiv \psi + H\left(\frac{\delta\rho_i}{\dot{\rho}_i}\right) \ . \tag{4.12}$$

[4] Defined as scales on which gradient terms can be neglected.

If one evaluates ζ_σ for the curvaton on unperturbed ($\psi = 0$) hypersurfaces when the curvaton starts to oscillate, one gets

$$\zeta_\sigma = \frac{1}{3}\frac{\delta\rho_\sigma}{\rho_\sigma} \ . \tag{4.13}$$

Following [145], we write the total curvature perturbation in the following form

$$\zeta = (1-f)\zeta_r + f\zeta_\sigma \ , \tag{4.14}$$

where the relative contribution of the curvaton to the total curvature is given by

$$f = \frac{3\rho_\sigma}{4\rho_r + 3\rho_\sigma} \ . \tag{4.15}$$

From equation (4.14) and (4.15), one sees that the curvaton perturbation, ζ_σ, is initially of the isocurvature type, since $\rho_\sigma/\rho_r \to 0$ in the early-time limit, which implies $f \to 0$.

Until the curvaton decays, the matter (curvaton) and radiation can be treated as two separate fluids satisfying each their own energy conservation equation

$$\dot{\rho}_i = -3H\left(\rho_i + p_i\right) \ . \tag{4.16}$$

In [137], it was shown that ζ_i remains constant for adiabatic perturbations in any fluid whose energy-momentum is locally conserved. Thus, before the curvaton starts to decay, each ζ_i is constant on super-horizon scales. The evolution in the total curvature perturbation ζ on these scales are then due only to the change in f, and from equation (4.14) one obtains

$$\dot{\zeta} = \dot{f}\left(\zeta_\sigma - \zeta_r\right) = Hf(1-f)\left(\zeta_\sigma - \zeta_r\right) \ . \tag{4.17}$$

From this expression one can see how the curvature perturbation changes as the curvaton density grows with respect to the radiation, $\dot{f} > 0$. One can also easily show that this is consistent with the expression in equation (4.11).

In the curvaton scenario, the initial curvature perturbation in the radiation is typically very small $\zeta_r \approx 0$. Since ζ_r and ζ_σ are constant until the curvaton decays[5], it can be seen from equation (4.14) that the total curvature perturbation at the time of the decay is

$$\zeta \simeq f_{dec}\zeta_\sigma = \frac{1}{3} f_{dec} \left.\frac{\delta\rho_\sigma}{\rho_\sigma}\right|_{dec} \ . \tag{4.18}$$

All evaluated at the time of the decay, as indicated by the subscript *dec*. Thus, if the curvaton dominates completely when it decays, we find

$$\zeta \simeq \left.\frac{\delta\rho_\sigma}{\rho_\sigma}\right|_{dec} \ , \tag{4.19}$$

as it also was assumed in[6] [1].

[5]Here we assume for simplicity that the curvaton decays instantaneously.
[6]On superhorizon scales $\zeta \simeq \mathcal{R}$.

4.2 Decaying Axion as the Origin of Initial Density Perturbations

In this section we shall show that the Kalb-Ramond axion or one of the other axion fields that appears in the effective action of string theory, can act as a curvaton field and lead to the observed adiabatic CMB perturbations. This was originally done in [1] and later investigated in more details in [2, 143, 144].

During the pre-big bang era, the curvaton (axion) field is constant and does not contribute to the background. However, the canonically normalized quantum fluctuation of the curvaton is amplified. In this epoch the curvaton field contributes to the total energy density like a cosmological constant. After the universe has entered the first radiation dominated era at conformal time $\tau = \tau_r$ and when the Hubble rate subsequently reaches $H_{osc} \simeq m$, the curvaton starts to oscillate in its potential. From this point on, the energy density of the curvaton will behave like matter and falls off like $\rho_\sigma \propto a^{-3}$. This implies that the curvaton will soon dominate energy density and at this point the Hubble rate is $H_{dom} \sim \Omega m \sigma_r^4$. Here Ω is a conformal factor and σ_r is the value of the curvaton field at the beginning of the first radiation dominated epoch. Finally the curvaton decays into radiation, and the curvaton fluctuations induces the initial adiabatic density perturbations in the cosmic fluid leading to the CMB anisotropies, as described in the previous section.

From requiring that the curvaton only dominates energy density after it has started to oscillate in its potential[7] we must require $\sigma_r/m_p < 1/\Omega$. Since the initial adiabatic density perturbation has a very blue spectrum ($n = 4$), we can also safely assume that the curvature perturbations at the beginning of the first radiation dominated era are negligible.

As discussed in section 2.2 the pre-big bang scenario is based on a four-dimensional effective tree-level string theory action, which in Einstein frame can be written as[8]

$$S_{eff} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[\mathcal{R} + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}(\nabla\beta)^2 + \mathcal{L}_{matter} \right] , \tag{4.20}$$

where $\kappa^2 = 8\pi/M_s^2$ and M_s is the string mass. Here ϕ and β are the universal four-dimensional moduli fields. In section 2.2.1, we discussed how this action can be derived from 10-dimensional string theory compactified on some 6-dimensional compact internal manifold. The matter Lagrangian, $\mathcal{L}_{matter}$, is composed of gauge fields and axions, which we assume do not to contribute to the background.

The solution in Einstein frame to the equations of motion for the background fields can in the spatially flat case be parameterized as [8] (see also equations (2.141),

[7] If this is not the case, we are not really considering a curvaton model, since the curvaton will then lead to some additional inflation [143], and would be more like a two field inflationary model.

[8] Below we use a slightly different convention for the moduli field β, as compared to equation (2.119). This is in order to be consistent with the notation in [1, 2]. The two definition are equivalent up to a trivial rescaling.

(2.120), (2.122))

$$a = a_r \left|\frac{\tau}{\tau_r}\right|^{1/2} \quad , \quad e^{\phi} = e^{\phi_r} \left|\frac{\tau}{\tau_r}\right|^{\sqrt{3}\cos\xi} \quad , \quad e^{\beta} = e^{\beta_r} \left|\frac{\tau}{\tau_r}\right|^{\sqrt{3}\sin\xi} . \tag{4.21}$$

As mentioned above, in the pre-big bang scenario we identify the curvaton with the axion field. It is convenient to split out the part of the matter Lagrangian that contains the axion field $\mathcal{L}_{matter} = \mathcal{L}_{gauge} + \mathcal{L}_{axion}$, where we will parameterize the axion Lagrangian in the following way

$$\int d^4x\sqrt{-g}\mathcal{L}_{axion} = \frac{1}{2}\int d\eta d^3x a^4 e^{l\phi} e^{m\beta} \left[(\nabla\sigma)^2 + m^2\sigma^2\right] . \tag{4.22}$$

The non-minimal coupling of the axion is parametrized by l, m. From the action in equation (2.106) one can read of the values of l, m for the Kalb-Ramond axion (the Poincaré dual of the NS-NS two form) in the string frame. In Einstein frame they would take the values $l = 2$, $m = 0$. However, in the case of a more general compactification, there will typically be many axion-like scalar fields present with different non-minimal couplings and thus l, m can in principle take a wide range of positive or negative value of order one.

We are interested in the inhomogeneous fluctuations of the curvaton (axion) field around a homogeneous background field $\sigma = \langle\sigma\rangle + \delta\sigma$. In the following σ denotes only the background value of the curvaton field. It is useful to define a pump field $S = a\exp(l\phi/2 + m\beta/2)$. Then the canonically normalized perturbation field is $\psi_k = S\delta\sigma(k)$ and the perturbation equation can be written as

$$\psi_k'' + (k^2 - \frac{S''}{S} + a^2m^2)\psi_k = 0 . \tag{4.23}$$

In the pre-big bang era we can ignore the mass term in the perturbation equation and the solution to the perturbation equation normalized to a vacuum fluctuation in the infinite past is (see also the analogous discussion in section 2.1.4)

$$\psi_k = \frac{\sqrt{\pi}}{2}\sqrt{\tau}H_\nu^{(2)}\left(|k\tau|\right) \quad , \qquad \nu = |\lambda - 1/2| \tag{4.24}$$

with $\lambda(\lambda-1)/\tau^2 = S''/S$. Note that ν, which determines the spectral index of the perturbations like in equation (2.68), now depends on the pre-big bang background dynamics through the non-trivial pump field $S = a\exp(l\phi/2+m\beta/2)$, so in principle it can take any value depending on the values of l, m and ξ.

Since the perturbation equation (4.23) is the same as the background equation of motion on scales where the gradient term can be neglected, one can argue that the ratio $\delta\sigma/\sigma$ stays approximately constant on superhorizon scales. One can explicitly verify this, if one calculates the amplitude of the curvaton perturbation outside the horizon in the post-big bang regime by matching its solution to the spectrum obtained at the onset of the first radiation dominated epoch as given in equation (4.24) [2].

Matching the solution of the perturbation equation in the radiation dominated era to the solution in equation (4.24) one then finds in the limit $\tau < \tau_{osc}$

$$\delta\sigma(k) = \delta\sigma_r(k) \equiv C(\nu) a_r^{-3/2} H_r^{-1/2} \left(\frac{k}{k_r}\right)^{-\mu} , \qquad \tau < \tau_{osc} \tag{4.25}$$

with

$$C(\nu) = \sqrt{2}\frac{\Gamma(\mu)2^{\mu}}{\Gamma(3/2)2^{3/2}} . \tag{4.26}$$

Thus, before the curvaton starts to oscillate in its potential (at $\tau = \tau_{osc}$) σ as well as $\delta\sigma(k)$ are constant as one would expect.

One can then follow the full analytical solution[9] to the epoch τ_{osc} where the curvaton starts to oscillate in its potential. In the limit $\tau > \tau_{osc}$ one obtains in this way [2]

$$\delta\sigma(k) = \frac{\tilde{A}}{a\sqrt{am}} \left(\frac{H_r}{m}\right)^{1/4} \left(\frac{k}{k_r}\right)^{-\nu} , \qquad \tau > \tau_{osc} \tag{4.27}$$

where

$$\tilde{A} = 2^{-3/4}\Omega^{-1}\sqrt{2}C(\mu)\sqrt{\Gamma(1/4)/\Gamma(3/4)} . \tag{4.28}$$

and $\Omega = \exp(l\phi_r/2 + mb_r/2)$.

It is easy to see that

$$\delta\sigma(k) = \left(\frac{a_{osc}}{a}\right)^{3/2} \delta\sigma_{osc}(k) , \qquad \tau > \tau_{osc} \tag{4.29}$$

where $\delta\sigma_{osc}$ is the curvaton amplitude at $\tau = \tau_{osc}$. Using the fact that the universe is radiation dominated until the curvaton starts to oscillate in its potential, one may use $m = H_{osc} = H_r(a_r/a_{osc})^2$ and obtain

$$\delta\sigma_{osc}(k) = \frac{\tilde{A}}{C(\nu)} \left(\frac{a_r}{a_{osc}}\right)^{3/2} \left(\frac{H_r}{m}\right)^{3/4} \delta\sigma_r(k) = \frac{\tilde{A}}{C(\nu)}\delta\sigma_r(k) . \tag{4.30}$$

In this way one arrives at [2]

$$\delta\sigma(k) = \left(\frac{a_{osc}}{a}\right)^{3/2} \frac{\tilde{A}}{C(\nu)}\delta\sigma_r(k) , \qquad \tau > \tau_{osc} . \tag{4.31}$$

As expected, the curvaton field fluctuations are constant until the curvaton starts to oscillate in the quadratic potential and the field fluctuations begin to fall off as $a^{-3/2}$, just like the background field σ.

[9] The full analytical solution and this discussion thereof is given in [2].

There are now two possible cases. The case with a non-vanishing background value of the curvaton field, and the case of vanishing background field. Only the first one is relevant to us[10]. In this case one can write

$$\rho_\sigma = \frac{1}{2}\Omega^2 m^2 \sigma^2 \qquad \delta\rho_\sigma(k) = m^2\Omega^2\sigma\delta\sigma(k) \ , \tag{4.32}$$

where $\Omega = \exp(l\phi/2 + m\beta/2)$. The curvaton density perturbation becomes

$$\delta_k \equiv \frac{\delta\rho_\sigma(k)}{\langle\rho_\sigma\rangle} = 2\frac{\delta\sigma(k)}{\sigma} \ . \tag{4.33}$$

So after the axion has started to oscillate in its potential one gets

$$\frac{\delta\sigma(k)}{\sigma} = \frac{\tilde{A}}{C(\nu)}\frac{\delta\sigma_r(k)}{\sigma_r} \tag{4.34}$$

using that $\sigma_{osc} = \sigma_r$ and that after the curvaton has started to oscillate in the potential its amplitude falls off like $a^{-3/2}$. It is interesting to note that even if the curvaton field fluctuations depend on Ω, the fluctuations in the curvaton energy density itself does not. This peculiarity has also been noted earlier, see for instance [175]. This also implies that the relation in equation (4.34) implicitly depends on Ω through $\tilde{A}$.

Finally

$$\frac{k^{3/2}\delta\sigma(k)}{\sigma} = k^{3/2}\frac{\tilde{A}}{C(\mu)}\frac{\delta\sigma_r(k)}{\sigma_r} = \tilde{A}\frac{H_r}{\sigma_r}\left(\frac{k}{k_r}\right)^{3/2-\nu} \quad , \qquad \tau > \tau_{osc} \tag{4.35}$$

which was obtained for the pre-big bang curvaton model in [1, 143] and for inflation in [5] up to the numerical factor $\tilde{A}$.

If the curvaton starts to dominate energy density only after is has started to oscillate in the potential we must require $\Omega\sigma_r < 1$ in Planck units. Otherwise a short era of inflation will arise [143]. Normalizing to the COBE observations, one sees from equation (4.18) that one must require

$$10^{-5} \simeq \mathcal{P}_\zeta^{1/2}(k_0) \simeq \frac{k_0^{3/2}\delta\rho_{\sigma_r}(k_0)}{\rho_{\sigma_r}} \tag{4.36}$$

which implies that a flat spectrum of CMB perturbations is only possible if [5]

$$H_r < 10^{-5} m_p \ . \tag{4.37}$$

This is more or less the same as the constrain on the Hubble scale during slow-roll inflation obtained in [176] from COBE observations. However for pre-big bang

[10] A vanishing background field leads to non-Gaussian perturbations, which are ruled out by observations [5, 145].

scenario the situation is more interesting. In the pre-big bang scenario we expect $M_s \simeq H_r$, hence, a flat spectrum would require

$$M_s < 10^{-5} m_p \tag{4.38}$$

as observed in [1, 143]. This is a very low value for the string scale. It looks like the amplitude of the curvaton field fluctuations is too large to be in agreement with the standard string scale and the COBE normalization. However, there are several ways how one can circumvent this problem. If the curvaton is to be identified with an axion with a periodic potential [11], the periodic potential can damp the density fluctuations to the right level [1]. For a quadratic curvaton potential it was suggested in [143] by Bozza, Gasperini, Giovannini, and Veneziano that a kink in the spectrum might also do the job of lowering the curvaton field fluctuations to the right level[12].

There also exists another interesting possibility; the curvaton might decay before it dominates the energy density[13]. The curvature perturbation is then given by equation (4.18), which yields

$$\zeta = \frac{\rho_\sigma}{4\rho_r + 3\rho_\sigma} \frac{\delta\rho_\sigma}{\rho_\sigma} \ . \tag{4.39}$$

If $\rho_r >> \rho_\sigma$ at the decay, we get from equation (4.39) and (4.34)

$$\zeta_{dec} \simeq \frac{1}{2} \left(\frac{\rho_\sigma}{\rho_r} \right) \bigg|_{dec} \frac{\delta\sigma_r(k)}{\sigma_r} \tag{4.40}$$

Thus, if the pre-big bang curvaton field decays when it only contributes $r = 10^{-3}$ of the total energy density, then by means of equation (4.40) we would get $M_s = 10^{-2} m_p$. This is within the current constrains on non-Gaussianity from COBE data, which yield $r > 6 \times 10^{-4}$ (see [145] for details on the non-Gaussian aspects of the curvature perturbations). In this case the curvaton can not be the axion suggested in [1], but must have a stronger interaction with the photons than what an axion can offer. This possibility is exciting since $r \simeq 10^{-3}$ can already be ruled out or be confirmed by the MAP and PLANCK satellites. Note, that it has been shown by Lyth, Ungarelli and Wands [145] that if $r << 1$ then CDM cannot be created before the curvaton decays or by the curvaton decay itself because large correlated or anti-correlated isocurvature perturbations are produced.

In the next two subsection we will see how respectively an intermediate string phase or a periodic axion potential might also damp the curvaton field fluctuations to the right level in order to be in agreement with a high string scale and the COBE normalization.

[11] One might fear that a periodic potential will lead to formation of topological defects. However, if the potential has only one degenerate minimum i.e. if the anomaly factor is one ($N = 1$), then the topological defects decay and do not pose a cosmological problem [177].

[12] In [143, 144] Bozza, Gasperini, Giovannini, and Veneziano solved the field perturbation equations in order to demonstrate explicitly the details of the curvaton mechanism in the pre-big bang scenario. They also did a more extensive investigation of the parameter space to obtain more exact predictions.

[13] This was first suggested by Lyth and Wands [5].

4.3 The Effect of the Intermediate String Phase

From section 2.2.4 it is clear that a non-perturbative stringy cosmological era is most likely needed in order to accommodate the gracefull exit. Thus it is reasonable to assume that there is an intermediate string phase $\tau_s < \tau < \tau_r$ during which the curvaton field fluctuations are frozen on super horizon scales. Let us parameterize our ignorance about the string phase in a generic fashion. We let $z_s = a_r/a_s$ denote the ratio of the scale factor at the end of the string phase and at the beginning of the string phase. Likewise, we denote by $g_r = \exp(\phi(\tau_r)/2)$ and $g_s = \exp(\phi(\tau_s)/2)$ the string coupling constant at respectively the end and the beginning of the string phase. In order to have enough inflation in the pre-big bang era, we must assume $1 < z_s < 10^{20}$ [178]. It is also natural to assume $g_r/g_s > 1$.

Since it is $\delta\sigma(k)$ which is constant on superhorizon scales and not the canonical normalized field ψ_k, we should match $\delta\sigma(k)$ at τ_s and τ_r. The matching implies that the curvaton fluctuation spectrum is multiplied by an additional factor of [2]

$$\frac{S_r}{S_s}\left(\frac{\tau_s}{\tau_r}\right)^{-\mu+1/2} = z_s^{-\mu+3/2}\left(\frac{H_r}{H_s}\right)^{-\mu+1/2}\left(\frac{g_r}{g_s}\right)^{l}\left(\frac{b_r}{b_s}\right)^{m} \tag{4.41}$$

where we defined $b(\tau) \equiv \exp(\beta(\tau)/2)$. When normalizing to the COBE observations, this leads to a non trivial dependence on l, m.

As an example of a model in which the parameters l, m are non-trivial, one can, as in [2], consider a non-linear sigma model in Einstein gravity where the scalar fields parameterize a $SL(3,\mathcal{R})/SO(3)$ coset [179]. In this model, the lowest order action in 4-D Einstein frame reads [8]

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}(\nabla\beta)^2 \right.\\ &\left. -\frac{1}{2}e^{\sqrt{3}b+\phi}\left(\nabla\sigma_3\right)^2 - \frac{1}{2}e^{-\sqrt{3}b+\phi}\left(\nabla\sigma_2\right)^2 - \frac{1}{2}e^{2\phi}\left(\nabla\sigma_1 - \sigma_3\nabla\sigma_2\right)^2\right], \end{aligned} \tag{4.42}$$

where $\kappa^2 = 8\pi/m_p^2$, ϕ is the 4-D dilaton, β is the moduli, σ_1 is the NS-NS (Kalb-Ramond) axion and σ_2, σ_3 are RR axions. This model has been discussed in great detail also in [8, 175, 179–181]. The action contains no potential terms, but one expects them to be generated non-perturbatively. The perturbations of the axion with the smallest tilt will dominate the energy density perturbation spectra. It has been demonstrated that there are regions of parameter space where either of the axions can be the dominating one and with a flat scale invariant spectrum [175]. The conformal factors for the three axion fields are [175]

$$\Omega_i^2 = \begin{cases} e^{2\phi} & for \quad \sigma_1 \\ e^{\phi-\sqrt{3}\beta} & for \quad \sigma_2 \\ e^{\phi+\sqrt{3}\beta} & for \quad \sigma_3 \end{cases} \tag{4.43}$$

From the definition of the conformal factor just below equation (4.32), we find for the third axion field $l = 1$, $m = \sqrt{3}$. For a flat spectrum with $\nu = 3/2$, we then obtain from equation (4.36)

$$10^{-5} > \frac{g_r}{g_s}\frac{H_s}{H_r}\left(\frac{b_r}{b_s}\right)^{\sqrt{3}}\frac{M_s}{m_p} \tag{4.44}$$

which, depending of the detailed dynamics of the string phase, can relax the bound in equation (4.38) if for instance[14] $b_s/b_r \sim g_r/g_s >> 1$, $H_s/H_r \lesssim 1$. But if we like in [182] take the intermediate string phase to be a period of constant curvature, frozen internal dimensions and linearly growing dilaton in the string frame, then in the Einstein frame the curvature scale will not be constant but vary like the string coupling $H_s/H_r \approx g_s/g_r$, so even a very long string phase of this kind will not improve on the bound in equation (4.38) for σ_3 and for the Kalb-Ramond axion σ_1 with $l = 2$ it gets worse. A dual dilaton phase (also considered in [182]) with frozen internal dimensions and decreasing curvature scale in the string frame might even be problematic since it would change the bound in equation (4.38) in the wrong direction. However, it is clear that an intermediate string phase of accelerated expansion (accelerated contraction in the Einstein frame) and increasing curvature scale (as considered in [183]) only have to be very short in order to relax the bound in equation (4.38), with the curvature scale growing only two orders of magnitude. If the internal dimensions are contracting, we can obtain a high string scale even with a constant curvature scale during the intermediate string phase (in string frame) with σ_3. If the internal dimensions contract fast enough the σ_3 field might even work as a curvaton if we have a decelerating dual dilaton phase.

The discussion above is of course just an example. Generally we expect as many as 15 different axion fields in a $SL(6,\mathcal{R})$ invariant model, which is the maximal invariance for toroidal compactification from 10 to 4 dimensions, all with different conformal factors [184]. Note that in the simplest case of sudden transition without an intermediate string phase, all the different axion fields have the same amplitude for modes crossing the Hubble scale at the start of the post-big bang era $k \approx k_r$. On larger scales the amplitudes are fixed by the different spectral tilts and the field with the smallest tilt yields the largest density perturbations. The increased symmetry group will enlarge the part of parameter space where the spectral tilt of the dominating axion is close to flat and where l, m has non trivial values. In the maximal case, the 14 other axions will have a more blue spectrum and only contribute fractionally to the density perturbations. It might be possible that some of the fields are stable and can act as cold dark matter, leading also to an isocurvature or entropy component in the density fluctuations. However, such scenarios depends on a more detailed understanding of how the non-perturbative potentials are generated.

In this light it might be possible to obtain a flat spectrum in the pre-big bang

[14] It is assumed that the extra dimensions are contracting in the pre-big bang phase as well as in the intermediate string phase such that $b_s >> b_r$.

curvaton scenario with the standard string scale, but a more detailed understanding especially of the nature of the graceful exit is still needed.

4.4 Periodic Potential

In the discussion above we implicitly assumed that axion perturbation spectrum is flat. If the spectrum is blue the bounds improve. The ratio between the astrophysical scales k_a^{-1} and the smallest scale k_r^{-1}, set by the string scale, is $k_a/k_r \sim 10^{-30}$. Now, if the spectral tilt is $3/2 - \nu \simeq 0.13$, we see from equation (4.35) and (4.36) that the bound in equation (4.38) becomes trivial $M_s \lesssim 0.1 m_p$. On the other hand, if the spectrum is red it will diverge on large scales (small k), and the bound in equation (4.38) will become worse. However, a periodic potential for the axion will damp the fluctuations on large scales, avoiding eventual divergences of the large scale axion field fluctuations [1, 185–188].

It is possible that in the non-perturbative regime or at some early point during the post-big bang era the axion acquires a periodic potential due to instanton effects [185, 189]. This happens at some temperature T_Γ whence the axion mass starts to build up. As soon as the axion mass is of the order of the Hubble rate $m \simeq H$, the axion field starts oscillating in the potential. Given a periodic potential of the type

$$V(\psi) = \frac{1}{2} V_0 \left(1 - \cos\left(\frac{\psi}{\psi_0}\right)\right) , \tag{4.45}$$

where $\psi \equiv \langle\sigma\rangle + \delta\sigma$ and $\delta\psi(k) \equiv k^{3/2}\delta\sigma(k)$, and assuming that the axion field is Gaussian distributed with a zero average and a variance $\langle\psi^2(\vec{x})\rangle$, the average density is given by

$$\rho_\psi = \langle V(\psi(\vec{r}))\rangle \approx \frac{1}{2} V_0 \left(1 - \exp\left[-\frac{1}{2}\frac{\langle\psi^2(\vec{x})\rangle}{\psi_0^2}\right]\right) . \tag{4.46}$$

The periodicity of the potential will damp the fluctuations which are larger than the period of the potential. For a positively tilted (blue) spectrum, the damping vanishes and the potential effectively reduces to a quadratic potential. In this section we will therefore only consider the case of a negative tilted (red) axion spectrum ($\gamma \equiv 3/2 - \nu < 0$).

The relative density fluctuations of the axion field as the potential is turned on is for a negative tilted spectrum given by [1, 185]

$$\frac{(\delta\rho_\psi)_k}{\rho_\psi} \approx \frac{1}{\sqrt{2}}\frac{\delta\psi_k}{\psi_0} \exp\left[-\frac{1}{2}\int_k^{k_r} \frac{dk}{k}\frac{|\delta\psi|^2}{\psi_0^2}\right] , \tag{4.47}$$

where the exponential damping due to the periodic potential is explicit. For the model independent axion, which is the case $l = 2$ ($l = 1$ in string frame), the fluctuations will be completely damped away by the exponential factor. But the case with $l = 1$ ($l = 0$ in string frame) is phenomenologically more interesting. Let

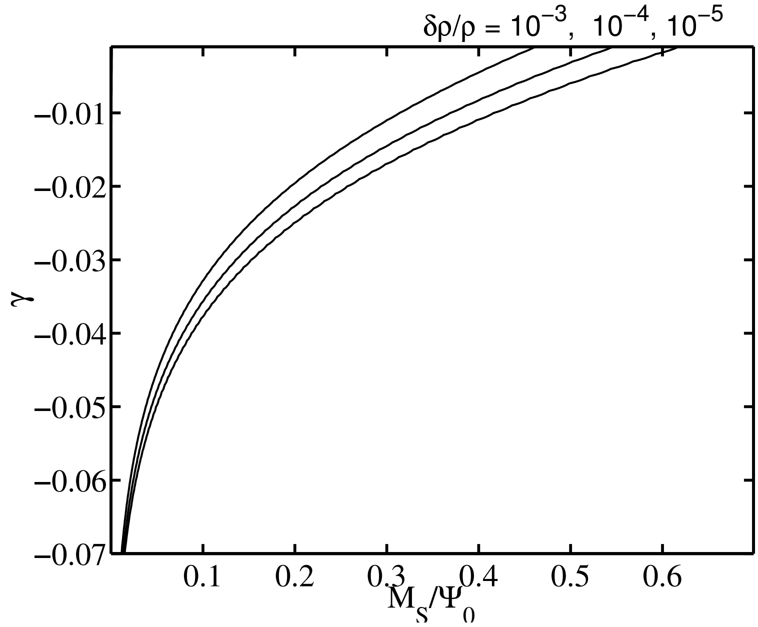

Figure 4.1: Levels of constant $(\delta\rho_a)_k/\rho_a$ as a function of M_s/ψ_0 on the horizontal axis and γ on the vertical axis. The three lines correspond to $(\delta\rho_a)_k/\rho_a = 10^{-3}$, 10^{-4}, and 10^{-5}. In the plot also $k_a/k_s \approx 10^{-30}$ and $k_{osc}/k_s \approx 10^{-7}$ are fixed.

us assume that the spectral tilt $\gamma = 3/2 - \nu$ of the axion field fluctuations is negative but very close to zero. In [1] it was shown that from equation (4.47) one obtains

$$\frac{(\delta\rho_\psi)_{k_a}}{\rho_\psi} = \frac{M_s}{\psi_0}\left(\frac{k_a}{k_s}\right)^\gamma \exp\left[-\frac{1}{2}\frac{M_s^2}{\psi_0^2}\left(\frac{-1}{2(\gamma^2-\gamma)}\left(\frac{k_{osc}}{k_r}\right)^{2\gamma} - \frac{1}{2\gamma}\left(\frac{k_a}{k_r}\right)^{2\gamma}\right)\right]. \tag{4.48}$$

Here k_{osc} is the comoving scale that has just entered the horizon when axion starts to oscillate at $\tau = \tau_{osc}$. As we shall see in the next section, we may take $k_{osc}/k_r \approx 10^{-7}$ within a few orders of magnitude.

In fig.(4.1) we have show a contour plot of $(\delta\rho_\psi)_k/\rho_\psi$ as given by equation (4.48) as a function of M_s/ψ_0 and γ. We take the lower cut off on the momentum to be $k_{min} \simeq 0$ such that $\rho_a = V_0/2$ from equation (4.46). For $(\delta\rho_\psi)_k/\rho_\psi = 10^{-4}$ we find that the right level of density fluctuations are obtained with γ in a reasonable range $-0.04 < \gamma < -10^{-5}$ when $0.1 < M_s/\psi_0 < 0.6$ (If $M_s/\psi_0 \gtrsim 1$ then for a negative γ the fluctuations are completely damped away for both $l = 2$ and $l = 1$).

From fig.(4.1) we also see that if ψ_0 differs only within a few orders of magnitude from M_s we get $\gamma \gtrsim -0.05$ which correspond, when the effect of the exponential damping is taken into account, to a bound on the spectral tilt of the adiabatic density fluctuations, Δn, in the range $0 < \Delta n \lesssim 1$. This can be verified using equation (4.51). It is potentially in agreement with experimental bounds [190].

If we require $(\delta\rho_\psi)_k/\rho_\psi \sim 10^{-4}$ and set $\gamma = -0.01$, we obtain $M_s/\psi_0 \approx 0.4$.

It is likely that string theory axions can have a decay constant $\psi_0 \approx 10^{17}$ GeV, which would lead to a present string coupling in the lower limit of the theoretically favoured range $g_s \sim 0.1 - 0.01$. In any case, the possibility of realizing this idea depends on the compactification.

When M_s/ψ_0 is very small, less than 10^{-4}, the damping switches off. This means that even in this case γ can be close to zero with a small but negative spectral tilt $\Delta n = 2\gamma$ (see equation (4.50) and (4.51)).

Finally we note that if $e^{\beta_s} \neq 1$, the ratio M_s/ψ_0 either increases or decreases, depending on the sign of m. It is interesting to note that m can have both signs. In models with an $SL(3,\mathbb{R})$-invariant effective action there are three axions with respectively $m = 0$, $m = -\sqrt{3}$, and $m = \sqrt{3}$ as can be seen from equation (4.43).

Let us now estimate the spectral index of the CMB fluctuations corresponding to a slightly negative value for the axion tilt γ. The density perturbations normalized in a logarithmic $(\delta\rho_\psi)_k/\rho$ are related to the Furrier component of the axion (curvaton) energy density in equation (4.33) by

$$k^{-3/2}\frac{(\delta\rho_\psi)_k}{\rho} = \delta_k \ . \tag{4.49}$$

So if we define ξ as the effective tilt of the axion density perturbations

$$\frac{(\delta\rho_\psi)_k}{\rho} \propto k^{\xi} \ , \tag{4.50}$$

then the spectral index of the perturbations are given by $n = 2\xi + 1$ and the spectral tilt $\Delta n \equiv n - 1$ is given by

$$\Delta n \equiv 2\xi \tag{4.51}$$

For a positive tilt with $\xi = \gamma \sim 0.13$, as mentioned in the beginning, we would get a spectral tilt $\Delta n = 0.26$. The resulting index is below $n = 1.32$ which is the upper limit given by WMAP [190]. For a small negative tilt $\gamma > -0.05$ the damping factor in equation (4.47) implies that this gives rise to a slightly positive tilted (blue) spectrum of density fluctuations $0.3 \lesssim \Delta n \lesssim 1$. Only if the ratio M_s/ψ_0 is very small, such that the damping switches off, it is possible to get a negative tilt.

Since the axion potential is highly non-linear, it is not trivial to solve the perturbation equation mode-by-mode and to compute the perturbations after the onset of the potential. For a quadratic axion potential, valid for small field values, one finds that at low frequencies the spectral tilt of the energy density is unchanged after the onset of the potential as discussed in section 4.2. However, it would be important to study the evolution of the field fluctuations in the non-linear regime on superhorizon scales after the potential is generated and the axion mass has become larger than the Hubble rate, in this way one could check whether the evolution of the fluctuations on superhorizon scales in this case affects the spectral tilt.

To understand what happens after the potential is turned on at least qualitatively, we divide the field ψ in its large scale component, which at scales $l = k^{-1}$

behaves as a constant classical field $\psi_c(l^{-1})$, and its short wave length part $\delta\psi$ corresponding to momenta $k \geq l^{-1}$,

$$\psi = \psi_c(l^{-1}) + \delta\psi \tag{4.52}$$

where

$$\psi_c(l^{-1}) = \psi_c + \tilde{\psi}(l^{-1}) \ . \tag{4.53}$$

Here $\psi_c(l^{-1})$ includes the classical scale independent field ψ_c together with a contribution $\tilde{\psi}(l^{-1})$ from all the fluctuations with momenta smaller than k. The dispersion of the long-wave perturbations $\tilde{\sigma}_k = \sqrt{\sigma^2 - \sigma_k^2}$, is given by

$$\tilde{\sigma}_k^2 = \int_{k_{min}}^{k} d\ln k \, |\delta\psi(k)|^2 \ . \tag{4.54}$$

If $\tilde{\sigma}_k$ is comparable to ψ_c then $\psi_c(l^{-1})$ will indeed depend on x on scales larger than l. Thus, the classical field $\psi_c(l^{-1})$ will be uniformly distributed among all possible values and the average energy density will be given by $\rho_\psi \simeq (1/2)V_0$ as in equation (4.46).

The constant axion field $\psi_c(l^{-1})$ starts to oscillate near the nearest minimum of the potential and after a few oscillations, as the amplitude of the coherent oscillations decreases, the potential effectively becomes quadratic. This is equivalent to the statement that the axions will behave like non-relativistic matter when they oscillate coherently in the potential. At this point the energy density will scale as $\rho_\psi \sim V_0(\psi^2/\psi_0^2)$ and the density perturbations as $(\delta\rho_\psi)_k \sim V_0(\psi\delta\psi)/\psi_0^2$. Hence,

$$\frac{\rho_\psi(k)}{\rho_\psi} \sim \frac{\delta V\,(\psi\,(k))}{V\,(\psi)} \sim \frac{\delta\psi(k)}{\psi} \tag{4.55}$$

where ψ can be identified with $\psi_c(l^{-1})$. This is valid if $\psi_c(l^{-1})$ is larger than the short-wave length dispersion σ_k which will be the case for a negatively and approximately flat spectrum.

Since $\psi_c(l^{-1})$ evolves in the same way as $\delta\psi(k)$, we note after the point when we can treat the potential as being quadratic, the ratio in equation (4.55) remains fixed up to some scale-independent factor that does not destroy the scale invariance of the spectrum as also discussed in section 4.2. However, a computer simulation would be needed to check whether something non-trivial happens in the short era in between the topological damping sets in and the linear approximation is valid.

Also, since $\delta\rho_\psi \sim \delta\psi(k)$ in equation (4.55) and (4.47), the density fluctuations will be Gaussian.

4.5 Reheat Temperature and Entropy Production

From requiring that the energy density contributed by the curvaton at the beginning of the radiation dominated epoch is less than critical, we obtain an upper bound on the curvaton mass and the reheat temperature. From

$$\rho_\sigma < \rho_c \quad , \qquad \tau = \tau_r \tag{4.56}$$

we have

$$\frac{1}{2}\Omega^2 m^2 \sigma_r^2 < 3 m_p^2 H_r^2 \tag{4.57}$$

which leads to

$$10^{-5} \simeq \mathcal{P}_\zeta^{1/2} \simeq \frac{k^{3/2}\delta\rho_r}{\rho_r} > \frac{m}{m_p} \ . \tag{4.58}$$

Above $\rho_c = 3/(8\pi)H^2M_p^2$ is the critical energy density.

Let us now check that the axions have a chance of dominating the energy density, as was assumed in the previous section. To estimate the life time of the massive axions, we parameterize the interaction between the axion and the gauge fields as

$$(\sigma/M')F\tilde{F} \ , \tag{4.59}$$

where $M' \sim \pi^2\psi_0$ generally depends on the compactification [139, 189]. A typical axion lifetime is [139, 189]

$$\tau \approx M'^2/m^3 \ , \tag{4.60}$$

where the axion mass m is given by

$$m \approx 10\frac{\Lambda^3}{M_p M'} \ . \tag{4.61}$$

With $M' \sim 10^{18}$ GeV we get $m = 10^6$ GeV and thus $k_{osc}/k_s \sim 10^{-7}$ as claimed in the previous section. Like in [189] we will assume that $\Lambda \approx 10^{14}$ GeV. Defining $R \equiv M'/M_p$ one finds that

$$\tau \approx 10^{-3}R^5\frac{M_p^8}{\Lambda^9} \approx R^5 \cdot 10^{23}\ \text{GeV}^{-1} \approx R^5 \cdot 10^{-2}\ \text{sec} \ . \tag{4.62}$$

The average energy density in the non-relativistic part of the axion field after the potential is turned on reads

$$\rho_\sigma = \langle V(\psi(\vec{r}))\rangle \approx \frac{1}{2}V_0 \ , \tag{4.63}$$

which implies that the relative energy density of the axions is

$$\Omega_\sigma = \frac{\rho_\sigma}{\rho_c} = \frac{4\pi}{3}\frac{V_0}{M_p^2H^2} \ . \tag{4.64}$$

To evaluate Ω_σ at the time the potential is turned on, we use the fact that at this time the Hubble paremeter H has the same order of magnitude as the axion mass. Typically we can also take [139, 189]

$$V_0 \approx \frac{\Lambda^6}{M_p^2} \tag{4.65}$$

so that we obtain

$$\Omega_\sigma \approx 4\pi \frac{\Lambda^6}{M_p^4 m^2} \simeq 10^{-1} \frac{M'^2}{M_p^2} = 10^{-1} R^2 \ , \tag{4.66}$$

Thus the axions do not dominate energy density at this point. The axion behaves as non-relativistic dust and soon starts to dominate the energy density. Let us evaluate the reheat temperature T_{RH} as the axion decays into photons. Since $H \approx 1.66 g_*^{1/2} T^2/M_p$ so that $H^2(t=\tau) \approx \frac{1}{4}\tau^{-2} \approx 2.8 g_* T_{RH}^4/M_p^2$ which implies that

$$T_{RH} = 0.55 g_*^{-1/4} \frac{M_p^{1/2}}{\tau^{1/2}} \tag{4.67}$$

we obtain

$$T_{RH} \sim R^{-5/2} 10^{-2} \text{ GeV} \ . \tag{4.68}$$

If we define T_{osc} as the temperature of the universe when $m \approx H$, we find

$$T_{osc} \approx \sqrt{\frac{m M_p}{5 g_*^{1/2}}} \approx \sqrt{\frac{\Lambda^3}{R M_p}} \sim R^{-1/2} 10^{11} \text{ GeV} \ . \tag{4.69}$$

We denote the temperature as the axions decay by T_D. Noting that $T_D < T_{RH}$ and $T_{osc} > 10^{11}$ GeV, we see that the temperature difference between T_{osc} and T_{RH} is almost 10 orders of magnitude. During a matter dominated epoch the temperature scales like the inverse scale factor squared, so the density of matter with respect to radiation will grow approximately 5 orders of magnitude. We can conclude that indeed the universe will have time to become matter dominated after the axion mass is turned on and before it decays.

In [139] the entropy production associated with the decay of the axion was calculated as

$$\Delta s \simeq 10^{15} \left(\frac{A_0}{M'}\right)^2 \left(\frac{\Lambda}{10^{14} \text{ GeV}}\right)^{-3} R^4 \Omega_r^{-3/4} \tag{4.70}$$

where $A_0 \sim M'/\pi$ is the initial displacement of the axion VEV and $\Omega_r \equiv N_r/N_{tot}$. N_r is the number of degrees of freedom in spin 0 and 1 fields charged under the gauge groups of the observable sector. N_{tot} is the total number of degrees of freedom in spin 0, 1, and 2. We expect $0.01 \lesssim \Omega_r \lesssim 1$. By tuning the parameters slightly, it is possible to obtain a reasonable amount of entropy production of 8 to 10 orders of magnitude in order to dilute dangerous relics [139].

Let us choose as a typical example $R \sim 10^{-2}$. In this case $T_{RH} = 10^3$ GeV, but one needs to tune the other parameters in order to have enough entropy production to dilute dangerous relics. If $R \sim 1$, then one is about to get in trouble with a too low reheat temperature $T_{RH} = 10$ MeV, but there is plenty of entropy production.

Chapter 5

Summary

We have essentially explored two possible ways, in which *fundamental physics beyond Einstein gravity* could influence our understanding of cosmology. Based on [3], we have investigated the effects of fundamental physics on the standard inflationary scenario. We have also explored how adiabatic density perturbations can be generated in an alternative scenario called pre-big bang cosmology. The latter discussion was based on [1, 2]. Whether fundamental physics will affect cosmology in an observable way is still an open question. One consequence of the pre-big bang scenario seems to be that the observed CMB perturbations stems from the curvaton, as first pointed out in [1]. Of course, the curvaton is not a unique signature of pre-big bang in the sense that it can also be incorporated in a more standard inflationary model. In this case it forms some non-minimal inflationary model where the inflaton is not responsible for the generation of cosmic perturbations. On the other hand, inflation is only a paradigm that must be incorporated in some grander theory. Pre-big bang can be viewed as such kind of a realization of inflation in a more fundamental theory namely string theory, where both the inflaton and the curvaton naturally appears as respectively the dilaton and the Kalb-Ramond axion. Quoting Kolb and Turner: "*The inflaton should spring forth from some grander theory and not vice-versa*" [27]. If observations in the future tends to suggest that the CMB perturbations originates from the curvaton, the present author will take it as a small indication that a pre-big bang like scenario might be the right description of our universe.

We have also explored trans-Planckian effects in standard inflationary scenarios. It is still not clear whether such effects will be observable, but certainly an understanding of trans-Planckian physics is needed for the consistency of the perturbation theory, unless the inflationary scale is very low and inflation has lasted just the needed amount of e-foldings. The model of the dynamics of perturbation modes in the trans-Planckian regime, which has been discussed here based on [3], seems to yield a natural way of fixing the initial vacuum of the perturbation modes. In addition, it leads to the new feature of trans-Planckian damping, the cosmological consequences of which it would be interesting to investigate in more detail in the future.

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