Fractal Space-Time, Non-Differentiable Geometry and Scale Relativity*

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Abstract

In this review paper, we recall the successive steps that we have followed in the construction of the scale-relativity theory. The aim of this theory is to derive the physical behavior of a non-differentiable and fractal space-time and of its geodesics (with which particle trajectories are identified), under the constraint of the principle of the relativity of scales. Various levels of description of scale-laws, from the simplest scale-invariant laws to the log-Lorentzian laws of special scale-relativity, are considered. In particular, we describe new laws of 'scale dynamics' , that correspond to distortions of a structure in the scale-space respectively to the simple scale invariant case. Then we study the effects induced by the existence of internal fractal structures on the motion in standard space. We find that the main consequence is the transformation of classical dynamics in a quantum dynamics. The various mathematical tools of quantum mechanics (complex wave functions, then spinors and bi-spinors) are constructed as manifestations of the non-differentiable geometry. Then the Schrödinger, Klein-Gordon, Pauli and Dirac equations are successively derived as integrals of the equation of the geodesics of various space-time geometries, that correspond to more and more profound levels of description of the structures issued from non-differentiability. We also recall how one can construct a purely geometric theory of gauge fields in this framework: the gauge transformations are re-interpreted as scale transformations in such fractal space-times, and their associated charges are the conservative quantities that originates from the new internal scale symmetries. Finally we tentatively suggest a new development of the theory, in which quantum laws would hold also in the scale-space: in such an approach, one naturally defines a new conservative quantity, named 'complexergy', which measures the complexity of a system as regards its internal hierarchy of organization. Several examples of applications of these proposals in various sciences, and of their experimental and observational tests, are presented at each step of the exposition.

1 Introduction

The theory of scale-relativity is an attempt to extend today's theories of relativity, by applying the principle of relativity not only to motion transformations, but also

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to scale transformations of the reference system. Recall that, in the formulation of Einstein [1], the principle of relativity consists in requiring that 'the laws of nature be valid in every systems of coordinates, whatever their state'. Since Galileo, this principle had been applied to the states of position (origin and orientation of axes) and of motion of the system of coordinates (velocity, acceleration). These states are characterized by their relativity, namely, they are never definable in a absolute way, but only in a relative way. This means that the state of any system (including reference systems) can be defined only with regard to another system.

We have suggested that the observation scale (i.e., in other words, the resolution at which a system is observed or experimented) should also be considered as characterizing the state of reference systems, in addition to position, orientation and motion. It is an experimental fact known for long that the scale of a system can only be defined in a relative way: namely, only scale ratios do have a physical meaning, never absolute scales. This led us to propose that the principle of relativity should be generalized in order to apply also to relative scale transformations of the reference system, i.e., dilations and contractions of space-time resolutions. Note that, in this new approach, one reinterprets the resolutions, not only as a property of the measuring device and / or of the measured system, but more generally as a property that is intrinsic to the geometry of space-time itself: in other word, space-time is considered to be fractal. But here, we connect the fractal geometry with relativity, so that the resolutions are assumed to characterize the state of scale of the reference system in the same way as velocity characterizes its state of movement. The principle of relativity of scale then consists in requiring that 'the fundamental laws of nature apply whatever the state of scale of the coordinate system'.

It is clear that the present state of fundamental physics is far from coming under such a principle. In particular, in today's view there are two physics, a quantum one toward the small scales and a classical one toward large scales. The principle of scale-relativity amounts to requiring a re-unification of these laws, by writing them under a unique more fundamental form, which become respectively the usual quantum laws and classical laws depending on the relative state of scale of the reference system.

There are other motivations for adding such a first principle to fundamental physics. It allows one to generalize the current description of the geometry of space-time. The present description (curved geometry) is usually reduced to at least two-time differentiable manifolds (even though singularities are possible at certain particular points and events). So a way of generalization of current physics consists in trying to abandon the hypothesis of differentiability of space-time coordinates. This means to consider general continuous manifolds, which may be differentiable or non-differentiable. These manifolds include as a sub-set the usual differentiable ones, and therefore all the Riemannian geometries that subtend Einstein's generalized relativity of motion. Then in such an approach, the standard classical physics will be naturally recovered, in limits which will be studied throughout the present contribution.

But new physics is also expected to emerge as a manifestation of the new non-differentiable geometry. One can prove (as recalled in the present contribution) that a continuous and non-differentiable space is fractal, in Mandelbrot's general definition of this concept [2, 3], namely, the coordinates acquire an explicit dependence on resolutions and diverge when the resolution interval tends to zero [4, 6].

This leads one to apply the concept of fractality not only to objects in a given space, but to the geometry of space-time itself (which means to define it in an internal way though its metrics invariants). Hence the tool of fractals, whose universality has now been recognized in almost every sciences (see e.g.the volumes [8, 90], other volumes of these series and references therein) may also be destined for playing a central role in fundamental physics. Note however that (as will be recalled in this paper), non-differentiable space-time do also have new properties that are irreducible to the sole fractality.

One of the first fundamental consequences of the non-differentiability and fractality of space-time is the non-differentiability and fractality of its geodesics, while one of the main feature of space-time theories is their ability to identify the trajectories of 'free' particles with the space-time geodesics. Now the introduction of non-differentiable trajectories in physics dates back to pioneering works by Feynman in the framework of quantum mechanics [9]. Namely, Feynman has demonstrated that the typical quantum mechanical paths that contribute in a dominant way to the path integral are fractal non-differentiable curves of fractal dimension 2 [10, 11].

In the fractal space-time approach, one is therefore naturally led to consider the reverse question: does quantum mechanics itself find its origin in the fractality and non-differentiability of space-time ? Such a suggestion, first made twenty years ago [12, 11], has been subsequently developed by several authors [13, 14, 15, 16, 17, 4, 5, 18].

The introduction of non-differentiable trajectories was also underlying the various attempts of construction of a stochastic mechanics [19, 20]. But stochastic mechanics is now known to have problems of self-consistency [21, 26], and, moreover, to be in contradiction with quantum mechanics [22]). The proposal that is developed here, even if it shares some common features with stochastic mechanics, due to the necessity to use a statistical description as a consequence of the non-differentiability, is fundamentally different and is not subjected to the same difficulties [26].

Remark that, in the new approach, we do not have to assume that trajectories are fractal and non-differentiable, since this becomes a consequence of the fractality of space-time itself. Indeed, one of the main advantages of a space-time theory is that the equation of motion of particles has not to be added to the field equations: it is a direct consequence of them, since the particles are expected to follow the space-time geodesics. As we shall see, the Schrödinger equation (and more generally the Klein-Gordon, Pauli and Dirac equations) are, in the scale-relativity approach, re-expressions of the equation of geodesics [4, 23, 24]. Note that we consider here only geodesical curves (of topological dimension 1), but it is also quite possible to be more general and to consider subspaces of larger topological dimensions (fractal strings [7], fractal membranes, etc...).

The present review paper is mainly devoted to the theoretical aspects of the scale-relativity approach. We shall first develop at various levels the description of the laws of scale that come under the principle of scale-relativity (Sections 3 and 4). Examples of applications of these laws in various domains (astrophysics, high energy physics, sciences of life) will be briefly given, with references of more detailed studies for the interested readers.

Then we shall recall how the description of the effects on motion of the internal fractal and non-differentiable structures of 'particles' lead to write a geodesics equation that is integrated in terms of quantum mechanical equations (Sections

5-6). The Schrödinger equation is derived in the (motion) non-relativistic case, that corresponds to a space-time of which only the spatial part is fractal. Looking for the motion-relativistic case amounts to work in a full fractal space-time, in which the Klein-Gordon equation is derived. Finally the Pauli and Dirac equation are derived as integrals of geodesics equations when accounting for the breaking of the reflexion symmetry of space differential elements that is expected from non-differentiability.

Now, the three minimal conditions under which this result is obtained (infinity of trajectories, each trajectory is fractal, breaking of differential time reflexion invariance) may be achieved in more general systems than only the microscopic realm. As a consequence, fundamental laws that share some properties in common with the standard quantum mechanics of microphysics but not all, may apply to different realms. Some examples of applications of these new quantum mechanics (which are not based on the Planck constant $\hbar$) in the domains of gravitation and of sciences of life will be given.

The next Section 7 is devoted to the account of the interpretation of the nature of gauge transformation and of gauge fields in the scale-relativity framework (only the simple case of a U(1) electromagnetic like gauge field will be considered). We attribute the emergence of such field to the effects of coupling between scale and motion. In other words, the internal resolutions becomes themselves 'fields' which are functions of the coordinates. Some applications in the domain of elementary particle physics will be briefly given.

We finally consider hints of a new tentative extension of the theory (Section 8), in which quantum mechanical laws are written in the scale space. In this case the internal relative fractal structures become described by probability amplitudes, from which a probability density of their 'position' in scale space can be deduced. A new quantized conservative quantity, that we have called 'complexergy', is defined (it plays for scale laws the same role as played by energy for motion laws), whose increase corresponds to an increase of the number of hierarchically imbricated levels of organisations in the system under consideration.

2 Structure of the theory

2.1 Successive levels of development of the theory

The theory of scale relativity is constructed by completing the standard laws of classical physics (laws of motion in space, i.e. of displacement in space-time) by new scale laws (in which the space-time resolutions are considered as variables intrinsic to the description). We hope such a stage of the theory to be only provisional, and the motion and scale laws to be treated on the same footing in the final theory. However, before reaching such a goal, one must realize that the various possible combinations of scale laws and motion laws already lead to a large number of sub-sets of the theory to be developed. Indeed, three domains of the theory are to be considered:

(1) *Scale-laws*: description of the internal fractal structures of trajectories in a non-differential space-time at a given point / event;

(2) *Induced effects of scale laws on the equations of motion*: generation of quantum mechanics as mechanics in a nondifferentiable space-time;

(3) *Scale-motion coupling*: effects of dilations induced by displacements, that we interpret as gauge fields (only the case of the electromagnetic field has been

considered up to now) [25, 23, 92].

Now, concerning the first step (1) alone, several levels of the description of scale laws can be considered. These levels are quite parallel to that of the historical development of the theory of motion laws:

(1.i) *Galilean scale-relativity*: standard laws of dilation, that have the structure of a Galileo group (fractal power law with constant fractal dimension). When the fractal dimension of trajectories is $D_F = 2$, the induced motion laws are that of standard quantum mechanics [4, 23].

(1.ii) *Special scale-relativity*: generalization of the laws of dilation to a Lorentzian form [16]. The fractal dimension itself becomes a variable, and plays the role of a fifth dimension, that we call 'djinn'. An impassable length-time scale, invariant under dilations, appears in the theory; it replaces the zero, owns all its physical properties (e.g., an infinite energy-momentum would be needed to reach it), and plays for scale laws the same role as played by the velocity of light for motion.

(1.iii) *Scale-dynamics*: while the first two cases correspond to "scale freedom", one can also consider distorsion from strict self-similary that would come from the effect of a "scale-force" or "scale-field" [26, 27].

(1.iv) *General scale-relativity*: in analogy with the field of gravitation being ultimately attributed to the geometry of space-time, a future more profound description of the scale-field could be done in terms of geometry of the five-dimensional space-time-djinn and its couplings with the standard classical space-time. (This case will not be fully considered in this paper: however, the third step involving scale-motion couplings and leading to a new interpretation of gauge fields is expected, in the end, to become part of a general theory of scale-relativity.).

(1.v) *Quantum scale-relativity*: the above cases assume differentiability of the scale transformations. If one assumes them to be continuous but, as we have assumed for space-time, non-differentiable, one is confronted for scale laws to the same conditions that lead to quantum mechanics in space-time. One may therefore attempt to construct a new quantum mechanics in scale-space, thus achieving a kind of 'third quantization'.

The possible complication of the theory becomes apparent when one realizes that these various levels of the description of scale laws lead to different levels of induced dynamics (2) and scale-motion coupling (3), and that other sublevels are to be considered, depending on the status of motion laws (non-relativistic, special-relativistic, general-relativistic).

In the present contribution, we recall the various possible developments of scale laws (1.i-1.v). Then we consider the induced effects on motion (2) of the simplest self-similar scale laws (1.i), that lead to transform classical mechanics into a quantum mechanics. (For hints on possible generalizations, see [23]. The scale-motion coupling laws are analysed in two cases: Galilean scale laws (1.i) and their Lorentzian generalization (1.ii). Some examples of applications of the various levels of the theory in various sciences (gravitation, particle physics, sciences of life) are briefly considered at the end of each section.

2.2 First principles

Let us briefly recall the fundamental principles that underly, since the work of Einstein [1], the foundation of theories of relativity. We shall express them here under a general form that transcends particular theories of relativity, namely, they can be applied to any state of the reference system (origin, orientation, motion, scale,...). The basic principle is the principle of relativity, that requires that the laws of physics should be of such a nature that they apply to any reference system. In other words, it means that physical quantities are not defined in an absolute way, but are instead relative to the state of the reference system. It is subsequently implemented in physics by three related and interconnected principles and their related tools:

(1) The principle of covariance, that requires that the equations of physics keep their form under changes of the state of reference systems. As remarked by Weinberg [28], it should not be interpreted in terms of simply giving the most general (arbitrarily complicated) form to the equations, which would be meaningless. It rather means that, knowing that the fundamental equations of physics have a simple form in some particular coordinate systems, they will keep this simple form whatever the system. With this meaning in mind, two levels of covariance can be defined: (i) Strong covariance, according to which one recovers the simplest possible form of the equations, which is the Galilean form they have in the vacuum devoid of any force. For example, the equations of motion in general relativity take the free inertial form $Du^{\mu} = 0$, in terms of Einstein's covariant derivative, so they come under strong covariance. (ii) Weak covariance, according to which the equations keep a same, simple form under any coordinate transformation. A large part of the general relativity theory is only weakly covariant: for example, the Einstein's field equations have a source term, and the gravitational field (the Christoffel symbols) are not tensors.

(2) The equivalence principle is a more specific statement of the principe of relativity, when it is applied to a given physical domain. In general relativity, it states that a gravitational field is locally equivalent to a field of acceleration, i.e., it expresses that the very existence of gravitation is relative on the choice of the reference systems, and it specifies the nature of the coordinate systems that absorb it. In scale relativity, one may make a similar proposal and set generalized equivalence principles according to which the quantum behavior is locally equivalent to a fractal and non-differentiable motion, while the gauge fields are locally equivalent to expansion / contraction fields on the internal resolutions (scale variables).

(3) The geodesics principle states that the free trajectories are the geodesics of space-time. It plays a very important role in a geometric / relativity theory, since it means that the fundamental equation of dynamics is completely determined by the geometry of space-time, and therefore has not to be set as an independent equation. Moreover, in such a theory the action identifies (up to a constant) with the fundamental length-invariant, so that the stationary action principle and the geodesics principle become identical.

One of the main tools by which these principles are implemented is the covariant derivative. This tool includes in an internal way the effects of geometry through a new definition of the derivative, contrarily to the standard field approach whose effects are considered to be externally applied on the system. In general relativity, it amounts to substract the geometric effects to the total increase of a vector, leaving only the inertial part $DA_{\mu} = dA_{\mu} - \Gamma^{\rho}_{\mu\nu} A_{\rho} dx^{\nu}$. One of the most remarkable results of general relativity is that the three principles (strong covariance, equivalence and geodesics / least action principle) lead to the same inertial form for the motion equation, $Du^{\mu} = 0$ (see e.g. [84]). As we shall recall in this paper, the theory of scale-relativity attempts to follow a similar line of thought, and to construct new covariant tools adapted to the new problem posed here, i.e., the description of a non-differentiable and fractal geometry coming under the principle of the relativity of scales.

3 On the relation between non-differentiability and fractality

3.1 Beyond differentiability

One of the main questions that is asked concerning the emergence of fractals in natural and physical sciences is the reason for their universality [2, 3]. We mean here by 'universality' that an explicitly scale-dependent behavior (not only self-similarity) has been found in a wide class of very different situations in almost every sciences. While particular causes may be found for their origin by a detailed description of the various systems where they appear (chaotic dynamics, biological systems, life sciences social sciences, etc...) their universality nevertheless may call for a universal answer (i.e., for an identification of fundamental features which would be common to these various causes).

Our suggestion, which has been developed in [4, 23], is as follows. Since the time of Newton and Leibniz, the founders of the integro-differentiation calculus, one basic hypothesis which is put forward in our description of physical phenomena is that of differentiability. The strength of this hypothesis has been to allow physicists to write the equations of physics in terms of differential equations. However, there is neither a *a priori* principle nor definite experiments that impose the fundamental laws of physics to be differentiable. On the contrary, it has been shown by Feynman that typical quantum mechanical paths are non-differentiable [9].

The basic idea that underlines the theory of scale relativity is therefore to give up the hypothesis of differentiability of space-time (namely, to use a generalized description including differentiable and non-differentiable systems). In such a framework, the successes of present day differentiable physics could be understood as applying to domains where the approximation of differentiability (or integrability) was good enough, i.e. at scales such that the effects of nondifferentiability were smoothed out; but conversely, we expect the differential method to fail when confronted with truly nondifferentiable or nonintegrable phenomena, namely at very small and very large length scales (i.e., quantum physics and cosmology), and also for chaotic systems seen at very large time scales.

3.2 Continuity and non-derivabitity implies fractality

A new 'frontier' of mathematical physics amounts to construct a continuous but nondifferentiable physics. Set in such terms, the project may seem extraordinarily difficult. Fortunately, there is a fundamental key which will be of great help in this quest, namely, the concept of scale transformations. Indeed, the main consequence of continuity and nondifferentiability is explicit scale-dependence (and divergence) [4, 23, 26]. One can prove that the length of a continuous and nowhere-differentiable curve is dependent on resolution ε, and, further, that $\mathcal{L}(\varepsilon) \to \infty$ when $\varepsilon \to 0$, i.e. that this curve is fractal (in a general meaning). The scale divergence of continuous and almost nowhere-differentiable curves is a direct consequence of Lebesgue's theorem, which states that *a curve of finite length is almost everywhere differentiable.* Let us recall the demonstration of this fundamental property.

Consider a continuous but nondifferentiable function $f(x)$ between two points $A_0[x_0, f(x_0)]$ and $A_\Omega[x_\Omega, f(x_\Omega)]$. Since f is non-differentiable, there exists a point A_1 of coordinates $[x_1, f(x_1)]$ with $x_0 < x_1 < x_\Omega$, such that A_1 is not on the segment $A_0 A_\Omega$. Then the total length $\mathcal{L}_1 = \mathcal{L}(A_0 A_1) + \mathcal{L}(A_1 A_\Omega) > \mathcal{L}_0 = \mathcal{L}(A_0 A_\Omega)$. We can now iterate the argument and find two coordinates x_{01} and x_{11} with $x_0 < x_{01} < x_1$ and $x_1 < x_{11} < x_\Omega$, such that $\mathcal{L}_2 = \mathcal{L}(A_0 A_{01}) + \mathcal{L}(A_{01} A_1) + \mathcal{L}(A_1 A_{11}) + \mathcal{L}(A_{11} A_\Omega) > \mathcal{L}_1 > \mathcal{L}_0$. By iteration we finally construct successive approximations $f_0, f_1, ...f_n$ of $f(x)$ whose lengths $\mathcal{L}_0, \mathcal{L}_1, ...\mathcal{L}_n$ increase monotonically when the 'resolution' $r \approx (x_\Omega - x_0) \times 2^{-n}$ tends to zero. In other words, continuity and nondifferentiability implies a monotonous scale dependence of f.

From Lebesgue's theorem, that states that 'a curve of finite length is almost everywhere differentiable', see e.g. [29], one deduces that if f is continuous and almost everywhere nondifferentiable, then $\mathcal{L}(\varepsilon) \to \infty$ when the resolution $\varepsilon \to 0$, i.e., f is *scale-divergent*. This theorem is also demonstrated in Ref. [4], p. 82, using non-standard analysis.

What about the reverse proposition: Is a continuous function whose length is scale-divergent between any two points (such that $x_A - x_B$ finite), i.e., $\mathcal{L}(r) \to \infty$ when $r \to 0$, non-differentiable? The answer is as follows [26]:

(i) If the length diverges as fast as a power law, i.e. $\mathcal{L}(\epsilon) \propto (\lambda/\epsilon)^\delta$, or faster than a power law (e.g., exponential divergence $\mathcal{L}(\epsilon) \propto \exp(\lambda/\epsilon)$, etc...), then the function is certainly nondifferentiable. It is interesting to see that the standard (self-similar, power-law) fractal behavior plays a critical role in this theorem: it gives the limiting behavior beyond which non-differentiability is ensured.
(ii) In the intermediate domain of slower divergences (for example, logarithmic divergence, $\mathcal{L}(\epsilon) \propto \ln(\lambda/\epsilon), \ln(\ln(\lambda/\epsilon))$, etc...), the function may be either differentiable or nondifferentiable.

This can be proved by looking at the way the length increases and the slope changes under successive zooms of a constant factor ρ. There are two ways by which the divergence can occur: either by a regular increase of the length (due to the regular appearance of new structures at all scales that continuously change the slope), or by the existence of jumps (in this case, whatever the scale, there will always exist a smaller scale at which the slope will change). The power law corresponds to a continuous length increase, $\mathcal{L}(\rho\epsilon) = \mu\mathcal{L}(\epsilon)$, then to a continuous and regular change of slope when $\epsilon \to 0$: therefore the function is nondifferentiable in this case. Divergences slower than power laws may correspond to a regular length increase, but with a factor μ which becomes itself scale-dependent: $\mathcal{L}(\rho\epsilon) = \mu(\epsilon)\mathcal{L}(\epsilon)$ with $\mu(\epsilon) \to 0$ when $\epsilon \to 0$. In this case, some functions can be differentiable, if they are such that new structures indeed appear at all scales (and could then be named "fractal" under the general definition initially given by Mandelbrot [2] to this word), but these structures become smaller and smaller with decreasing scale, so that a slope can finally be defined in the limit $\epsilon \to 0$. Some other functions diverging slower than power laws are not differentiable, e.g. if there always exists a scale smaller than any given scale such that an important change of slope occurs: in this case, the slope limit may not exist in the end.

3.3 Explicit scale dependence on resolution

This result is the key for a description of nondifferentiable processes in terms of differential equations: We introduce explicitly the resolutions in the expressions of the main physical quantities, and, as a consequence, in the fundamental equations of physics. This means that a physical quantity f, usually expressed in terms of space-time variables x, i.e., $f = f(x)$, must be now described as also depending on resolutions, $f = f(x, \varepsilon)$. In other words, rather than considering only the strictly nondifferentiable mathematical object $f(x)$, we shall consider its various approximations obtained from smoothing it or averaging it at various resolutions:

$$f(x, \varepsilon) = \int_{-\infty}^{+\infty} \Phi(x, y, \varepsilon) f(x + y) dy \tag{1}$$

where $\Phi(x, y, \varepsilon)$ is a smoothing function centered on x, for example a step function of width $\approx 2\varepsilon$, or a Gaussian of standard error $\approx \varepsilon$. (This can be also seen as a wavelet transformation, but using a filter that is not necessarily conservative). Such a point of view is particularly well adapted to applications in physics: any real measurement is always performed at finite resolution (see [4, 13] for additional comments on this point). In this framework, $f(x)$ becomes the limit when $\varepsilon \to 0$ of the family of functions $f(x, \varepsilon)$. But while $f(x, 0)$ is nondifferentiable, $f(x, \varepsilon)$, which we have called a 'fractal function' [4], is now differentiable for all $\varepsilon \neq 0$.

The problem of the physical description of the process where the function f intervenes is now shifted. In standard differentiable physics, it amounts to finding differential equations implying the derivatives of f, namely $\partial f / \partial x, \partial^2 f / \partial x^2$, that describe the laws of displacement and motion. The integro-differentiable method amounts to performing such a local description, then integrating to get the global properties of the system under consideration. Such a method has often been called 'reductionist', and it was indeed adapted to most classical problems where no new information appears at different scales.

But the situation is completely different for systems implying fractals and non-differentiability at a fundamental level, like the space-time of microphysics itself as suggested here. At high energies, the properties of quarks, of nucleons, of the nucleus, of atoms are interconnected but not reducible one to the other. In living systems, the scales of DNA bases, chromosomes, nuclei, cells, tissues, organs, organisms, then social scales, do co-exist, are related one with another, but are certainly not reducible to one particular scale, even the smaller one. In such cases, new, original information may exist at different scales, and the project to reduce the behavior of a system at one scale (in general, the large one) from its description at another scale (the smallest, described by the limit $\delta x \to 0$ in the framework of the standard differentiable tool) seems to lose its meaning and to be hopeless. Our suggestion consists precisely to give up such a hope, and of introducing a new frame of thought where all scales co-exit simultaneously in a scale-space, but are connected together via scale-differential equations. As we shall see, the solutions of the scale equations that come under the principle of scale-relativity are able to describe not only continuous scaling behavior on some ranges of scales, but also the existence of sudden transitions at some particular scale.

Indeed, in non-differentiable physics, $\partial f(x)/\partial x = \partial f(x, 0)/\partial x$ does not exist any longer. But the physics of the given process will be completely described if we succeed in knowing $f(x, \varepsilon)$ for all values of ε, which *is* differentiable when $\varepsilon \neq 0$, and can be the solution of differential equations involving $\partial f(x, \varepsilon)/\partial x$ but also $\partial f(x, \varepsilon)/\partial \ln \varepsilon$. More generally, if one seeks nonlinear laws, one expects the equations of physics to take the form of second order differential equations, which will then contain, in addition to the previous first derivatives, operators like $\partial^2/\partial x^2$ (laws of motion), $\partial^2/\partial(\ln \varepsilon)^2$ (laws of scale), but also $\partial^2/\partial x \partial \ln \varepsilon$, which corresponds to a coupling between motion laws and scale laws.

4 Scale laws

4.1 Scale invariance and Galilean scale-relativity

Consider a non-differentiable (fractal) curvilinear coordinate $\mathcal{L}(x, \varepsilon)$, that depends on some parameter x and on the resolution ε. Such a coordinate generalizes to non-differentiable and fractal space-times the concept of curvilinear coordinates introduced for curved Riemannian space-times in Einstein's general relativity [4]. $\mathcal{L}(x, \varepsilon)$, being differentiable when $\varepsilon \neq 0$, can be the solution of differential equations involving the derivatives of $\mathcal{L}$ with respect to both x and ε.

4.1.1 Differential dilation operator

Let us apply an infinitesimal dilation $\varepsilon \to \varepsilon' = \varepsilon(1 + d\rho)$ to the resolution. Being, at this stage, interested in pure scale laws, we omit the x dependence in order to simplify the notation and we obtain, at first order,

$$\mathcal{L}(\varepsilon') = \mathcal{L}(\varepsilon + \varepsilon\, d\rho) = \mathcal{L}(\varepsilon) + \frac{\partial \mathcal{L}(\varepsilon)}{\partial \varepsilon}\, \varepsilon\, d\rho = (1 + \tilde{D}\, d\rho)\, \mathcal{L}(\varepsilon), \tag{2}$$

where $\tilde{D}$ is, by definition, the dilation operator. The identification of the two last members of this equation yields

$$\tilde{D} = \varepsilon \frac{\partial}{\partial \varepsilon} = \frac{\partial}{\partial \ln \varepsilon}\ . \tag{3}$$

This well-known form of the infinitesimal dilation operator, obtained above by the 'Gell-Mann-Levy method' (that allows one to find the currents corresponding to a given symmetry [32]), shows that the "natural" variable for the resolution is $\ln \varepsilon$, and that the expected new differential equations will indeed involve quantities as $\partial \mathcal{L}(x, \varepsilon)/\partial \ln \varepsilon$. The renormalization group equations, in the multi-scale-of-length approach proposed by Wilson [30, 31], already describe such a scale dependence. The scale-relativity approach allows one to suggest more general forms for these scale groups (and their symmetry breaking).

4.1.2 Simplest differential scale law

The simplest renormalization group-like equation states that the variation of $\mathcal{L}$ under an infinitesimal scale transformation $d \ln \varepsilon$ depends only on $\mathcal{L}$ itself. We thus write

$$\frac{\partial \mathcal{L}(x, \varepsilon)}{\partial \ln \varepsilon} = \beta(\mathcal{L}). \tag{4}$$

Still looking for the simplest form of such an equation, we expand $\beta(\mathcal{L})$ in powers of $\mathcal{L}$. We obtain, to the first order, the linear equation (in which a and b are independent of ε at this level of the analysis, but may depend on x):

$$\frac{\partial \mathcal{L}(x, \varepsilon)}{\partial \ln \varepsilon} = a + b\mathcal{L}\ , \tag{5}$$

of which the solution is (see Fig. 1)

$$\mathcal{L}(x, \varepsilon) = \mathcal{L}_0(x) \left[1 + \zeta(x) \left(\frac{\lambda}{\varepsilon}\right)^{-b}\right], \tag{6}$$

where $\lambda^{-b}\zeta(x)$ is an integration constant and $\mathcal{L}_0 = -a/b$.

Let us now define, following Mandelbrot [2, 3], a scale dimension $\delta = D_F - D_T$, (where D_F is the fractal dimension, defined here in terms of covering dimension, and D_T the topological dimension) as:

$$\delta = \frac{d \ln \mathcal{L}}{d \ln(\lambda/\varepsilon)} \; . \tag{7}$$

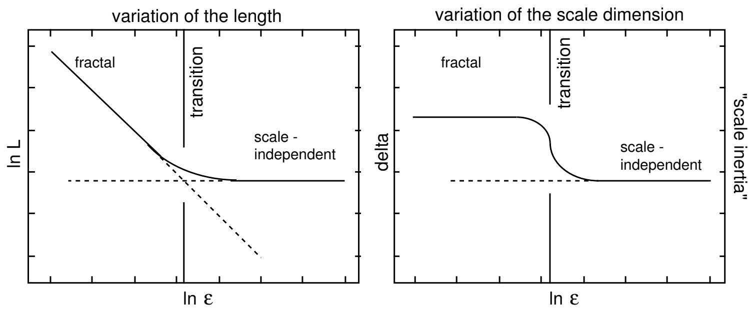

Figure 1: Scale dependence of the length and of the effective scale-dimension in the case of 'inertial' scale laws (which are solutions of the simplest, first order scale-differential equation): toward the small scale one gets a scale-invariant law with constant fractal dimension, while the explicit scale-dependence is lost at scales larger than some transition scale λ.

In the asymptotic regime $\varepsilon << \lambda$, $\delta = -b$ is constant, and one obtains a power-law dependence on resolution that reads

$$\mathcal{L}(x, \varepsilon) = \mathcal{L}_0(x) \left(\frac{\lambda}{\varepsilon}\right)^{\delta} . \tag{8}$$

Anticipating on the following, one can also define a variable 'effective' or 'local' scale dimension from the derivative of the complete solution (6), that jumps from zero to its constant asymptotic value at the transition scale λ (see Fig. 1 and the following figures):

$$\delta_{\text{eff}} = \frac{\delta}{1 + (\varepsilon/\lambda)^{\delta}}. \tag{9}$$

4.1.3 Galilean relativity of scales

Let us now check that such a simple self-similar scaling law does come under the principle of relativity extended to scale transformations of the resolutions. The above quantities transform, under a scale transformation $\varepsilon \to \varepsilon'$, as

$$\ln \frac{\mathcal{L}(\varepsilon')}{\mathcal{L}_0} = \ln \frac{\mathcal{L}(\varepsilon)}{\mathcal{L}_0} + \delta(\varepsilon) \ln \frac{\varepsilon}{\varepsilon'} \; , \tag{10}$$

$$\delta(\varepsilon') = \delta(\varepsilon). \tag{11}$$

These transformations have exactly the mathematical structure of the Galileo group (applied here to scale rather than motion), as confirmed by the dilation composition law, $\varepsilon \to \varepsilon' \to \varepsilon''$, which writes

$$\ln \frac{\varepsilon''}{\varepsilon} = \ln \frac{\varepsilon'}{\varepsilon} + \ln \frac{\varepsilon''}{\varepsilon'} \; , \tag{12}$$

and is therefore similar to the law of composition of velocities. Since the Galileo group of motion transformations is known to be the simplest group that implements the principle of relativity, the same is true for scale transformations.

4.1.4 Scale transition

However, it is important to note that Eq. (6) gives, in addition, a transition from a fractal to a non-fractal behavior at scales larger than some transition scale λ. In other words, contrarily to the case of motion laws, for which the invariance group is universal, the scale group symmetry is broken beyond some (relative) transition scale.

Indeed, Eq.(5) is more general that the mere renormalisation argument, under which the constant a would vanish. Two arguments lead us to keep this constant:

(i) A research of generality (we look for the most general among the simplest laws).

(ii) The fact that the new scale laws do not take the place of motion-displacement laws, but instead must be combined with them. Starting from a strictly scale-invariant law ($a = 0$), and adding a translation in standard position space ($\mathcal{L} \rightarrow \mathcal{L} + \mathcal{L}_0$), we indeed recover the broken solution ($a \neq 0$, which is asymptotically scale-dependent (in a scale-invariant way) and independent of scale beyond some transition.

The scale symmetry is therefore spontaneously broken by the very existence of the standard space-time symmetries (here, the translations, that are part of the full Poincaré group of space-time transformations including also the rotations and the Lorentz boosts). The symmetry breaking is not achieved here by a suppression of one law to the profit of the other, but instead by a domination of each law (scale vs motion) over the other respectively toward the small and large scales. Since the transition is itself relative (on the state of motion of the reference system) this implies that one can jump from a behavior to the other by a change of the reference system. As we shall see in what follows, this transition plays an important role in the fractal space-time approach to quantum mechanics, since we identifiy it with the Einstein-de Broglie scale, and therefore the fractal-non fractal transition with the quantum-classical transition [4].

4.1.5 Scale relativity versus scale invariance

Let us briefly be more specific about the way the scale-relativity viewpoint differs from 'scaling' or simple 'scale invariance'. In the standard concept of scale invariance, one considers scale transformations of the coordinate,

$$X \rightarrow X' = q \times X, \tag{13}$$

then one looks for the effect of such a transformation on some function $f(X)$. It is scaling when

$$f(qX) = q^{\alpha} \times f(X) \tag{14}$$

The scale relativity approach involves a more profound level of description, since the coordinate X is now explicitly resolution-dependent, i.e. $X = X(\varepsilon)$. Therefore we now look for a scale transformation of the resolution,

$$\varepsilon \rightarrow \varepsilon' = \rho\varepsilon, \tag{15}$$

which implies a scale transformation of the position variable

$$X(\rho\varepsilon) = \rho^{-\delta} X(\varepsilon). \tag{16}$$

But now the scale factor on the variable gets a physical meaning which goes beyond a trivial change of units. It corresponds to a coordinate measured on a fractal curve of fractal dimension $D = 1 + \delta$ at two different resolutions. Finally, one can also consider again a scaling function of a fractal coordinate:

$$f(\rho^{-\delta} X) = \rho^{-\alpha\delta} \times f(X). \tag{17}$$

In the framework of the analogy with the laws of motion and displacement, the dilation (13) is the equivalent of a static translation $x' = x + a$. Indeed, it reads in logarithmic form

$$\ln \frac{X'}{\lambda} = \ln \frac{X}{\lambda} + \ln q, \tag{18}$$

Note that it can also be generalized to four different dilations on the four coordinates, $\ln(X'_\mu/\lambda) = \ln(X_\mu/\lambda) + \ln q_\mu$. One jumps from static translation $x' = x + a$ to motion by introducing a time dependent translation $a = -vt$, so that one obtains the Galileo law of coordinate transformation, $x' = x - vt$. The passage from a simple dilation law $\ln X' = \ln X + \ln q$ to the law of scale transformation of a fractal self-similar curve, $\ln X' = \ln X - \delta \times \ln \rho$ is therefore of the same nature. In other words, fractals are to scale invariance what motion is to static translations.

These 'scale-translations' should not be forgotten when constructing the full scale-relativistic group of transformations (in similarity with the Poincaré group, that adds four space-time translations to the Lorentz group of rotation and motion in space (i.e., rotation in space-time).

It is also noticeable here that such a scale-relativity group will be different and larger than a conformal group, for the two reasons outlined in this section:

(i) The conformal group adds to the Poincaré group a global dilatation and an inversion (that leads to four special conformal transformations when combined with translations), yielding a 15 parameter group. But these transformations are applied to the coordinates without specification of their physical cause. In scale relativity, the cause is the fractality, i.e. the resolution dependence of the coordinates. For example, the symmetric element in a resolution transformation is $\ln(\lambda/\varepsilon') = -\ln(\lambda/\varepsilon)$, which is nothing but a resolution inversion $\varepsilon' = \lambda^2/\varepsilon$. A fractal coordinate which is resolution-dependent as a power law, $L(\varepsilon) = (\lambda_0/\varepsilon)^\delta$, is therefore itself transformed by an inversion, namely $L(\varepsilon') = L_1/L(\varepsilon)$, where $L_1 = (\lambda_0/\lambda)^\delta$.

(ii) Ultimately we need to define four independent resolution transformations on the four coordinates. Such a transformation does not preserve the angles and it therefore goes beyond the conformal group (see Sec. 4.3.6).

4.2 Special scale-relativity

4.2.1 Theory

The question that we shall now address is that of finding the laws of scale transformations that meet the principle of scale relativity. Up to now, we have characterized typical scale laws as the simplest possible laws, namely, those which are solutions of the simplest form of linear scale differential equations: this reasoning has provided us with the standard, power-law, fractal behavior with constant fractal dimension in the asymptotic domain. But are the simplest possible laws those chosen by nature? Experience in the construction of the former physical theories suggests that the correct and general laws are simplest among those which satisfy some fundamental principle, rather than those which are written in the simplest way: anyway, these last laws are often approximations of the correct, more general laws. Good examples of such relations between theories are given by Einstein's motion special relativity, of which the Galilean laws of inertial motion are low velocity approximations, and by Einstein's general relativity, which includes Newton's theory of gravitation as an approximation. In both cases, the correct laws are constructed from the requirement of covariance, rather than from the too simple requirement of invariance.

The theory of scale relativity [16, 4] proceeds along a similar reasoning. The principle of scale relativity may be implemented by requiring that the equations of physics be written in a covariant way under scale transformations of resolutions. Are the standard scale laws (those described by renormalization-group-like equations, or by a fractal power-law behavior) scale-covariant ? They are usually described (far from the transition to scale-independence) by asymptotic laws such as $\mathcal{L} = \mathcal{L}_0(\lambda/\varepsilon)^\delta$, with δ a *constant* scale-dimension (which may differ from the standard value $\delta = 1$ by an anomalous dimension term [32]). This means that, as we have recalled hereabove, a scale transformation $\varepsilon \to \varepsilon'$ can be written:

$$\ln \frac{\mathcal{L}(\varepsilon')}{\mathcal{L}_0} = \ln \frac{\mathcal{L}(\varepsilon)}{\mathcal{L}_0} + \mathbb{V}\ \delta(\varepsilon), \tag{19}$$

$$\delta(\varepsilon') = \delta(\varepsilon),$$

where we have set:

$$\mathbb{V} = \ln(\varepsilon/\varepsilon'). \tag{20}$$

The choice of a logarithmic form for the writing of the scale transformation and the definition of the fundamental resolution parameter $\mathbb{V}$ is justified by the expression of the dilatation operator $\tilde{D} = \partial/\partial \ln \varepsilon$. The relative character of $\mathbb{V}$ is evident: in the same way that only velocity differences have a physical meaning (Galilean relativity of motion), only $\mathbb{V}$ differences have a physical meaning (relativity of scales). We have then suggested [16] to characterize this relative resolution parameter $\mathbb{V}$ as a 'state of scale' of the coordinate system, in analogy with Einstein's formulation of the principle of relativity [1], in which the relative velocity characterizes the state of motion of the reference system.

Now in such a frame of thought, the problem of finding the laws of linear transformation of fields in a scale transformation $\mathbb{V} = \ln \rho$ ($\varepsilon \to \varepsilon'$) amounts to finding four quantities, $a(\mathbb{V}), b(\mathbb{V}), c(\mathbb{V})$, and $d(\mathbb{V})$, such that

$$\ln \frac{\mathcal{L}'}{\mathcal{L}_0} = a(\mathbb{V})\ \ln \frac{\mathcal{L}}{\mathcal{L}_0} + b(\mathbb{V})\, \delta, \tag{21}$$

$$\delta' = c(\mathbb{V})\ \ln \frac{\mathcal{L}}{\mathcal{L}_0} + d(\mathbb{V})\, \delta.$$

Set in this way, it immediately appears that the current 'scale-invariant' scale transformation law of the standard form (Eq. 19), given by $a = 1, b = \mathbb{V}, c = 0$ and $d = 1$, corresponds to the Galileo group.

This is also clear from the law of composition of dilatations, $\varepsilon \to \varepsilon' \to \varepsilon$", which has a simple additive form,

$$\mathbb{V}" = \mathbb{V} + \mathbb{V}'. \tag{22}$$

However the general solution to the 'special relativity problem' (namely, find a, b, c and d from the principle of relativity) is the Lorentz group [33]. In particular, we have proved [16] that, for two variables, only 3 axioms were needed (linearity, internal composition law and reflection invariance) Then we have suggested to replace the standard law of dilatation, $\varepsilon \to \varepsilon' = \varrho\varepsilon$ by a new Lorentzian relation [16]. However, while the relativistic symmetry is universal in the case of the laws of motion, this is not true for the laws of scale. Indeed, physical laws are no longer dependent on resolution for scales larger than the classical-quantum transition (identified with the fractal-nonfractal transition in our approach) which has been analysed above. This implies that the dilatation law must remain Galilean above this transition scale.

For simplicity, we shall consider in what follows only the one-dimensional case. We define the resolution as $\varepsilon = \delta x = c\delta t$, and we set $\lambda_0 = c\tau_{dB} = \hbar c/E$. In its rest frame, λ_0 is thus the Compton length of the system or particle considered, i.e., in the first place the Compton length of the electron (this will be better justified in Section 6). The new law of dilatation reads, for $\varepsilon < \lambda_0$ and $\varepsilon' < \lambda_0$

$$\ln \frac{\varepsilon'}{\lambda_0} = \frac{\ln(\varepsilon/\lambda_0) + \ln \varrho}{1 + \ln \varrho \, \ln(\varepsilon/\lambda_0)/\ln^2(\mathbf{\Lambda}/\lambda_0)}. \tag{23}$$

This relation introduces a fundamental length scale $\mathbf{\Lambda}$, that we have identified (toward the small scales) with the Planck length (currently $1.6160(11) \times 10^{-35}$ m),

$$\mathbf{\Lambda} = l_{\mathbb{P}} = (\hbar G/c^3)^{1/2}. \tag{24}$$

But, as one can see from Eq.(23), if one starts from the scale $\varepsilon = \mathbf{\Lambda}$ and apply any dilatation or contraction ϱ, one gets back the scale $\varepsilon' = \mathbf{\Lambda}$, whatever the initial value of λ_0 (i.e., whatever the state of motion, since λ_0 is Lorentz-covariant under *velocity* transformations). In other words, $\mathbf{\Lambda}$ is now interpreted as a limiting lower length-scale, impassable, invariant under dilatations and contractions. In the simplified case of a transformation from $\mathcal{L}_0$ to $\mathcal{L}$(see [16, 4] for general expressions), the length measured along a fractal coordinate, that was previously scale-dependent as $\ln(\mathcal{L}/\mathcal{L}_0) = \delta_0 \ln(\lambda_0/\varepsilon)$ for $\varepsilon < \lambda_0$ becomes in the new framework (see Fig. 2)

$$\ln(\mathcal{L}/\mathcal{L}_0) = \frac{\delta_0 \ln(\lambda_0/\varepsilon)}{\sqrt{1 - \ln^2(\lambda_0/\varepsilon)/\ln^2(\lambda_0/\mathbf{\Lambda})}}. \tag{25}$$

The main new feature of scale relativity respectively to the previous fractal or scale-invariant approaches is that the scale dimension δ and the fractal dimension $D_F = 1 + \delta$, which were previously constant ($D_F = 2, \delta = 1$), are now explicitly varying with scale (see Fig. 2), following the law (given once again in the simplified case when we start from the referece scale $\mathcal{L}_0$):

$$\delta(\varepsilon) = \frac{\delta_0}{\sqrt{1 - \ln^2(\lambda_0/\varepsilon)/\ln^2(\lambda_0/\mathbf{\Lambda})}}. \tag{26}$$

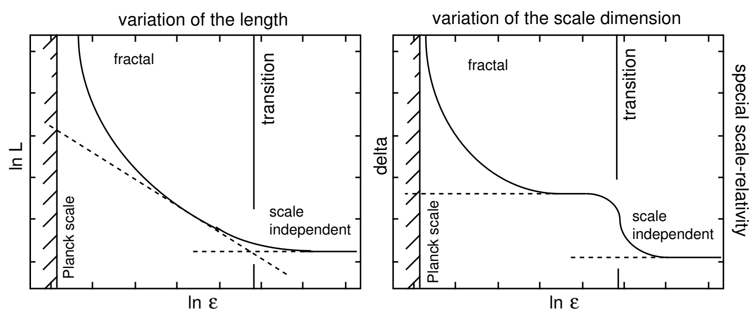

Figure 2: Scale dependence of the length and of the effective scale-dimension in the case of scale-relativistic Lorentzian scale laws.

These new laws corresponds to a Minkowskian scale metrics invariant that reads:

$$d\sigma^2 = d\delta^2 - \frac{(d \ln \mathcal{L})^2}{\mathbb{C}^2}. \tag{27}$$

For a more complete development of special relativity (including its implications as regards new conservative quantities), see Refs. [16, 4, 23].

4.2.2 Applications

The theory of special scale-relativity has many consequences in two domains of physics (we shall not develop them further in the present contribution for lack of place: we refer the interested reader to the quoted references, in particular the two review papers [23, 35]).

(i) *High energy and elementary particle physics* [16, 4, 23, 34] (with applications to primeval cosmology [23, 35]). For example, one can apply this description to the 'internal' structures of the electron (identified with the fractal geodesics of a fractal space-time). This means that the fractal dimension jumps from $D_F = 1$ to $D_F = 2$ at the electron Compton scale $\lambda_0 = \lambda_e = \hbar/m_e c$ (see the construction of quantum laws in what follows), then begins to vary with scale. Its variation is first very slow (quadratic in log of scale):

$$D_F(\varepsilon) = 2(1 + \frac{1}{4}\frac{\mathbb{V}^2}{\mathbb{C}_0^2} + ...), \tag{28}$$

where $\mathbb{V} = \ln(\lambda_0/\varepsilon)$ and $\mathbb{C}_0 = \ln(\lambda_0/l_{\mathbb{P}})$, $l_{\mathbb{P}}$ being the Planck length-scale. Then it tends to infinity at very small scales when $\mathbb{V} \to \mathbb{C}_0$, i.e., $\varepsilon \to \mathbf{\Lambda} = l_{\mathbb{P}}$.

The new status of the Planck length-scale (and time-scale), identified with a minimal scale, invariant under dilations and contractions, implies new relations between length-time scales and momentum-energy scales [4, 23]. These relations involve new log-Lorentz factors that allow to solve some remaining problems of the standard model, such as the divergence problem of masses and charges, and the hierarchy problem between the GUT scale and the electroweak scale [23], and to suggest new methods for understanding the mass and charge spectrum of elementary particles [46, 23].

(ii) *Cosmology* [4, 23, 35]. We have suggested that log-Lorentz dilation transformations were also relevant at very large scales. In this case the invariant scale

$\mathbf{\Lambda}$ becomes a maximal length-scale, invariant under dilations, that we have identified with the length-scale $\mathbf{\Lambda} = \Lambda_c^{-1/2}$ that can be constructed from the cosmological constant Λ_c (which is the inverse of the square of a length). Such an identification brings new light about the nature and the value of the cosmological constant: its value theoretically predicted in [4], $\Lambda_{c,\text{pred}} = 1.36 \times 10^{-56}$ cm^{-2} is supported by its recent determination from observational measurements: $\Lambda_{c,\text{obs}} = (1.29 \pm 0.23) \times 10^{-56}$ cm^{-2} [36]. This new approach also leads to a description of the large-scale Universe in which the fractal dimension of the distribution of matter is increasing with scale (following the law of Eq. 26, reaching the value 3 (uniformity) at a scale of about 750 Mpc, in agreement with observational data.

Another application of these new laws to turbulence has also been suggested by Dubrulle and Graner [37], but with a different interpretation of the variables.

4.3 Generalized scale laws

4.3.1 Discrete scale invariance, complex dimension and log-periodic behavior

A correction to pure scale invariance is potentially important, namely the log-periodic correction to power laws that is provided, e.g., by complex exponents or complex fractal dimensions [7]. Sornette et al. (see [38] and references therein) have shown that such a behavior provides a very satisfactory and possibly predictive model of the time evolution of many critical systems, including earthquakes and market crashes [39]. More recently, it has been applied to the analysis of major event chronology of the evolutionary tree of life [40, 41], of human development [43] and of the main economic crisis of western and Precolumbian civilizations [41, 44].

Let us show how one can recover log-periodic corrections from requiring scale covariance of the scale differential equations [27]. Consider a scale-dependent function $\mathcal{L}(\varepsilon)$, (it may be for example the length measured along a fractal curve). In the applications to temporal evolution quoted above, the scale variable is identified with the time interval $|t - t_c|$, where t_c is the date of crisis. Assume that $\mathcal{L}$ satisfies a renormalization-group-like first order differential equation,

$$\frac{d\mathcal{L}}{d\ln\varepsilon} - \nu\mathcal{L} = 0, \tag{29}$$

whose solution is a power law $\mathcal{L}(\varepsilon) \propto \varepsilon^\nu$. Now looking for corrections to this law, we remark that simply jumping to a complex value of the exponent ν would lead to large log-periodic fluctuations rather than to a controlable correction to the power-law. So let us assume that the right-hand side of Eq. 29 actually differs from zero

$$\frac{d\mathcal{L}}{d\ln\varepsilon} - \nu\mathcal{L} = \chi. \tag{30}$$

We can now apply the scale-covariance principle and require that the new function χ be solution of an equation which keeps the same form as the initial equation

$$\frac{d\chi}{d\ln\varepsilon} - \nu'\chi = 0. \tag{31}$$

Setting $\nu' = \nu + \eta$, we find that $\mathcal{L}$ must be solution of a second-order equation

$$\frac{d^2\mathcal{L}}{(d\ln\varepsilon)^2} - (2\nu + \eta)\frac{d\mathcal{L}}{d\ln\varepsilon} + \nu(\nu + \eta)\mathcal{L} = 0. \tag{32}$$

It writes $\mathcal{L}(\varepsilon) = a\varepsilon^\nu(1 + b\varepsilon^\eta)$, and finally, the choice of an imaginary exponent $\eta = i\omega$ yields a solution whose real part includes a log-periodic correction:

$$\mathcal{L}(\varepsilon) = a\,\varepsilon^\nu\,[1 + b\cos(\omega \ln \varepsilon)]. \tag{33}$$

Adding a constant term provides a transition to scale independence at large scales (see Fig. 3).

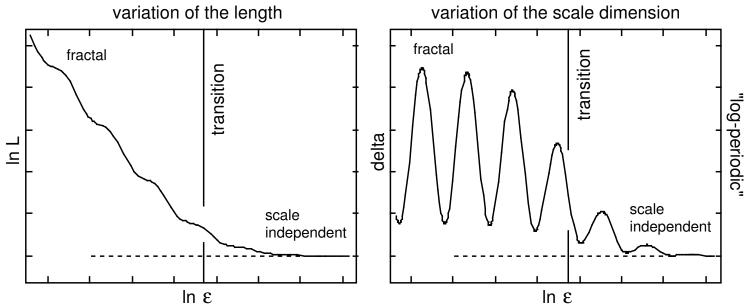

Figure 3: Scale dependence of the length and of the scale dimension in the case of a log-periodic behavior with fractal / nonfractal transition, $\mathcal{L}(\varepsilon) = \mathcal{L}_0[1 + (\lambda/\varepsilon)^\nu\, e^{b\cos(\omega\ln(\varepsilon/\lambda))}]$.

Let us now give another physically meaningful way to obtain an equivalent behavior, that does not make use of imaginary exponents. Define a log-periodic local scale dimension:

$$\delta = \frac{\partial \ln \mathcal{L}}{\partial \ln \varepsilon} = \nu - b\,\omega \sin(\omega \ln \varepsilon) \tag{34}$$

It leads after integration to a scale-divergence that reads

$$\mathcal{L}(\varepsilon) = a\,\varepsilon^\nu\, e^{b\cos(\omega\ln\varepsilon)} \tag{35}$$

whose first order expansion is (33). Such a law is a solution of a scale stationary wave equation:

$$\frac{\partial^2}{(\partial \ln \varepsilon)^2} \ln \frac{\mathcal{L}}{\mathcal{L}_0} + \omega^2 \ln \frac{\mathcal{L}}{\mathcal{L}_0} = 0, \tag{36}$$

where $\mathcal{L}_0 = a\varepsilon^\nu$ is the strictly self-similar solution. Hence the log-periodic behavior can be viewed as a stationary wave in the scale-space (this prepares Sec. 9, in which we tentatively introduce a quantum wave scale equation). Note that these solutions can apply to fractal lengths only for $b\omega < \nu$, since the local. scale dimension should remain positive: this behavior is typical of what is observed when measuring the resolution-dependent length of fractal curves of the von Koch type which are built by iteration and, strictly, have only discrete scale invariance instead of a full continuous scale invariance. But such laws also apply to other kind of variables (for example market indices or ion concentration near earthquake zones, see [38]) for which local decreases are relevant.

We give in Fig. 4 an example of application of such log-periodic laws to the analysis of the chronology of species evolution (see more detail in Refs. [40, 41, 42]). One finds either an acceleration toward a critical date T_c or a deceleration from a critical date, T_c depending on the considered lineage.

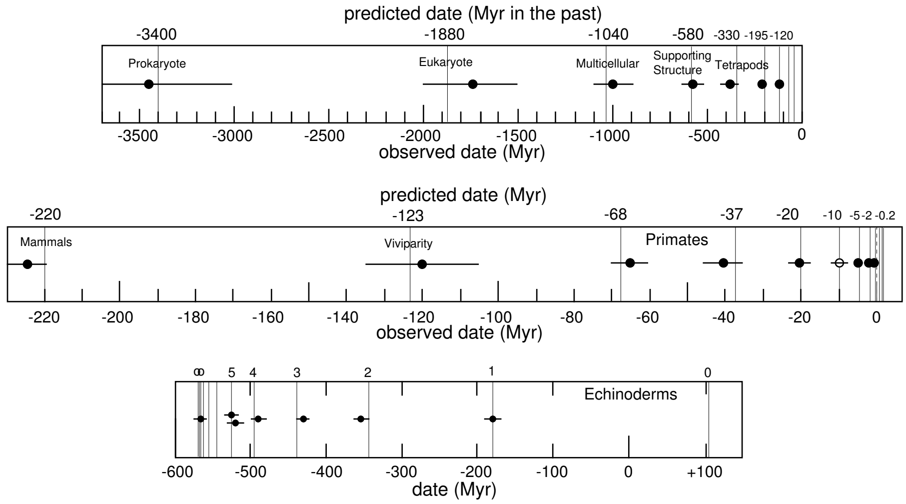

Figure 4: Three typical examples of log-periodic chronological laws in species evolution (from Refs.[40, 41, 42]). *Up*: main trunk of the 'tree of life', from the apparition of life to homeothermy and viviparity (last two plotted points). The dates are best-fitted by an accelerating log-periodic law $T_n = T_0 + (T_c - T_0) \times g^{-n}$, with $T_c = -30 \pm 60$ Myr and $g = 1.83$. *Middle*: comparison of the dates of the major leaps in the evolution of primates including the hominids (plus the two preceding main events: apparition of mammals and viviparity), with an accelerating log-periodic law. The best fit gives $T_c = 2.0 \pm 0.4$ Myr and $g = 1.78 \pm 0.01$. For the 14 dates from the origin of life to the appearance of the Homo sapiens bauplan, one finds $T_c = 2.1 \pm 1.0$ Myr and $g = 1.76 \pm 0.01$. The probability is less than 10^{-4} to obtain such a fit by chance. *Down*: comparison with a decelerating log-periodic law of the dates of the major leaps in the evolution of echinoderms. The best fit yields $T_c = -575 \pm 25$ Myr and $g = 1.67 \pm 0.02$ (origin excluded). This means that the critical time from which the deceleration starts is their date of apparition (within uncertainties). All results are statistically highly significant (see the quoted references for details about the data and their analysis).

4.3.2 Lagrangian approach to scale laws

The Lagrangian approach can be used in the scale space in order to obtain physically relevant generalizations of the above simplest (scale-invariant) laws. In this aim, we are led to reverse the definition and meaning of the variables. Namely, the scale dimension δ becomes a primary variable that plays, for scale laws, the same role as played by time in motion laws. We have suggested to call 'djinn' this variable scale dimension.

The resolution, ε, can therefore be defined as a derived quantity in terms of the fractal coordinate $\mathcal{L}$ and of the scale dimension or djinn, δ

$$\mathbb{V} = \ln\left(\frac{\lambda}{\varepsilon}\right) = \frac{d\ln\mathcal{L}}{d\delta} \ . \tag{37}$$

A scale Lagrange function $\tilde{L}(\ln\mathcal{L}, \mathbb{V}, \delta)$ is introduced, from which a scale action is constructed

$$\tilde{S} = \int_{\delta_1}^{\delta_2} \tilde{L}(\ln\mathcal{L}, \mathbb{V}, \delta)\ d\delta. \tag{38}$$

The application of the action principle yields a scale Euler-Lagrange equation that writes

$$\frac{d}{d\delta}\frac{\partial \tilde{L}}{\partial \mathbb{V}} = \frac{\partial \tilde{L}}{\partial \ln \mathcal{L}} \, . \tag{39}$$

In analogy with the physics of motion, in the absence of any "scale-force" (i.e., $\partial\tilde{L}/\partial \ln \mathcal{L} = 0$), the Euler-Lagrange equation becomes

$$\partial\tilde{L}/\partial \mathbb{V} = \text{const} \Rightarrow \mathbb{V} = \text{const}. \tag{40}$$

which is the equivalent for scale of what inertia is for motion. The simplest possible form for the Lagrange function is a quadratic dependence on the "scale velocity", (i.e., $\tilde{L} \propto \mathbb{V}^2$). The constancy of $\mathbb{V} = \ln(\lambda/\varepsilon)$ means that it is independent of the djinn δ. Equation (37) can therefore be integrated to give the usual power law behavior, $\mathcal{L} = \mathcal{L}_0(\lambda/\varepsilon)^\delta$. This reversed viewpoint has several advantages which allow a full implementation of the principle of scale relativity:

(i) The djinn δ is given its actual status of a fifth dimension or "scale time" and the logarithm of the resolution, $\mathbb{V}$, its status of "scale velocity" (see Eq. 37). This is in accordance with its scale-relativistic definition, in which it characterizes the "state of scale" of the reference system, in the same way as the velocity $v = dx/dt$ characterizes its state of motion.

(ii) This leaves open the possibility of generalizing our formalism to the case of four independent space-time resolutions, $\mathbb{V}^\mu = \ln(\lambda^\mu/\varepsilon^\mu) = d\ln\mathcal{L}^\mu/d\delta$. This amount to jump to a five-dimensional geometric description in terms of a space-time-djinn. Note in this respect that the genuine nature of resolutions is tensorial, $\varepsilon_\mu^\nu = \varepsilon_\mu\varepsilon^\nu = \rho_{\mu\lambda}\varepsilon^\nu\varepsilon^\lambda$ and involves correlation coefficients, in analogy with variance-covariance matrices.

(iii) Scale laws more general than the simplest self-similar ones can be derived from more general scale Lagrangians [26].

4.3.3 Scale 'dynamics'

The whole of our previous discussion indicates to us that the scale invariant behavior corresponds to 'freedom' (i.e. scale forec-free behavior) in the framework of a scale physics. However, in the same way as there exists forces in nature that imply departure from inertial, rectilinear uniform motion, we expect most natural fractal systems to also present distorsions in their scale behavior respectively to pure scale invariance. This means taking non-linearity in scale into account. Such distorsions may be, as a first step, attributed to the effect of a scale "dynamics", i.e. of a "scale-field". (Caution: at this level of description, this is only an analog of dynamics, which acts on the scale axis, on the internal structures of the system under consideration, not in space-time. See what follows for the effects of coupling with space-time displacements).

In this case the Lagrange scale-equation takes the form of Newton's equation of dynamics:

$$F = \mu \frac{d^2 \ln \mathcal{L}}{d\delta^2}, \tag{41}$$

where μ is a 'scale-mass', which measures how the system resists to the 'scale-force', and where $\Gamma = d^2 \ln \mathcal{L}/d\delta^2 = d\ln(\lambda/\varepsilon)/d\delta$ is the 'scale-acceleration'.

We shall now attempt to define physical, generic, scale-dynamical behaviors which could be common to very different systems. For various systems the scale-force may have very different origins, but in all cases where it has the same form (constant, harmonic oscillator, etc...), the same kind of scale behavior would be obtained. It is also worthwhile to remark that such a 'Newtonian' approach is itself considered to be only an intermediate step while waiting for a fully developped general scale-relativity. Thus the scale-forces are expected to be finally recovered as approximations of the manifestations of the geometry of the scale-space.

4.3.4 Constant scale-force

Let us first consider the case of a constant scale-force. The potential is $\varphi = F \ln \mathcal{L}$, and Eq. 41 writes

$$\frac{d^2 \ln \mathcal{L}}{d\delta^2} = G, \tag{42}$$

where $G = F/\mu = cst$. It is easily integrated in terms of a parabolic solution (which is the equivalent for scale laws of parabolic motion in a constant field):

$$\mathbb{V} = \mathbb{V}_0 + G\delta \quad ; \quad \ln \mathcal{L} = \ln \mathcal{L}_0 + \mathbb{V}_0 \delta + \frac{1}{2} G\delta^2. \tag{43}$$

However the physical meaning of this result is not clear under this form. This is due to the fact that, while in the case of motion laws we search for the evolution of the system with time, in the case of scale laws we search for the dependence of the system on resolution, which is the directly measured observable. We find, after redefinition of the integration constants:

$$\delta = \delta_0 + \frac{1}{G} \ln \left(\frac{\lambda}{\varepsilon} \right) \quad ; \quad \ln \left(\frac{\mathcal{L}}{\mathcal{L}_0} \right) = \frac{1}{2G} \ln^2 \left(\frac{\lambda}{\varepsilon} \right). \tag{44}$$

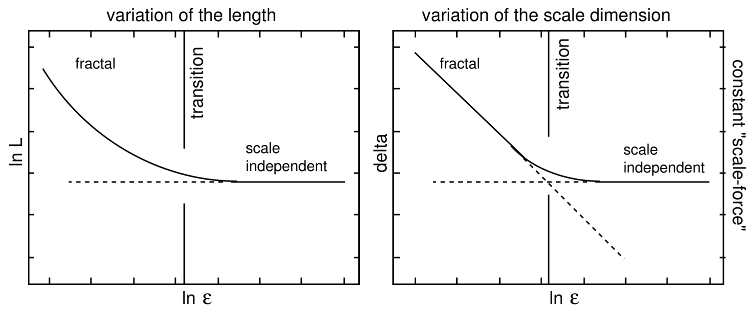

Figure 5: Scale dependence of the length and of the effective scale-dimension in the case of a constant 'scale-force'.

The scale dimension δ becomes a linear function of resolution (the same being true, as a consequence, of the fractal dimension $D_F = 1 + \delta$), and the $(\log \mathcal{L}, \log \varepsilon)$ relation is now parabolic instead of linear (see Fig. 5). There are several physical situations where, after careful examination of the data, the power-law models were clearly rejected since no constant slope could be defined in the $(\log \mathcal{L}, \log \varepsilon)$ plane. In the several cases where a clear curvature appears in this plane (e.g., turbulence, sand piles, ...), the physics could come under such a 'scale-dynamical' description. In these cases it might be of interest to identify and study the scale-force responsible for the scale distorsion (i.e., for the deviation to standard scaling)..

4.3.5 Scale harmonic oscillator

Another interesting case of scale-potential is that of a repulsive harmonic oscillator, $\varphi = -(\ln \mathcal{L}/\delta)^2/2$. It is solved as

$$\ln \frac{\mathcal{L}}{\mathcal{L}_0} = \delta \sqrt{\ln^2 \left(\frac{\lambda}{\varepsilon}\right) - \frac{1}{\delta^2}}. \tag{45}$$

For $\varepsilon << \lambda$ it gives the standard Galilean case $\mathcal{L} = \mathcal{L}_0(\lambda/\varepsilon)^\delta$, but its large-scale behavior is particularly interesting, since it does not permit the existence of resolutions larger than a scale

$$\lambda_{\max} = \lambda \times e^{-1/\delta}, \tag{46}$$

where λ is the fractal / non-fractal transition scale for the asymptotic domain (see Figure 6). In other words, the transition scale is replaced by a smaller scale at which the effective fractal dimension becomes (formally) infinite.

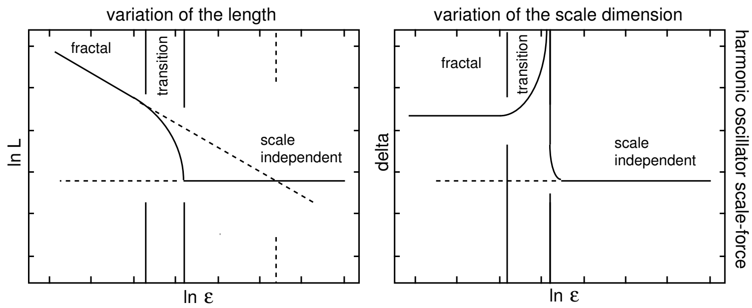

Figure 6: Scale dependence of the length and of the scale dimension in the case of a harmonic oscillator scale-potential.

We hope such a behavior to provide a model of confinement in QCD. Indeed, the gauge symmetry group of QCD, SU(3), is the dynamical symmetry group of a 3-dimensional isotropic harmonic oscillator, while gauge invariance can be re-interpreted in scale relativity as scale invariance on space-time resolutions (see following section).

This suggestion is re-inforced by the following results and remarks:

(i) QCD is precisely characterized by the property of 'asymptotic freedom, which means that quarks become free at small scales while the strong coupling constant increases toward large scales. It may become formally infinite at the confinement scale.

(ii) There is increasing evidence for an internal fractal structure of the proton, more generally of hadrons [45].

(iii) Free u and d quark masses (i.e., the masses they would have in the absence of confinement) are far smaller than their effective mass in the proton and neutron. This means that their Compton length $\lambda_c = \hbar/mc$ is larger than the confinement scale (of order 1 Fermi). This is exactly what is expected in the above model. Indeed, as we shall see in what follows, the fractal/non fractal transition is identified in rest frame with the Compton length of a particle. The above equation (46) can therefore be interpreted as a relation between the free quark masses, the confinement scale and an internal fractal dimension, that may be tested in the future.

(iv) A 3-dimensional sphere is scale space $(\ln X, \ln Y, \ln Z)$ becomes, when viewed in terms of direct variables (X, Y, Z) and for large values of the variables, a triad (see Fig.7). Such a behavior may provide a model of color and quarks, in which the three quarks in hadrons would have no real separated existence, but could be identified with the three extremities of such a '3-tip string'. Their appearance could therefore be the mere result of a change of reference system (i.e., of relativity), provided the genuine physical variables for the description of intra-hadron physics be the scale variables $(\ln X, \ln Y, \ln Z)$, in terms of which there is pure isotropy on the scale-sphere, while our measurement devices work in terms of the (X, Y, Z) variables.

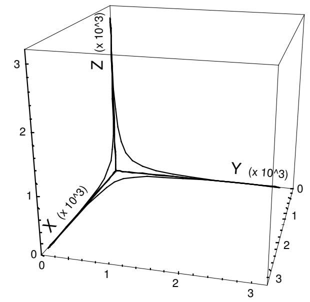

Figure 7: 'Three-tip string'. Three dimensional sphere in scale space, $(\ln X)^2 + (\ln Y)^2 + (\ln Z)^2 = A^2$, plotted in terms of direct variables (X, Y, Z), for $A = 8$.

A possible other interpretation of this scale harmonic oscillator model is possible. If we interpret the variable ε in the above equations as a distance r from a center, then one describes a system in which the effective fractal dimension of trajectories diverges at some distance $r = \lambda_{\max}$ from the center, is larger than 1 in the inner region and becomes 1 (i.e., non-fractal in the outer region). Since the increase of the fractal dimension of a curve corresponds to the increase of its 'thickness' (see [4] p. 80 for an example of a fractal curve whose fractal dimension varies with position), such a model can be interpreted as describing a system in which the inner and outer domains are separated by a wall or membrane. Such a solution of non-linear scale equations may therefore have biological applications.

4.3.6 Toward a generalized theory of scale-relativity

The above approach in terms of 'scale dynamics' is actually intended to be a provisional description. Indeed, in analogy with Einstein's general relativity of motion, in which Newton's gravitational 'force' becomes a mere manifestation of space-time curvature, we hope that the scale dynamical forces introduced hereabove are only intermediate and practical concepts, which should ultimately be recovered as mere manifestation of the fractality of space-time.

We shall be led in future developments of the theory to look for the general non-linear scale laws that satisfy the principle of scale relativity, and also to treat scale-laws and motion laws on the same footing. We have suggested that, in this purpose, a change of representation was necessary.

Namely, recall that our first proposal [13] was to work in a 'fractal space-time' representation, involving four coordinates that are explicit functions of the four space-time resolutions, $\{X(\varepsilon_x), Y(\varepsilon_y), Z(\varepsilon_z), T(\varepsilon_t)\}$. Since the resolutions are interpreted as characterizing the state of scale of the reference system, this eight variable representation can be viewed as the equivalent of a 'phase-space' representation. Now, much work remains to be done, since we have mainly considered, up to now, the simplified case of only one resolution variable. We shall be led in future works to define four different transformations on the four space-time resolutions.

Moreover, in this representation gauge transformations emerge as a manifestation of the coupling between scale and motion (see the following Section 7). Namely, we have seen that the coordinates are generally split into two terms, a scale-dependent part and a scale-independent part that is identified with the classical non-fractal coordinates $\{x, y, z, t\}$. In other words, the classical differentiable space-time is recovered as a large-scale degeneracy of the microscopic fractal space-time. This allows one to consider resolutions that now vary with the classical coordinates, and therefore to have a more profound description in terms of fractal coordinates $X^\mu[\varepsilon(x, y, z, t)]$. As we shall see, such an approach leads to a new interpretation of gauge invariance and of gauge fields (we shall only consider here the simple Abelian gauge field case of the electromagnetic type).

Now, in the case when the fractal dimension is no longer constant, we have proposed another representation in terms of a five-dimensional 'spacetime-djinn', in which the scale dimension becomes itself a fifth dimension. Note that the coordinates themselves are not scale-dependent in an essential way, but only coordinate intervals: indeed, as emphasized previously and will be further developed in the following, they can be separated in two parts, a classical scale-independent part and a fractal fluctuation with vanishing mathematical expectation. Only the fractal fluctuation depends on scale. Therefore the spacetime-djinn combines space and time intervals with the djinn, i.e., it is defined in terms of the variables $\{dX, dY, dZ, dT, \delta\}$. It itself involves two levels:

(i) A 'Galilean' scale-relativity description, in which the djinn δ, though possibly variable, remains a parameter which is separated from the four space-time coordinates, i.e. $\{dX(\delta),\ dY(\delta),\ dZ(\delta),\ dT(\delta)\}$. It allows one to implement the identification of resolutions with 'scale-velocities',

$$\ln\left(\frac{\lambda^\mu}{\varepsilon^\mu}\right) = \frac{d\ln|dX^\mu|}{d\delta}. \tag{47}$$

In this framework, the scale-motion coupling laws introduced to account for gauge transformations are recovered as second order derivatives involving the djinn and the coordinates ($\partial^2/\partial\delta\partial x$), and they therefore appear on the same footing as motion accelerations ($\partial^2/\partial t^2$) and 'scale-accelerations' ($\partial^2/\partial\delta^2$).

(ii) A fully covariant scale-relativistic description in terms of five fractal variables χ^α, with $\alpha = 0$ to 4 and signature $(+,-,-,-,-)$. Its simplified two-dimensional version has already been explicitly given in the above description of special-scale relativity, involving a log-Lorentzian law of dilation. In this representation, the four dilations on the four space-time resolutions are identified with rotations in the spacetime-djinn, i.e., to log-Lorentz scale-boosts. In this respect the final group of relativity (motion + scale) is beyond the conformal group, since it contains independent dilations of the four coordinates (fractal parts), which do not conserve angles. It is expected to be at least a combination of the Poincaré group for the 4-dimensional standard space-time ('classical' variables) and of a group of generators $\chi_\alpha \partial_\beta$ (including 'scale-rotations' $(\chi_\alpha \partial_\beta - \chi_\beta \partial_\alpha)/2$). Moreover, as already remarked, the true nature of the resolutions is tensorial instead of vectorial. All these points will be taken into account in the future developments of the theory [47].

5 Fractal space and induced quantum mechanics

5.1 Introduction

Let us now consider an essential part of the theory of scale relativity, namely, the description of the effects in standard space-time that are induced by the internal fractal structures in scale-space. The previous Sections were devoted to pure scale laws, i.e., to the description of the scale dependence of fractal trajectories at a given point of space-time. The question now addressed is: what are the consequences on motion of the internal fractal structures of space (more generally, of space-time)? This is a huge question that cannot be solved in one time. We therefore proceed by first studying the induced effects of the simplest scale laws (namely, self-similar laws of fractal dimension 2 for trajectories) under restricted conditions (only fractal space, then fractal space and time, breaking of symmetry on time, then also on space). As recalled in the following Sections, we successively recover in this way more and more profound levels of quantum mechanical laws: namely, non-relativistic quantum mechanics (Schrödinger equation), relativistic quantum mechanics without spin (Klein-Gordon equation) and for spinors (Dirac equation).More complicated situations (constant fractal dimension differing of 2, variable fractal dimension, special scale-relativistic log–Lorentzian behavior, etc...), that may lead to scale-relativistic corrections to standard quantum mechanics, have been tentatively considered in previous works [46, 23], but will not be recalled here.

5.2 Infinite number of geodesics

Strictly, the non-differentiability of the coordinates means that the velocity

$$V = \frac{dX}{dt} = \lim_{dt \to 0} \frac{X(t+dt) - X(t)}{dt} \tag{48}$$

is undefined. Namely, when dt tends to zero, either the ratio dX/dt tends to infinity, or it fluctuates without reaching any limit. However, as recalled in the introduction, continuity and non-differentiability imply an explicit dependence on scale of the various physical quantities. As a consequence, the velocity, V is itself re-defined as an explicitly resolution-dependent function $V(t, dt)$. In the simplest case, we expect that it is solution of a scale differential equation like Eq. 5, i.e.

$$V = v + w = v \left[1 + \zeta \left(\frac{\tau}{dt} \right)^{1-1/D_F} \right]. \tag{49}$$

This means that the velocity is now the sum of two independent terms of different orders of differentiation, since their ratio v/w is, from the standard viewpoint, infinitesimal. In analogy with the real and imaginary parts of a complex number, we have suggested [66] to call v the 'classical part' of the velocity, $v = \mathcal{C}\ell\langle V \rangle$, (see below the definition of the classical part operator $C\ell\langle\ \ \rangle$) and w its 'fractal part'. The new component w is an explicitely scale-dependent fractal fluctuation (which would be infinite from the standard point of view where one makes $dt \to \infty$) and τ and ζ are chosen such that $\mathcal{C}\ell\langle\zeta\rangle = 0$ and $\mathcal{C}\ell\langle\zeta^2\rangle = 1$.

The above description strictly applies for an individual fractal trajectory. Now, one of the direct geometric consequences of the non-differentiability and of the subsequent fractal character of space itself (not only of the trajectories) is that there is an infinity of fractal geodesics relating any couple of points of this fractal space [4]. This can be easily understood already at the level of fractal surfaces, which can be described in terms of a fractal distribution of conic points of positive and negative infinite curvature (see [4], Sec. 3.6 and 3.10). As a consequence, we are led to replace the velocity $V(t, dt)$ on a particular geodesic by the velocity field $V[x(t), t, dt]$ of the whole infinite ensemble of geodesics. Moreover, this fundamental and irrepressible loss of information of purely geometric origin means the giving-up of determinism at the level of trajectories and leads to jump to a statistical and probabilistic description. But here, contrarily to the view of standard quantum mechanics, the statistical nature of the physical tool is not set as a foundation of physics, but derived from geometric properties.

We have therefore suggested [13] that the description of a quantum mechanical particle, including its property of wave-particle duality, could be reduced to the geometric properties of the set of fractal geodesics that corresponds to a given state of this "particle". In such an interpretation, we do not have to endow the "particle" with internal properties such as mass, spin or charge, since the "particle" is not identified with a point mass which would follow the geodesics, but its "internal" properties can now be defined as global geometric properties of the fractal geodesics themselves. As a consequence, any measurement is interpreted as a sorting out (or selection) of the geodesics: as an example, if the "particle" has been observed at a given position with a given resolution, this means that the geodesics which pass through this domain have been selected [4, 13].

5.3 'Classical part' and 'fractal part' of differentials

The transition scale appearing in Eq. (49) yields two distinct behaviors of the system (i.e., the 'particle', identified with an infinite family of geodesics of the fractal space) depending on the resolution at which it is considered. Equation (49) multiplied by dt gives the elementary displacement, dX, of the system as a sum of two infinitesimal terms of different orders

$$dX = dx + d\xi. \tag{50}$$

The variable

$$dx = \mathcal{C}\ell\langle dX\rangle \tag{51}$$

is defined as the "classical" part of the full deplacement dX. By 'classical', we do not mean that this is necessarily a variable of classical physics (for example, as we shall see hereafter, thedx will become two-valued due to non-differentiability, which is clearly not a classical property). We mean that it remains differentiable, and therefore come under classical differentiable equations.

Here $d\xi$ represents the fractal fluctuations or "fractal part" of the deplacement dX : due to the definitive loss of information implied by the non-differentiability, we have no other choice than to represent it in terms of a stochastic variable. We therefore write:

$$dx = v\ dt, \tag{52}$$

$$d\xi = \eta\sqrt{2\mathcal{D}}(dt^2)^{1/2D}, \tag{53}$$

which becomes, for $D = 2$,

$$d\xi = \eta\sqrt{2\mathcal{D}}dt^{1/2}, \tag{54}$$

where $2\mathcal{D} = \tau_0 = \tau v^2$, and where η is a stochastic variable such that $< \eta >= 0$ and $< \eta^2 >= 1$. Owing to Eq. (49), we identify τ with the Einstein transition scale, $\tau = \hbar/E = \hbar/\frac{1}{2}mv^2$. Therefore, as we shall see further on, $2\mathcal{D} = \tau_0$ is a scalar quantity which can be identified with the Compton scale (up to fundamental constants), $\hbar/mc$, i.e., its physical meaning is the mass of the particle itself.

Now, the Schrödinger, Klein-Gordon and Dirac equations give results applying to measurements performed on quantum objects, but achieved with classical devices, in the differentiable "large-scale" domain. The microphysical scale at which the physical systems under study are considered induces the selection of a bundle of geodesics, corresponding to the scale of the systems (see above), while the measurement process implies a smoothing out of the geodesic bundle coupled to a transition from the non-differentiable "small-scale" to the differentiable "large-scale" domain. We are therefore led to define an operator $\mathcal{C}\ell\langle\quad\rangle$, which we apply to the fractal variables or functions each time we are drawn to the classical domain where the dt behavior dominates.

5.4 Discrete symmetry breaking

One of the most fundamental consequences of the non-differentiable nature of space (more generally, of space-time) is the breaking of a discrete symmetry, namely, of the reflection invariance on the differential element of (proper) time . As we shall see in what follows, it implies a two-valuedness of velocity which can be subsequently shown to be the origin of the complex nature of the quantum tool.

The derivative with respect to the time t of a differentiable function f can be written twofold

$$\frac{df}{dt} = \lim_{dt\to 0} \frac{f(t+dt) - f(t)}{dt} = \lim_{dt\to 0} \frac{f(t) - f(t-dt)}{dt}\ . \tag{55}$$

The two definitions are equivalent in the differentiable case. In the non-differentiable situation, both definitions fail, since the limits are no longer defined. In the new framework of scale relativity, the physics is related to the behavior of the function during the "zoom" operation on the time resolution δt, identified with the differential element dt. The nondifferentiable function $f(t)$ is replaced by an explicitly scale-dependent fractal function $f(t, dt)$, which therefore a function of two variables, t (in space-time) and dt (in scale-space). Two functions f'_+ and f'_- are therefore defined as explicit functions of the two variables t and dt

$$f'_+(t, dt) = \frac{f(t + dt, dt) - f(t, dt)}{dt}\ , \tag{56}$$

$$f'_-(t, dt) = \frac{f(t, dt) - f(t - dt, dt)}{dt}\ . \tag{57}$$

One passes from one definition to the other by the transformation $dt \leftrightarrow -dt$ (differential time reflection invariance), which was therefore an implicit discrete symmetry of differentiable physics. When applied to fractal space coordinates $x(t, dt)$, these definitions yield, in the non-differentiable domain, two velocity fields instead of one, that are fractal functions of the resolution, $V_+[x(t), t, dt]$ and $V_-[x(t), t, dt]$. In order to go back to the classical domain and derive the classical velocities appearing in Eq. (52), we smooth out each fractal geodesic in the bundles selected by the zooming process with balls of radius larger than τ. This amounts to carry out a transition from the non-differentiable to the differentiable domain and leads to define two classical velocity fields which are now resolution-independent: $V_+[x(t), t, dt > \tau] = \mathcal{C}\ell\langle V_+[x(t), t, dt]\rangle = v_+[x(t), t]$ and $V_-[x(t), t, dt > \tau] = \mathcal{C}\ell\langle V_-[x(t), t, dt]\rangle = v_-[x(t), t]$. The important new fact appearing here is that, after the transition, there is no longer any reason for these two velocity fields to be the same. While, in standard mechanics, the concept of velocity was one-valued, we must introduce, for the case of a non-differentiable space, two velocity fields instead of one, even when going back to the classical domain. In recent papers, Ord [48] also insists on the importance of introducing 'entwined paths' for understanding quantum mechanics.

A simple and natural way to account for this doubling consists in using complex numbers and the complex product. As we recall hereafter, this is the origin of the complex nature of the wave function of quantum mechanics, since this wave function can be identified with the exponential of the complex action that is naturally introduced in this framework. We shall now demonstrate that the choice of complex numbers to represent the two-valuedness of the velocity is a simplifying and "covariant" choice (in the sense of the principle of covariance, according to which the form of the equations of physics should be conserved under all transformations of coordinates).

5.5 'Covariant' total derivative operator

We are now lead to describe the elementary displacements for both processes, $dX_\pm$, as the sum of a $\mathcal{C}\ell$ part, $dx_\pm = v_\pm \, dt$, and a fluctuation about this $\mathcal{C}\ell$ part, $d\xi_\pm$, which is, by definition, of zero classical part, $\mathcal{C}\ell\langle d\xi_\pm\rangle = 0$

$$\begin{aligned} dX_+(t) &= v_+ \; dt + d\xi_+(t), \\ dX_-(t) &= v_- \; dt + d\xi_-(t). \end{aligned} \tag{58}$$

Considering first the large-scale displacements, large-scale forward and backward derivatives, d_+/dt and d_-/dt, are defined, using the $\mathcal{C}\ell$ part extraction procedure. Applied to the position vector, x, they yield the twin large-scale velocities

$$\frac{d_+}{dt} x(t) = v_+ \; , \qquad \frac{d_-}{dt} x(t) = v_- \; . \tag{59}$$

As regards the fluctuations, the generalization to three dimensions of the fractal behavior of Eq. (53) writes (for $D_F = 2$)

$$\mathcal{C}\ell\langle d\xi_{\pm i} \; d\xi_{\pm j}\rangle = \pm 2 \; \mathcal{D} \; \delta_{ij} \; dt \qquad i, j = x, y, z, \tag{60}$$

as the $d\xi(t)$'s are of null $\mathcal{C}\ell$ part and mutually independent. The Krönecker symbol, δ_{ij}, in Eq. (60), implies indeed that the $\mathcal{C}\ell$ part of every crossed product $\mathcal{C}\ell\langle d\xi_{\pm i} \; d\xi_{\pm j}\rangle$, with $i \neq j$, is null.

5.5.1 Origin of complex numbers in quantum mechanics

We now know that each component of the velocity takes two values instead of one. This means that each component of the velocity becomes a vector in a two-dimensional space, or, in other words, that the velocity becomes a two-index tensor. The generalization of the sum of these quantities is straighforward, but one also needs to define a generalized product.

The problem can be put in a general way: it amounts to find a generalization of the standard product that keeps its fundamental physical properties.

From the mathematical point of view, we are here exactly confronted to the well-known problem of the doubling of algebra (see, e.g., Ref. [79]). Indeed, the effect of the symmetry breaking $dt \leftrightarrow -dt$ (or $ds \leftrightarrow -ds$) is to replace the algebra $\mathcal{A}$ in which the classical physical quantities are defined, by a direct sum of two exemplaries of $\mathcal{A}$, i.e., the space of the pairs (a, b) where a and b belong to $\mathcal{A}$. The new vectorial space $\mathcal{A}^2$ must be supplied with a product in order to become itself an algebra (of doubled dimension). The same problem is asked again when one takes also into account the symmetry breakings $dx^\mu \leftrightarrow -dx^\mu$ and $x^\mu \leftrightarrow -x^\mu$ (see [66]): this leads to new algebra doublings. The mathematical solution to this problem is well-known: the standard algebra doubling amounts to supply $\mathcal{A}^2$ with the complex product. Then the doubling $\mathbb{R}^2$ of $\mathbb{R}$ is the algebra $\mathbb{C}$ of complex numbers, the doubling $\mathbb{C}^2$ of $\mathbb{C}$ is the algebra $\mathbb{H}$ of quaternions, the doubling $\mathbb{H}^2$ of quaternions is the algebra of Graves-Cayley octonions. The problem with algebra doubling is that the iterative doubling leads to a progressive deterioration of the algebraic properties. Namely, one loses the order relation of reals in the complex plane, while the quaternion algebra is non-commutative, and the octonion algebra is also non-associative. But an important positive result for physical applications is that the doubling of a metric algebra is a metric algebra [79].

These mathematical theorems fully justify the use of complex numbers, then of quaternions, in order to describe the successive doublings due to discrete symmetry breakings at the infinitesimal level, which are themselves more and more profound consequences of space-time non-differentiability.

However, we give in what follows complementary arguments of a physical nature, which show that the use of the complex product in the first algebra doubling have a simplifying and covariant effect (we use here the word "covariant" in the original meaning given to it by Einstein [1], namely, the requirement of the form invariance of fundamental equations).

In order to simplify the argument, let us consider the generalization of scalar quantities, for which the product law is the standard product in $\mathbb{R}$.

The first constraint is that the new product must remain an internal composition law. We also make the simplifying assumption that it remains linear in terms of each of the components of the two quantities to be multiplied. A general bilinear product writes:

$$c^k = a^i \, \omega_{ij}^k \, b^j, \tag{61}$$

and it is therefore defined by eight numbers in the case (considered here) of two-valuedness.

The second physical constraint is the requirement to recover the classical variables and the classical product at the classical limit. The mathematical equivalent of this constraint is the requirement that $\mathcal{A}$ still be a sub-algebra of $\mathcal{A}^2$. Therefore we identify $a_0 \in \mathcal{A}$ with $(a_0, 0)$ and we set $(0, 1) = \alpha$. This allows us to write the new two-dimensional vectors in the simplified form $a = a_0 + a_1\alpha$, so that the product now writes

$$c = (a_0 + a_1\alpha)\,(b_0 + b_1\alpha) = a_0b_0 + a_1b_1\alpha^2 + (a_0b_1 + a_1b_0)\alpha. \tag{62}$$

The problem is now reduced to find α^2, which is now defined by only two coefficients

$$\alpha^2 = \omega_0 + \omega_1\alpha. \tag{63}$$

Let us now come back to the beginning of our construction. We have introduced two elementary displacements, each of them made of two terms, a $\mathcal{C}\ell$ part and a fractal part (see Eq. (58))

$$\begin{aligned} dX_+(t) &= v_+ \; dt + d\xi_+(t), \\ dX_-(t) &= v_- \; dt + d\xi_-(t). \end{aligned} \tag{64}$$

Let us first consider the two values of the 'classical' part of the velocity. Instead of considering them as a vector of a new plane, (v_+, v_-), we shall use the above construction for defining them as a number of the doubled algebra [67]. Namely, we first replace (v_+, v_-) by the equivalent twin velocity field $[(v_+ + v_-)/2, (v_+ - v_-)/2]$, then we define the number:

$$\mathcal{V} = \left(\frac{v_+ + v_-}{2} - \alpha\, \frac{v_+ - v_-}{2} \right). \tag{65}$$

This choice is motivated by the facta that, at the classical limit when $v = v_+ = v_-$, the real part identifies with the classical velocity v and the new 'imaginary' part vanishes. The same operation can be made for the fractal parts. One can define velocity fluctuations $w_+ = d\xi_+/dt$ and $w_- = d\xi_-/dt$, so that we define a new number of the doubled algebra:

$$\mathcal{W} = \left(\frac{w_+ + w_-}{2} - \alpha\, \frac{w_+ - w_-}{2} \right). \tag{66}$$

Finally the total velocity (classical part and fractal fluctuation) reads:

$$\mathcal{V} + \mathcal{W} = \left(\frac{v_+ + v_-}{2} - \alpha\, \frac{v_+ - v_-}{2} \right) + \left(\frac{w_+ + w_-}{2} - \alpha\, \frac{w_+ - w_-}{2} \right). \tag{67}$$

We shall see in what follows that a Lagrange function can be introduced in terms of the new two-valued tool, that leads to a conserved form for the Euler-Lagrange equations. In the end, the Schrödinger equation is obtained as their integral. Now, from the covariance principle, the Lagrange function in the Newtonian case should strictly be written:

$$\mathcal{L} = \frac{1}{2} m \, \mathcal{C}\ell\langle(\mathcal{V} + \mathcal{W})^2\rangle = \frac{1}{2} m \; \left(\mathcal{C}\ell\langle\mathcal{V}^2\rangle + \mathcal{C}\ell\langle\mathcal{W}^2\rangle \right) \tag{68}$$

We have $\mathcal{C}\ell\langle\mathcal{W}\rangle = 0$, by definition, and $\mathcal{C}\ell\langle\mathcal{V}\mathcal{W}\rangle = 0$, because they are mutually independent. But what about $\mathcal{C}\ell\langle\mathcal{W}^2\rangle$? The presence of this term would greatly complicate all the subsequent developments toward the Schrödinger equation, since it would imply a fundamental divergence of non-relativistic quantum mechanics. Let us expand it:

$$\begin{aligned} 4\,\mathcal{C}\ell\langle\mathcal{W}^2\rangle &= \mathcal{C}\ell\langle[(w_+ + w_-) - \alpha\,(w_+ - w_-)]^2\rangle \\ &= \mathcal{C}\ell\langle(w_+^2 + w_-^2)(1 + \alpha^2) - 2\alpha(w_+^2 - w_-^2) + 2w_+w_-(1 - \alpha^2)\rangle. \end{aligned} \tag{69}$$

Since $\mathcal{C}\ell\langle w_+^2\rangle = \mathcal{C}\ell\langle w_-^2\rangle$ and $\mathcal{C}\ell\langle w_+ w_-\rangle = 0$ (they are mutually independent), we finally find that $\mathcal{C}\ell\langle \mathcal{W}^2\rangle$ can vanish only provided

$$\alpha^2 = -1, \tag{70}$$

namely, $\alpha = \pm i$, the imaginary. Therefore we have shown that the choice of the complex product in the algebra doubling plays an essential physical role, since it allows to suppress what would be additional infinite terms in the final equations of motion. The two solutions $+i$ and $-i$ have equal physical meaning, since the final equation of Schrödinger (demonstrated in the following) and the wave function are physically invariant under the transformation $i \to -i$ provided it is applied to both of them, as is well known in standard quantum mechanics.

5.5.2 Complex velocity

We now combine the two derivatives to obtain a complex derivative operator, that allows us to recover local differential time reversibility in terms of the new complex process [4]:

$$\frac{\acute{d}}{dt} = \frac{1}{2}\left(\frac{d_+}{dt} + \frac{d_-}{dt}\right) - \frac{i}{2}\left(\frac{d_+}{dt} - \frac{d_-}{dt}\right) . \tag{71}$$

Applying this operator to the position vector yields a complex velocity

$$\mathcal{V} = \frac{\acute{d}}{dt}x(t) = V - iU = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2} . \tag{72}$$

The minus sign in front of the imaginary term is chosen here in order to obtain the Schrödinger equation in terms of ψ. The reverse choice would give the Schrödinger equation for the complex conjugate of the wave function $\psi^\dagger$, and would be therefore physically equivalent.

The real part, V, of the complex velocity, $\mathcal{V}$, represents the standard classical velocity, while its imaginary part, $-U$, is a new quantity arising from non-differentiability. At the usual classical limit, $v_+ = v_- = v$, so that $V = v$ and $U = 0$.

5.5.3 Complex time-derivative operator

Contrary to what happens in the differentiable case, the total derivative with respect to time of a fractal function $f(x(t), t)$ of integer fractal dimension contains finite terms up to higher order [68]

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dX_i}{dt} + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{dX_i dX_j}{dt} + \frac{1}{6}\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\frac{dX_i dX_j dX_k}{dt} + ... \tag{73}$$

Note that it has been shown by Kolwankar and Gangal [69] that, if the fractal dimension is not an integer, a fractional Taylor expansion can also be defined, using the local fractional derivative (however, see [70] about the physical relevance of this tool).

In our case, a finite contribution only proceeds from terms of D_F-order, while lesser-order terms yield an infinite contribution and higher-order ones are negligible. Therefore, in the special case of a fractal dimension $D_F = 2$, the total derivative writes

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dX_i}{dt} + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{dX_i dX_j}{dt} . \tag{74}$$

Let us now consider the '$\mathcal{C}\ell$ part' of this expression. By definition, $\mathcal{C}\ell\langle dX\rangle = dx$, so that the second term is reduced to $v.\nabla f$. Now concerning the term $dX_i dX_j/dt$, it is usually infinitesimal, but here its $\mathcal{C}\ell$ part reduces to $\mathcal{C}\ell\langle d\xi_i \; d\xi_j\rangle/dt$. Therefore, thanks to Eq. (60), the last term of the $\mathcal{C}\ell$ part of Eq. (74) amounts to a Laplacian, and we obtain

$$\frac{df}{dt_\pm} = \left(\frac{\partial}{\partial t} + v_\pm.\nabla \pm \mathcal{D}\Delta\right) f \; . \tag{75}$$

Substituting Eqs. (75) into Eq. (71), we finally obtain the expression for the complex time derivative operator [4]

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta \; . \tag{76}$$

This is one of the main results of the theory of scale relativity. Indeed, the passage from standard classical (i.e., almost everywhere differentiable) mechanics to the new non-differentiable theory can now be implemented by replacing the standard time derivative d/dt by the new complex operator $\acute{d}/dt$ [4] (being cautious with the fact that it involves a combination of first order and second order derivatives). In other words, this means that $\acute{d}/dt$ plays the role of a "covariant derivative operator", namely, we shall write in its term the fundamental equations of physics under the same form they had in the differentiable case.

It should be remarked, before going on with this construction, that we use here the word 'covariant' in analogy with the covariant derivative $D_j A^k = \partial_j A^k + \Gamma^k_{jl} A^l$ replacing $\partial_j A^k$ in Einstein's general relativity. But one shoud be cautious with this analogy, since the two situations are different. Indeed, the problem posed in the construction of general relativity was that of a new geometry, in a framework where the differential calculus was not affected. Therefore the Einstein covariant derivative amounts to substracting the new geometric effect $-\Gamma^k_{jl} A^l$ in order to recover the mere inertial motion, for which the Galilean law of motion $Du^k/ds = 0$ naturally holds [84]. Here there is an additional difficulty: the new effects come not only from the geometry (see Sec. 7 for a scale-covariant derivative acting in the same way as that of general relativity) but also from the non-differentiability and its consequences on the differential calculus.

Therefore the true status of the new derivative is actually an extension of the concept of total derivative. Already in standard physics, the passage from the free Galileo-Newton's equation to its Euler form was a case of conservation of the form of equations in a more complicated situation, namely, $dv/dt = 0 \rightarrow (d/dt)v = (\partial/\partial t + v.\nabla)v = 0$. In the fractal and non-differentiable situation considered here, the three consequences (infinity of geodesics, fractality and two-valuedness) lead to three new terms in the total derivative operator, namely $V.\nabla$, $-iU.\nabla$ and $-i\mathcal{D}\Delta$.

5.6 Covariant mechanics induced by scale laws

Let us now summarize the main steps by which one may generalize the standard classical mechanics using this covariance. We are now looking to motion in the standard space. In what follows, we therefore consider only the 'classical parts' of the variables, which are differentiable and independent of resolutions. The effects of the internal non-differentiable structures are now contained in the covariant derivative. We assume that the '$\mathcal{C}\ell$ part' of the mechanical system under consideration can be characterized by a Lagrange function that keeps the usual form but now in terms of the complex velocity, $\mathcal{L}(x, \mathcal{V}, t)$, from which an action $\mathcal{S}$ is defined

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(x, \mathcal{V}, t) dt. \tag{77}$$

In this expression, we have combined the forward and backward velocities in terms of a unique complex velocity. We have already given arguments, in the previous section, according to which this choice is a simplifying and covariant choice. We shall now support this conclusion by demonstrating hereafter that it indeed allows us to conserve the standard form of the Euler-Lagrange equations.

In a general way, the Lagrange function is expected to be a function of the variables x and their time derivatives $\dot{x}$. We have found that the number of velocity components $\dot{x}$ is doubled, so that we are led to write

$$L = L(x, \dot{x}_+, \dot{x}_-, t). \tag{78}$$

Instead, we have made the choice to write the Lagrange function as $L = L(x, \mathcal{V}, t)$. We now justify this choice by the covariance principle. Re-expressed in terms of $\dot{x}_+$ and $\dot{x}_-$, the Lagrange function writes

$$L = L\left(x, \frac{1-i}{2}\,\dot{x}_+ + \frac{1+i}{2}\,\dot{x}_-, t\right). \tag{79}$$

Therefore we obtain

$$\frac{\partial L}{\partial \dot{x}_+} = \frac{1-i}{2}\frac{\partial L}{\partial \mathcal{V}} \quad ; \quad \frac{\partial L}{\partial \dot{x}_-} = \frac{1+i}{2}\frac{\partial L}{\partial \mathcal{V}}, \tag{80}$$

while the new covariant time derivative operator writes

$$\frac{\acute{d}}{dt} = \frac{1-i}{2}\frac{d_+}{dt} + \frac{1+i}{2}\frac{d_-}{dt}\,. \tag{81}$$

Let us write the stationary action principle in terms of the Lagrange function of Eq. (78)

$$\delta S = \delta \int_{t_1}^{t_2} L(x, \dot{x}_+, \dot{x}_-, t)\; dt = 0. \tag{82}$$

It becomes

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}\,\delta x + \frac{\partial L}{\partial \dot{x}_+}\,\delta \dot{x}_+ + \frac{\partial L}{\partial \dot{x}_-}\,\delta \dot{x}_-\right) dt = 0. \tag{83}$$

Since $\delta \dot{x}_+ = d_+(\delta x)/dt$ and $\delta \dot{x}_- = d_-(\delta x)/dt$, it takes the form

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}\,\delta x + \frac{\partial L}{\partial \mathcal{V}}\left[\frac{1-i}{2}\frac{d_+}{dt} + \frac{1+i}{2}\frac{d_-}{dt}\right]\delta x\right) dt = 0, \tag{84}$$

i.e.,

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}\,\delta x + \frac{\partial L}{\partial \mathcal{V}}\,\frac{\acute{d}}{dt}\,\delta x\right) dt = 0 \qquad \text{QED.} \tag{85}$$

The subsequent demonstration of the Lagrange equations from the stationary action principle relies on an integration by part. This integration by part cannot be performed in the usual way without a specific analysis, because it involves the new covariant derivative.

The first point to be considered is that such an operation involves the Leibniz rule for the covariant derivative operator $\acute{d}/dt$. Since $\acute{d}/dt = \partial/dt + \mathcal{V} \, . \, \nabla - i\mathcal{D}\Delta$ is a linear combination of first and second order derivatives, the same is true of its Leibniz rule. This implies the appearance of an additional term in the expression for the derivative of a product (see [74]):

$$\frac{\acute{d}}{dt}\left(\frac{\partial L}{\partial \mathcal{V}} \, . \, \delta x\right) = \frac{\acute{d}}{dt}\left(\frac{\partial L}{\partial \mathcal{V}}\right) . \, \delta x + \frac{\partial L}{\partial \mathcal{V}} \, . \, \frac{\acute{d}}{dt}\delta x - 2i\,\mathcal{D}\,\nabla\left(\frac{\partial L}{\partial \mathcal{V}}\right) . \, \nabla\delta x. \tag{86}$$

Since $\delta x(t)$ is not a function of x, the additional term vanishes. Therefore the above integral becomes

$$\int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial x} - \frac{\acute{d}}{dt}\frac{\partial L}{\partial \mathcal{V}}\right)\delta x \; + \frac{\acute{d}}{dt}\left(\frac{\partial L}{\partial \mathcal{V}} \, . \, \delta x\right)\right] dt = 0. \tag{87}$$

The second point is concerned with the integration of the covariant derivative itself. We define a new integral as being the inverse operation of the covariant derivation, i.e.,

$$\fint \acute{d}f = f \tag{88}$$

in terms of which one obtains

$$\fint_{t_1}^{t_2} \acute{d}\left(\frac{\partial L}{\partial \mathcal{V}} \, . \, \delta x\right) = \left[\frac{\partial L}{\partial \mathcal{V}} \, . \, \delta x\right]_{t_1}^{t_2} = 0, \tag{89}$$

since $\delta x(t_1) = \delta x(t_2) = 0$ by definition of the variation principle. Therefore the action integral becomes

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{\acute{d}}{dt}\frac{\partial L}{\partial \mathcal{V}}\right)\delta x\, dt = 0. \tag{90}$$

And finally we obtain generalized Euler-Lagrange equations that read

$$\frac{\acute{d}}{dt}\frac{\partial L}{\partial \mathcal{V}} = \frac{\partial L}{\partial x}. \tag{91}$$

Therefore, thanks to the transformation $d/dt \rightarrow \acute{d}/dt$, they take exactly their standard classical form. This result reinforces the identification of our tool with a "quantum-covariant" representation, since, as we have shown in previous works and as we recall in what follows, this Euler-Lagrange equation can be integrated in the form of a Schrödinger equation.

Since we now consider only the 'classical parts' of the variables (while the effects on them of the fractal parts are included in the covariant derivative) the basic symmetries of classical physics hold. From the homogeneity of standard space one defines a generalized complex momentum given by

$$\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathcal{V}}. \tag{92}$$

If we now consider the action as a functional of the upper limit of integration in Eq. (77), the variation of the action from a trajectory to another nearby trajectory yields a generalization of another well-known relation of standard mechanics:

$$\mathcal{P} = \nabla \mathcal{S}. \tag{93}$$

As concerns the generalized energy, its expression involves an additional term [74, 49]: namely it write for a Newtonian Lagrange function and in the absence of exterior potential, $\mathcal{E} = (1/2)m(\mathcal{V}^2 - 2i\,\mathcal{D}\,\mathrm{div}\mathcal{V})$.

5.7 Newton-Schrödinger Equation

5.7.1 Geodesics equation

Let us now specialize our study, and consider Newtonian mechanics, i.e., the general case when the structuring external scalar field is described by a potential energy Φ. The Lagrange function of a closed system, $L = \frac{1}{2}mv^2 - \Phi$, is generalized, in the large-scale domain, as $\mathcal{L}(x, \mathcal{V}, t) = \frac{1}{2}m\mathcal{V}^2 - \Phi$. The Euler-Lagrange equations keep the form of Newton's fundamental equation of dynamics

$$m\frac{\hat{d}}{dt}\mathcal{V} = -\nabla\Phi, \tag{94}$$

which is now written in terms of complex variables and complex operators.

In the case when there is no external field (and when the field is gravitational, see following section), the covariance is explicit, since Eq. (94) takes the form of the equation of inertial motion, i.e., of a geodesics equation,

$$\hat{d}\mathcal{V}/dt = 0, \tag{95}$$

This is analog to Einstein's general relativity, where the equivalence principle of gravitation and inertia leads to a strong covariance principle, expressed by the fact that one may always find a coordinate system in which the metric is locally Minkowskian. This means that, in this coordinate system, the covariant equation of motion of a free particle is that of inertial motion $Du_\mu = 0$ in terms of the general-relativistic covariant derivative D and four-vector u_μ. The expansion of the covariant derivative subsequently transforms this free-motion equation in a local geodesic equation in a gravitational field.

The covariance induced by scale effects leads to an analogous transformation of the equation of motions, which, as we show below, become after integration the Schrödinger equation, (then the Klein-Gordon and Dirac equations in the motion-relativistic case), which we can therefore consider as the integral of a geodesic equation.

In both cases, with or without external field, the complex momentum $\mathcal{P}$ reads

$$\mathcal{P} = m\mathcal{V}, \tag{96}$$

so that, from Eq. (93), the complex velocity $\mathcal{V}$ appears as a gradient, namely the gradient of the complex action

$$\mathcal{V} = \nabla\mathcal{S}/m. \tag{97}$$

5.7.2 Complex wave function

We now introduce a complex wave function ψ which is nothing but another expression for the complex action $\mathcal{S}$

$$\psi = e^{i\mathcal{S}/\mathcal{S}_0}. \tag{98}$$

The factor $\mathcal{S}_0$ has the dimension of an action (i.e., an angular momentum) and must be introduced for dimensional reasons. We show in what follows, that, when this formalism is applied to microphysics, S_0 is nothing but the fundamental constant $\hbar$. The function ψ is related to the complex velocity appearing in Eq. (97) as follows

$$\mathcal{V} = -i\,\frac{\mathcal{S}_0}{m}\,\nabla(\ln\psi). \tag{99}$$

5.7.3 Schrödinger equation

We have now at our disposal all the mathematical tools needed to write the fundamental equation of dynamics of Eq. (94) in terms of the new quantity ψ. It takes the form

$$i\mathcal{S}_0 \frac{\acute{d}}{dt}(\nabla \ln \psi) = \nabla \Phi. \tag{100}$$

Now one should be aware that $\acute{d}$ and ∇ do not commute. However, as we shall see in the following, there is a particular choice of the arbitrary constant $\mathcal{S}_0$ for which $\acute{d}(\nabla \ln \psi)/dt$ is nevertheless a gradient.

Replacing $\acute{d}/dt$ by its expression, given by Eq. (76), yields

$$\nabla \Phi = i\mathcal{S}_0 \left(\frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta \right) (\nabla \ln \psi), \tag{101}$$

and replacing once again $\mathcal{V}$ by its expression in Eq. (99), we obtain

$$\nabla \Phi = i\mathcal{S}_0 \left[\frac{\partial}{\partial t} \nabla \ln \psi - i \left\{ \frac{\mathcal{S}_0}{m} (\nabla \ln \psi.\nabla)(\nabla \ln \psi) + \mathcal{D}\Delta(\nabla \ln \psi) \right\} \right]. \tag{102}$$

Consider now the remarkable identity [4]

$$(\nabla \ln f)^2 + \Delta \ln f = \frac{\Delta f}{f} , \tag{103}$$

which proceeds from the following tensorial derivation

$$\begin{aligned} \partial_\mu \partial^\mu \ln f + \partial_\mu \ln f \partial^\mu \ln f &= \partial_\mu \frac{\partial^\mu f}{f} + \frac{\partial_\mu f}{f} \frac{\partial^\mu f}{f} \\ &= \frac{f \partial_\mu \partial^\mu f - \partial_\mu f \partial^\mu f}{f^2} + \frac{\partial_\mu f \partial^\mu f}{f^2} \\ &= \frac{\partial_\mu \partial^\mu f}{f} . \end{aligned} \tag{104}$$

When we apply this identity to ψ and take its gradient, we obtain

$$\nabla \left(\frac{\Delta \psi}{\psi} \right) = \nabla[(\nabla \ln \psi)^2 + \Delta \ln \psi]. \tag{105}$$

The second term in the right-hand side of this expression can be transformed, using the fact that ∇ and Δ commute, i.e.,

$$\nabla \Delta = \Delta \nabla. \tag{106}$$

The first term can also be transformed thanks to another remarkable identity

$$\nabla(\nabla f)^2 = 2(\nabla f.\nabla)(\nabla f), \tag{107}$$

that we apply to $f = \ln \psi$. We finally obtain

$$\nabla \left(\frac{\Delta \psi}{\psi} \right) = 2(\nabla \ln \psi.\nabla)(\nabla \ln \psi) + \Delta(\nabla \ln \psi). \tag{108}$$

We recognize, in the right-hand side of this equation, the two terms of Eq. (102), which were respectively in factor of $\mathcal{S}_0/m$ and $\mathcal{D}$. Therefore, the particular choice

$$\mathcal{S}_0 = 2m\mathcal{D} \tag{109}$$

allows us to simplify the right-hand side of Eq. (102). This is more general than standard quantum mechanics, in which S_0 is restricted to the only value $S_0 = \hbar$. Eq. 109 is actually a generalization of the Compton relation (see next section): this means that the function ψ becomes a wave function only provided it is accompanied by a Compton-de Broglie relation. Without this condition, the equation of motion would remain of third order, with no general prime integral. Indeed, the simplification brought by this choice is twofold: (i) several complicated terms are compacted into a simple one; (ii) the final remaining term is a gradient, which means that the fundamental equation of dynamics can now be integrated in a universal way. The function ψ in Eq. (98) is therefore defined as

$$\psi = e^{i\mathcal{S}/2m\mathcal{D}}, \tag{110}$$

and it is solution of the fundamental equation of dynamics, Eq. (94), which we write

$$\frac{\hat{d}}{dt}\mathcal{V} = -2\mathcal{D}\nabla\left\{i\frac{\partial}{\partial t}\ln\psi + \mathcal{D}\frac{\Delta\psi}{\psi}\right\} = -\nabla\Phi/m. \tag{111}$$

Integrating this equation finally yields

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial}{\partial t}\psi - \frac{\Phi}{2m}\psi = 0, \tag{112}$$

up to an arbitrary phase factor which may be set to zero by a suitable choice of the ψ phase. Therefore the Schrödinger equation is the new form taken by the Hamilton-Jacobi / energy equation (see [49] on this point).

Arrived at that point, several steps have been already made toward the final identification of the function ψ with a wave function: it is complex, solution of a Schrödinger equation, so that its linearity is also ensured: namely, if ψ_1 and ψ_2 are solutions, $a_1\Psi_1 + a_2\psi_2$ is also a solution. Let us complete the proof by giving new insights about other basic axioms of quantum mechanics.

5.7.4 Compton length

In the case of standard quantum mechanics, as applied to microphysics, the necessary choice $\mathcal{S}_0 = 2m\mathcal{D}$ means that there is a natural link between the Compton relation and the Schrödinger equation. In this case, indeed, $\mathcal{S}_0$ is nothing but the fundamental action constant $\hbar$, while $\mathcal{D}$ defines the fractal/non-fractal transition (i.e., the transition from explicit scale- dependence to scale-independence in the rest frame), $\lambda = 2\mathcal{D}/c$. Therefore, the relation $\mathcal{S}_0 = 2m\mathcal{D}$ becomes a relation between mass and the fractal to scale-independence transition, which writes

$$\lambda_c = \frac{\hbar}{mc}\ . \tag{113}$$

We recognize here the definition of the Compton length. Its profound meaning (i.e., up to the fundamental constants $\hbar$ and c, it gives the inertial mass itself) is thus given, in our framework by the transition scale from fractality (at small scales) to scale-independence (at large scales). We note that this length-scale is to be understood as a structure of scale-space, not of standard space. The de Broglie length can now be easily recovered: the fractal fluctuation is a differential elements of order 1/2, i.e., it reads $< d\xi^2 >= \hbar dt/m$ in function of dt, and then $< d\xi^2 >= \lambda_x dx$ in function of the space differential elements. This implies $\lambda_x = \hbar/mv$, which is the non-relativistic expression for the de Broglie length.

We recover, in this case, the standard form of Schrödinger's equation

$$\frac{\hbar^2}{2m}\Delta\psi + i\hbar\frac{\partial}{\partial t}\psi = \Phi\psi. \tag{114}$$

5.7.5 Born postulate

The statistical meaning of the wave function (Born postulate) can now be deduced from the very construction of the theory (at least in the one-dimensional and stationary case). Even in the case of only one particle, the virtual geodesic family is infinite (this remains true even in the zero particle case, i.e., for the vacuum field). The particle properties are assimilated to those of a random subset of the geodesics in the family, and its probability to be found at a given position must be proportional to the density of the geodesics fluid. This density can easily be calculated in our formalism.

Indeed, we have found up to now two equivalent representations of the same equations: (i) a geodesics equation (in the free case) $\acute{d}\mathcal{V}/dt = 0$, depending on two variables V and U, i.e. the real and imaginary parts of the complex velocity; (ii) a Schrödinger equation (after integration) , depending on two variables $\sqrt{P}$ and θ, i.e. the modulus and the phase of the function ψ.

Now, a third mixed equivalent representation is possible (see Sec. 5.8.3), in terms of the couple of variables $(P,\ V)$. This opens the possibility to get a derivation of Born's postulate in this context. This question has already been considered by Hermann [72], who obtained numerical solutions of the equation of motion (94) in terms of a large number of explicit trajectories (in the case of a free particle in a box). He constructed a probability density from these trajectories and recovered in this way solutions of the Schrödinger equation without writing it and without using a wave function.

In function of these variables, the imaginary part of Eq. (112) writes

$$\frac{\partial P}{\partial t} + \text{div}(PV) = 0, \tag{115}$$

where V is identified, at the classical limit, with the classical velocity. This equation is recognized as an equation of continuity. This implies that $P = \psi\psi^\dagger$ is related to the probability density ρ (which is also subjected to an equation of continuity) by a relation $P = K\rho$, such that $dK/dt = \partial K/\partial t + V.\nabla K = 0$. In the one-dimensional stationary case this implies that K =cst, thus ensuring the validity of Born's postulate. The remarkable new feature that allows us to obtain such a result is that the continuity equation is not written as an additional a priori equation, but is now a part of our generalized equation of dynamics. This allows a final identification of the function ψ (i.e. the action after a change of variable) with a wave function.

5.7.6 Von Neumann postulate

The von Neumann postulate (i.e. the axiom of wave function collapse) is also easily recovered in such a geometric interpretation. Indeed, we may identify a measurement with a selection of the sub-sample of the geodesics family that keeps only the geodesics having the geometric properties corresponding to the measurement result. Therefore, just after the measurement, the system is in the state given by the measurement result.

As concerns others axioms of quantum mechanics, commutation relations and the Heisenberg representation, see Refs. [73, 74].

5.7.7 Schrödinger form of other fundamental equations of physics

The general method described above can be applied to any physical situation where the three basic conditions (namely, infinity of trajectories, each trajectory is a fractal curve of fractal dimension 2, breaking of differential time reflexion invariance) are achieved in an exact or in an approximative way. Several fundamental equations of classical physics can be transformed to take a generalized Schrödinger form under these conditions: namely, equation of motion in the presence of an electromagnetic field (see Sec. 7), the Euler and Navier-Stokes equations in the case of potential motion and for incompressible and isentropic fluids; the equation of the rotational motion of solids, the motion equation of dissipative systems; field equations (scalar field for one space variable). We cannot enter here into the detail of these generalizations, so we refer the interested reader to Ref. [26].

5.8 Application to gravitation

5.8.1 Curved and fractal space

Applications of the scale relativity theory to the problem of the formation and evolution of gravitational structures have been presented in several previous works [4, 23, 50, 26, 51, 52, 53, 54]. A recent review paper about the comparison between the theoretically predicted structures and observational data, from the scale of planetary systems to extragalactic scales, has been given in Ref. [56]. We shall only briefly sum up here the principles and methods used in such an attempt, then quote some of the main results obtained.

In its present acceptance, gravitation is understood as the various manifestations of the geometry of space-time at large scales. Up to now, in the framework of Einstein's theory, this geometry was considered to be Riemannian, i.e. curved. However, in the new framework of scale relativity, the geometry of space-time is assumed to be characterized not only by curvature, but also by fractality beyond a new relative time-scale and/or space-scale of transition, which is an horizon of predictibility for the classical deterministic description. As we shall see in what follows, fractality manifests itself, in the simplest case, in terms of the appearance of a new scalar field. We have suggested that this new field leads to spontaneous self-organization and may also be able to explain [34, 56], without additional matter, the various astrophysical effects which have been, up to now, tentatively attributed to unseen "dark" matter .

5.8.2 Gravitational Schrödinger equation

We shall briefly consider in what follows only the Newtonian limit. In this case the equation of geodesics keeps the form of Newton's fundamental equation of dynamics in a gravitational field, namely,

$$\frac{\bar{D}\mathcal{V}}{dt} = \frac{\acute{d}\,\mathcal{V}}{dt} + \nabla\left(\frac{\phi}{m}\right) = 0, \tag{116}$$

where ϕ is the Newtonian potential energy. As demonstrated hereabove, once written in terms of ψ, this equation can be integrated to yield a gravitational

Newton-Schrödinger equation :

$$\mathcal{D}^2 \triangle \psi + i\mathcal{D}\frac{\partial}{\partial t}\psi = \frac{\phi}{2m}\,\psi. \tag{117}$$

Since the imaginary part of this equation is the equation of continuity, and basing ourselves on our description of the motion in terms of an infinite family of geodesics, $P = \psi\psi^\dagger$ can be interpreted as giving the probability density of the particle position.

Note however that the situation and therefore the interpretation are different here from the application of the theory to the microphysical domain. The two main differences are:

(i) While in the microscopic realm elementary "particles" can be defined as the geodesics themselves (their defining properties such as mass, spin or charge being defined as internal geometric properties, see [23, 66]), in the macroscopic realm there does exist actual particles that follow the geodesics.

(ii) While differentiability is definitively lost toward the small scales in the microphysical domain, the macroscopic quantum theory is valid only beyond some time-scale transition (and/or space-scale transition) which is an horizon of predictibility. Therefore in this last case there is an underlying classical theory, which means that the quantum macroscopic approach is a hidden variable theory [26].

Even though it takes this Schrödinger-like form, equation (117) is still in essence an equation of gravitation, so that it must keep the fundamental properties it owns in Newton's and Einstein's theories. Namely, it must agree with the equivalence principle [50, 57], i.e., it is independent of the mass of the test-particle. In the Kepler central potential case ($\phi = -GMm/r$), GM provides the natural length-unit of the system under consideration. As a consequence, the parameter $\mathcal{D}$ takes the form:

$$\mathcal{D} = \frac{GM}{2w}, \tag{118}$$

where w is a fundamental constant that has the dimension of a velocity. The ratio $\alpha_g = w/c$ actually plays the role of a macroscopic gravitational coupling constant [57, 54]).

5.8.3 Formation and evolution of gravitational structures

Let us now compare our approach with the standard theory of gravitational structure formation and evolution. Instead of the Euler-Newton equation and of the continuity equation which are used in the standard approach, we write the only above Newton-Schrödinger equation. In both cases, the Newton potential is given by the Poisson equation. Two situations can be considered: (i) when the 'orbitals', which are solutions of the motion equation, can be considered as filled with the particles (e.g., planetesimals in the case of planetary systems formation, interstellar gas and dust in the case of star formation, etc...), the mass density ρ is proportional to the probability density $P = \psi\psi^\dagger$: this situation is relevant in particular for addressing problems of structure formation; (ii) another possible situation concerns test bodies which are not in sufficiently large number to change the matter density, but whose motion is nevertheless submitted to the Newton-Schrödinger equation: this case is relevant for the anomalous dynamical effects which have up to now been attributed to unseen, "dark" matter.

By separating the real and imaginary parts of the Schrödinger equation we get respectively a generalized Euler-Newton equation (written here in terms of the Newtonian potential energy ϕ) and a continuity equation:

$$m\,(\frac{\partial}{\partial t} + V \cdot \nabla)V = -\nabla(\phi + Q), \tag{119}$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0, \tag{120}$$

$$\Delta\phi = 4\pi G\rho\, m. \tag{121}$$

In the case $P \propto \rho$ this system of equations is equivalent to the classical one used in the standard approach of gravitational structure formation, except for the appearance of an extra potential energy term Q that writes:

$$Q = -2m\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}}. \tag{122}$$

The existence of this potential energy, which has been identified as such by Bohm in the microphysical case (but without an understanding of its origin, since it was derived from the a priori axioms of quantum mechanics) is, in our approach, readily demonstrated and understood: namely, it is the very manifestation of the fractality of space [49], in similarity with Newton's potential being a manifestation of curvature.

In the case (i) where actual particles achieve the probability density distribution (structure formation), we have $\rho = \rho_0 P$; then the Poisson equation (i.e., the field equation) becomes $\Delta\phi = 4\pi G m \rho_0 \psi\psi^\dagger$ and it is therefore strongly interconnected with the Schrödinger equation (i.e., the particle motion equation). An equation for matter alone can finally be written [26] (which has automatically its equivalent in an equation for the potential alone):

$$\Delta\left(\frac{\mathcal{D}^2\Delta\psi + i\mathcal{D}\partial\psi/\partial t}{\psi}\right) - 2\pi G\rho_0|\psi|^2 = 0. \tag{123}$$

This is a Hartree equation of the kind which is encountered in the description of superconductivity. We expect its solutions to provide us with general theoretical predictions for the structures (in position and velocity space) of self-gravitating systems at multiple scales [56]. This expectation is already supported by the observed agreement of several solutions with astrophysical observational data [4, 50, 54, 51, 52, 58, 59, 53].

Indeed, the theory has been able to predict in a quantitative way a large number of new effects in the domain of gravitational structures. Moreover, these predictions have been successfully checked in various systems on a large range of scales and in terms of a common fundamental gravitational coupling constant whose value averaged on these systems was found to be $w_0 = c\,\alpha_g = 144.7 \pm 0.7$ km/s [50]. New structures have been theoretically predicted, then checked by the observational data in a statistically significant way, for our solar system, including distances of planets [4, 51] and satellites [53], sungrazer comet perihelions [55], obliquities and inclinations of planets and satellites [58], exoplanets semi-major axes [50, 54] (see Fig. 10 and eccentricities [60], including planets around pulsars (for which a high precision is reached) [50, 59], double stars [52], planetary nebula

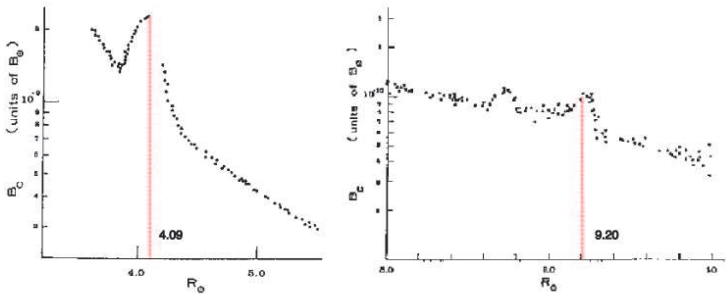

Figure 8: IR-dust density observed during the solar eclipse of January 1967 (adapted from MacQueen [61]). The scale-relativity approach leads to a hierarchical description of the Solar System, which is described by imbricated sub-systems that are solutions of gravitational Schrödinger equations. While the inner solar system is organized on a constant $w_0 = 144$ km/s, we expect the existence of an intramercurial subsystem organized on the basis of a new constant $w = 3 \times 144 = 432$ km/s. The Sun radius is in precise agreement with the peak of the fundamental level of this sequence: namely, one finds $n_\odot = 0.99$ with $R_\odot = 0.00465$ AU, that corresponds to a Keplerian velocity of 437.1 km/s. The next probability density peaks are predicted to lie at distances $4.09\, R_\odot$, and $9.20\, R_\odot = 0.043$ AU, which correspond respectively to Keplerian velocities of 432/2 and $432/3 = 144$ km/s. Since 1966, there has been several claims of detection during solar eclipses of IR thermal emission peaks from possible circumsolar dust rings which lie precisely at the predicted distances [56]. Several exoplanets have now been also found at similar relative distances a/M from their star (see Fig. 10).

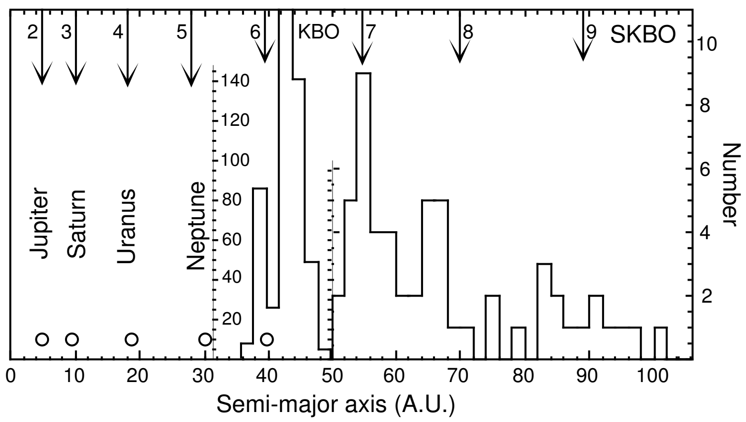

Figure 9: Distribution of the semi-major axis of Kuiper belt objects (KBO) and scattered Kuiper belt objects (SKBO), compared with the theoretical predictions of probability density peaks for the outer solar system (arrows) [56]. The whole inner solar system (whose density peak lies at the Earth distance, that corresponds to $n_i = 5$) can be identified with the fundamental $n_e = 1$ orbital of the outer solar system [51]. Therefore the outer solar system is expected to be organized according to a constant $w_e = 144/5 = 28.8$ km/s. Remark that the existence of probability density peaks for the Kuiper belt small planets at ≈ 40, 55, 70, 90 AU, etc..., has been theoretically predicted before the discovery of these objects [62].

[56] (see Fig. 12, binary galaxies [23], our local group of galaxies [56], clusters of galaxies and large scale structures of the universe [52, 56].

A full account of this new domain would be too long to be included in the present contribution. We give here only few typical examples of these effects (see the figures) and we refer the interested reader to the review paper Ref. [56] for

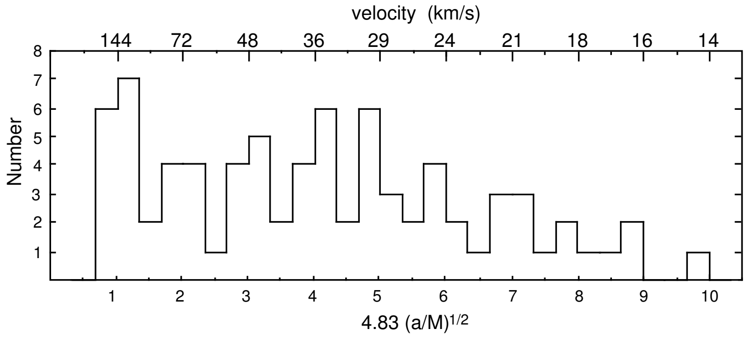

Figure 10: Observed distribution of the semi-major axes of recently discovered exoplanets and inner solar system planets, compared with the theoretical prediction from the scale-relativity / Schrödinger approach. Note that these predictions [4, 62] have been made before the first discovery of exoplanets. One expects the occurence of peaks of probability density for semimajor axes $a_n = GM(n/w_0)^2$, where n is integer, M is the star mass and $w_0 = 144.7 \pm 0.7$ km/s is a gravitational coupling constant (see [4, 50, 54]). For example, the velocity of Mercury is 48=144/3 km/s, of Venus 36=144/4 km/s, of the Earth 29=144/5 km/s and of Mars 24=144/6 km/s. The data supports the theoretical prediction in a statistically significant way (the probability to obtain such an agreement by chance is $P = 4 \times 10^{-5}$).

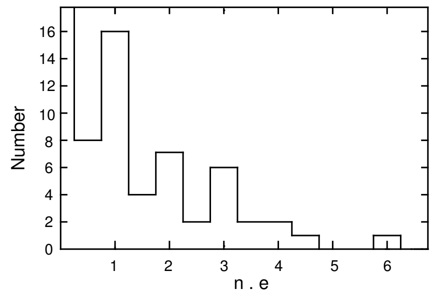

Figure 11: Observed distribution of the eccentricities of exoplanets. The theory predicts that the product of the eccentricity e by the quantity $\tilde{n} = 4.83(a/M)^{1/2}$, where a is the semi-major axis and M the parent star mass, should cluster around integers. The data support this theoretical prediction at a probability level $P = 10^{-4}$ [60, 56].

more detail.

5.8.4 Possible solution to the "dark matter" problem

In the case (ii) of isolated test particles, the density of matter ρ may be nearly zero while the probability density P does exist, but only as a virtual quantity that determines the potential Q, without being effectively achieved by matter. In this situation, even though there is no or few matter at the point considered (except the test particle that is assumed to have a very low contribution), the effects of the potential Q are real (since it is the result of the structure of the geodesics two-fluid). This situation is quite similar to the Newton potential in vacuum around a mass.

We have therefore suggested [34, 56, 35] that this extra scalar field, which is

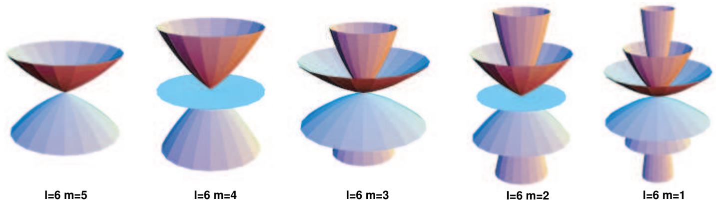

Figure 12: Example of morphologies predicted from solution of a macroscopic Schrödinger equation that describes an accretion or ejection process. This problem is similar to that of scattering in elementary particle physics (spherical ingoing or outgoing wave). Such morphologies are typical of those observed in 'planetary nebula', which are, despite their names, stars that eject their outer shells (see [56]).

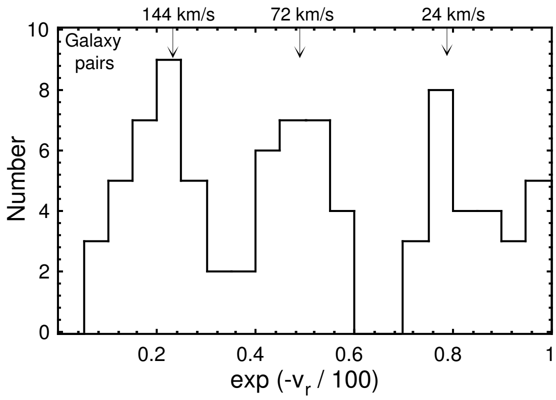

Figure 13: Deprojection of the intervelocity distribution of galaxy pairs [63, 56] from the Schneider-Salpeter catalog with precision redshifts [64]. The main probability peak is found to lie at 144 km/s (plus secondary peaks at 72=144/2 km/s and 24=144/6 km/s), in agreement with the exoplanet and inner solar system structuration (see Fig. 10).

a manifestation of the fractality of space, may be responsible for the various dynamical and lensing effects which are usually attributed to unseen "dark matter". Recall that up to now two hypotheses have been formulated in order to account for these effects (which are far larger than those due to visible matter): (i) The existence of a very large amount of unseen matter in the Universe: but, despite intense and continuous efforts, it has escaped detection. (ii) A modification of Newton's law of force: but such an ad hoc hypothesis seems impossible to reconcile with its geometric origin in general relativity, which lets no latitude for modification. In the scale-relativity proposal, there is no need for additional matter, and Newton's potential is unchanged since it remains linked to curvature, but there is an additional potential linked to fractality.

This interpretation is supported by the fact that, for a stationary solution of the gravitational Schrödinger equation, one gets the general relation:

$$\frac{\phi + Q}{m} = \frac{E}{m} = \text{cst}, \tag{124}$$

where E/m can take only quantized values (which are related to the fundamental gravitational coupling [57], $\alpha_g = w/c$).

This result can be applied, as an example, to the motion of bodies in the outer regions of spiral galaxies. In these regions there is practically no longer any visible matter, so that the Newtonian potential (in the absence of additional dark matter) is Keplerian. While the standard Newtonian theory predicts for the velocity of the halo bodies $v \propto \phi^{1/2}$, i.e. $v \propto r^{-1/2}$, we predict in our theory $v \propto |(\phi + Q)/m|^{1/2}$, i.e., $v =$ constant. More specifically, assuming that the gravitational Schrödinger equation is solved for the halo objects in terms of the fundamental level wave function, one finds $Q_{pred} = -(GMm/2r_B)(1 - 2\, r_B/r)$, where $r_B = GM/w_0^2$. This is exactly the result which is systematically observed in spiral galaxies (i.e., flat rotation curves) and which has motivated (among other effects) the assumption of the existence of dark matter. In other words, we suggest that the effects tentatively attributed to unseen matter are simply the result of the geometry of space-time. In this proposal, space-time is not only curved but also fractal beyond some given relative time and space-scales. While the curvature manifests itself in terms of the Newton potential, fractality would manifest itself in terms of the new scalar potential Q, and then finally in terms of the anomalous dynamics and lensing effects.

5.9 Application to sciences of life

Self-similar fractal laws have already been used as models for the description of a huge number of biological systems (lungs, blood network, brain, cells, vegetals, etc..., see e.g. [90, 8], previous volumes, and references therein).

The scale-relativistic tools may also be relevant for a description of behaviors and properties which are typical of living systems. Some examples have been given in [89, 40, 41, 42] (and are briefly recalled in the present review), concerning halieutics, morphogenesis, log-periodic branching laws and cell "membrane" models. As we shall see in what follows, scale relativity may also provide a physical and geometric framework for the description of additional properties such as formation, duplication, morphogenesis and imbrication of hierarchical levels of organization. This approach does not mean to dismiss the importance of chemical and biological laws in the determination of living systems, but on the contrary to attempt to establish a geometric foundation that could underlie them. Under such a view-point, biochemical processes would arise as a manifestation 'tool' of fundamental laws issued from first principles.

5.9.1 Morphogenesis

The Schrödinger equation can be viewed as a fundamental equation of morphogenesis. It has not been yet considered as such, because its unique domain of application was, up to now, the microscopic (molecular, atomic, nuclear and elementary particle) domain, in which the available information was mainly about energy and momentum. Such a situation is now changing thanks to field effect microscopy and atom laser trapping, which begin to allow the observation of quantum-induced geometric shapes at small scales.

However, scale-relativity extends the potential domain of application of Schrödinger-like equations to every systems in which the three conditions (infinite or very large number of trajectories, fractal dimension 2 of individual trajectories, local irreversibility) are fulfilled. Macroscopic Schrödinger equations can be defined, which are not based on Planck's constant $\hbar$, but on constants that are specific of each system (and may emerge from their self-organization). We have suggested that such an approach applies to gravitational structures at large scales. We have in particular been able by this way to recover complicated shapes such as those of planetary nebulae [56], that had up to now no detailed quantitative explanation.

Now the three above conditions seems to be particularly well adapted to the description of living systems. Let us give a simple example of such an application.

In living systems, morphologies are acquired through growth processes. One can attempt to describe such a growth in terms of an infinite family of virtual, fractal and locally irreversible, trajectories. Their equation can therefore be written under the form (94), then it can be integrated in a Schrödinger equation (112).

We now look for solutions describing a growth from a center. This problem is formally identical to the problem of planetary nebulae (which are stars that eject their outer shells), and, in the quantum point of view, to the problem of particle scattering. The solutions looked for correspond to the case of the outgoing spherical probability wave.

Depending on the potential, on the boundary conditions and on the symmetry conditions, a very large family of solutions can be obtained. Let us consider here only the simplest ones, i.e., central potential and spherical symmetry. The probability density distribution of the various possible values of the angles are given in this case by the spherical harmonics:

$$P(\theta, \varphi) = |Y_{lm}(\theta, \varphi)|^2. \tag{125}$$

These functions show peaks of probability for some angles, depending on the quantized values of the square of angular momentum L^2 (measured by the quantum number l) and of its projection L_z on axis z (measured by the quantum number m).

Finally a more probable morphology is obtained by sending matter at angles of maximal probability. The solutions obtained in this way, show floral 'tulip'-like shape (see Fig. 12 and Ref. [89]). Now the spherical symmetry is broken in the case of living systems. One jumps to cylindrical symmetry: this leads in the simplest case to introduce a periodic quantization of angle θ (measured by an additional quantum number k), that gives rise to a separation of discretized petals. Moreover there is a discrete symmetry breaking along axis z linked to orientation (separation of 'up' and 'down' due to gravity, growth from a stem). This results in floral shapes such as given in Fig. 14.

5.9.2 Formation, duplication and bifurcation

A fondamentally new feature of the scale-relativity approach concerning problems of formation is that the Schrödinger form taken by the geodesics equation can be interpreted as a general tendency for systems to which it applies to make structures, i.e., to self-organize. In the framework of a classical deterministic approach, the question of the formation of a system is always posed in terms of initial conditions. In the new framework, structures are formed whatever the initial conditions, in correspondance with the field, the boundary conditions and the symmetries, and in function of the values of the various conservative quantities that characterize the system.

A typical example is given by the formation of gravitational structures from a background medium of strictly constant density (Fig. 15). This problem has no classical solution: no structure can form and grow in the absence of large initial fluctuations. On the contrary, in the present quantum-like approach, the

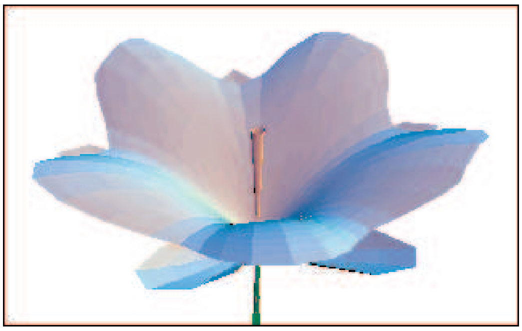

Figure 14: Morphogenesis of flower-like structure, solution of a Schrödinger equation that describes a growth process ($l = 5$, $m = 0$). The 'petals' and 'sepals' and 'stamen' are traced along angles of maximal probability density. A constant force of 'tension' has been added, involving an additional curvature of petals, and a quantization of the angle θ that gives an integer number of petals (here, $k = 5$).

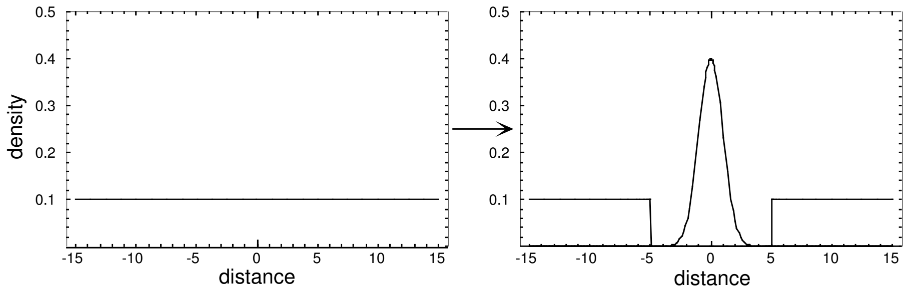

Figure 15: Model of formation of a structure from a background medium. The global harmonic oscillator potential defined by the background induces the formation of a local structure (see a 3D representation in Fig. 16), in such a way that the average density remains constant (i.e., the matter of the structure is taken from the medium).

stationary Schrödinger equation for an harmonic oscillator potential (which is the form taken by the gravitational potential in this case) does have confined stationary solutions. The 'fundamental level' solution ($n = 0$ is made of one object with Gaussian distribution (see Fig. 15), the second level ($n = 1$) is a pair of objects (see Fig. 16), then one obtains chains, trapezes, etc... for higher levels. It is remarkable that, whatever the scales (stars, clusters of stars, galaxies, clusters of galaxies) the zones of formation show in a systematic way this kind of double, aligned or trapeze-like structures [56].

Now these solutions may also be meaningful in other domains than gravitation, because the harmonic oscillator potential is encountered in a wide range of conditions. It is the general force that appears when a system is displaced from its equilibrium conditions, and, moreover, it describes an elementary clock. For these reasons, it is well adapted to an attempt of description of living systems, at first in a rough way [Chaline, Grou and Nottale, in preparation].

Firstly, such an approach could allow one to ask the question of the origin of life in a renewed way. This problem is the analog of the 'vacuum' (lowest energy) solutions, i.e. of the passage from a non-structured medium to the simplest, fundamental level structures. Provided the description of the prebiotic medium comes under the three above conditions (complete information loss on angles, position and time), we suggest that it could be subjected to a Schrödinger equation (with a coefficient $\mathcal{D}$ self-generated by the system itself). Such a possibility is supported by the symplectic formal structure of thermodynamics [65], in which the state equations are analogous to Hamilton-Jacobi equations. One can therefore contemplate the possibility of a future 'quantization' of thermodynamics, and then of the chemistry of solutions. In such a framework, the fundamental equations would describe a universal tendency to make structures. Moreover, a first result naturally emerges: due to the quantization of energy, we expect the primordial structures to appear at a given non-zero energy, without any intermediate step. But clearly much pluridisciplinary work is needed in order to implement such a working program.

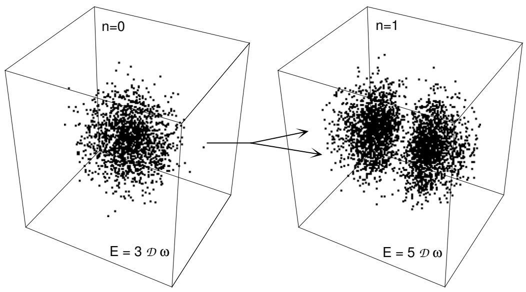

Figure 16: Model of duplication. The stationary solutions of the Schrödinger equation in a 3D harmonic oscillator potential can take only discretized morphologies in correspondence with the quantized value of the energy. Provided the energy increases from the one-object case ($E_0 = 3\mathcal{D}\omega$), no stable solution can exist before it reaches the second quantized level at $E_1 = 5\mathcal{D}\omega$. The solutions of the time-dependent equation show that the system jumps from the one object to the two-object morphology.

Secondly, the analogy can be pushed farther, since the passage from the fundamental level to the first excited level now provides us with a (rough) model of duplication (see Figs. 16 and 17). Once again, the quantization implies that, in case of energy increase, the system will not increase its size, but will instead be lead to jump from a one-object structure to a two-object structure, with no stable intermediate step between the two stationary solutions $n = 0$ and $n = 1$. Moreover, if one comes back to the level of description of individual trajectories, one finds that from each point of the initial one body-structure there exist trajectories that go to the two final structures. We expect, in this framework, that duplication needs a discretized and precisely fixed jump in energy.

Such a model can also be applied to the description of a branching process (Fig. 17), e.g. in the case of a tree growth when the previous structure remains instead of disappearing as in cell duplication.

Note finally that, though such a model is still too rough to claim that it describes biological systems, it may already improved by incorporating in it other results that are quoted elsewhere in this paper, in particular (i) the model of membrane through fractal dimension variable with the distance to a center (Sec. 4.3.5); (ii) the model of multiple hierarchical levels of organization depending on 'complexergy' (Sec. 8.4).

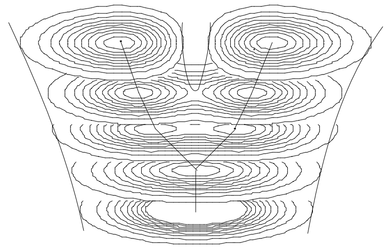

Figure 17: Model of branching / bifurcation. Successive solutions of the time-dependent 2D Schrödinger equation in an harmonic oscillator potential are plotted as isodensities. The energy varies from the fundamental level ($n = 0$) to the first excited level ($n = 1$), and as a consequence the system jumps from a one-object to a two-object morphology.

6 Fractal space-time and relativistic quantum mechanics

6.1 Klein-Gordon equation

6.1.1 Theory

Let us now come back to standard quantum mechanics, but in the motion-relativistic case (i.e., classical Minkowski space-time). We shall recall here how one can get the free and electromagnetic Klein-Gordon equations, as already presented in [25, 23].

Most elements of our approach as described hereabove remain correct, with the time differential element dt replaced by the proper time differential element, ds. Now not only space, but the full space-time continuum, is considered to be non-differentiable, and therefore fractal. We chose a metric of signature $(+, -, -, -)$. The elementary displacement along geodesics now writes (in the standard case $D_F = 2$):

$$dX^{\mu}_{\pm} = dx^{\mu}_{\pm} + d\xi^{\mu}_{\pm}. \tag{126}$$

Due to the breaking of the reflection symmetry ($ds \leftrightarrow -ds$) issued from non-differentiability, we still define two 'classical' derivatives, d_+/ds and d_-/ds, which, once applied to x^{μ}, yield two classical 4-velocities,

$$\frac{d_+}{ds} x^{\mu}(s) = v^{\mu}_{+} \quad ; \quad \frac{d_-}{ds} x^{\mu}(s) = v^{\mu}_{-}. \tag{127}$$

These two derivatives can be combined in terms of a complex derivative operator

$$\frac{\acute{d}}{ds} = \frac{(d_+ + d_-) - i\,(d_+ - d_-)}{2ds}, \tag{128}$$

which, when applied to the position vector, yields a complex 4-velocity

$$\mathcal{V}^{\mu} = \frac{\acute{d}}{ds} x^{\mu} = V^{\mu} - i\,U^{\mu} = \frac{v^{\mu}_{+} + v^{\mu}_{-}}{2} - i\,\frac{v^{\mu}_{+} - v^{\mu}_{-}}{2}. \tag{129}$$

We are lead to a stochastic description, due to the infinity of geodesics of the fractal space-time. This forces us to consider the question of the definition of a Lorentz-covariant stochasticity in space-time. This problem has been addressed by several authors in the framework of a relativistic generalization of Nelson's stochastic quantum mechanics. Two fluctuation fields, $d\xi_\pm^\mu(s)$, are defined, which have zero expectation ($< d\xi_\pm^\mu >= 0$), are mutually independent and such that

$$< d\xi_\pm^\mu \, d\xi_\pm^\nu >= \mp\lambda \, \eta^{\mu\nu} ds. \tag{130}$$

The constant λ is another writing for the coefficient $2\mathcal{D} = \lambda c$, with $\mathcal{D} = \hbar/2m$ in the standard quantum case: namely, it is the Compton length of the particle. This process makes sense only in $\mathbb{R}^4$, i.e. the "metric" $\eta^{\mu\nu}$ should be positive definite: indeed, the fractal fluctuations are of the same nature as uncertainties and 'errors', so that the space and the time fluctuations add quadratically. The sign corresponds to a choice of space-like fluctuations.

Dohrn and Guerra [76] introduce the above "Brownian metric" and a kinetic metric $g_{\mu\nu}$, and obtain a compatibility condition between them which reads $g_{\mu\nu}\eta^{\mu\alpha}\eta^{\nu\beta} = g^{\alpha\beta}$. An equivalent method was developed by Zastawniak [77], who introduces, in addition to the covariant drifts v_+^μ and v_-^μ, new drifts b_+^μ and b_-^μ (note that our notations are different from his). Serva [78] gives up Markov processes and considers a covariant process which belongs to a larger class, known as "Bernstein processes".

All these proposals are equivalent, and amount to transforming a Laplacian operator in $\mathbb{R}^4$ into a Dalembertian. Namely, the two $(+)$ and $(-)$ differentials of a function $f[x(s), s]$ can be written (we assume a Minkowskian metric for classical space-time):

$$d_\pm f/ds = (\partial/\partial s + v_\pm^\mu \, \partial_\mu \mp \frac{1}{2}\lambda \, \partial^\mu \partial_\mu) f. \tag{131}$$

In what follows, we shall only consider s-stationary functions, i.e., that are not explicitly dependent on the proper time s. In this case the covariant time derivative operator reduces to:

$$\frac{\acute{d}}{ds} = \left(\mathcal{V}^\mu + \frac{1}{2} i \, \lambda \, \partial^\mu\right) \partial_\mu. \tag{132}$$

Let us assume that the system under consideration can be characterized by an action $\mathcal{S}$, which is complex because the four-velocity is now complex. The same reasoning as in classical mechanics leads us to write $d\mathcal{S} = -mc\mathcal{V}_\mu dx^\mu$ (see [74] for another equivalent choice). The least-action principle applied on this action yields the equations of motion of a free particle, that takes the form of a geodesics equation, $\acute{d}\,\mathcal{V}_\alpha/ds = 0$. Such a form is also directly obtained from the 'strong covariance' principle and the generalized equivalence principle. We can also write the variation of the action as a functional of coordinates. We obtain the usual result (but here generalized to complex quantities):

$$\delta\mathcal{S} = -mc\mathcal{V}_\mu \delta x^\mu \;\Rightarrow\; \mathcal{P}_\mu = mc\mathcal{V}_\mu = -\partial_\mu \mathcal{S}, \tag{133}$$

where $\mathcal{P}_\mu$ is now a complex 4-momentum. As in the nonrelativistic case, the wave function is introduced as being nothing but a reexpression of the action:

$$\psi = e^{i\mathcal{S}/mc\lambda} \;\Rightarrow\; \mathcal{V}_\mu = i\lambda \, \partial_\mu (\ln \psi), \tag{134}$$

so that the equations of motion become:

$$\acute{d}\,\mathcal{V}_\alpha/ds = i\lambda \left(\mathcal{V}^\mu + \frac{1}{2} i\lambda\partial^\mu\right) \partial_\mu \mathcal{V}_\alpha = 0 \Rightarrow \left(\partial^\mu \ln \psi + \frac{1}{2}\partial^\mu\right) \partial_\mu \partial_\alpha \ln \psi = 0. \tag{135}$$

Now, by using the remarkable identity (104) established in [4], it reads:

$$\partial_\alpha(\partial_\mu\partial^\mu \ln\psi + \partial_\mu \ln\psi\, \partial^\mu \ln\psi) = \partial_\alpha \left(\frac{\partial_\mu\partial^\mu\psi}{\psi}\right) = 0. \tag{136}$$

So the equation of motion can finally be integrated in terms of the Klein-Gordon equation for a free particle:

$$\lambda^2\partial^\mu\partial_\mu\psi = \psi, \tag{137}$$

where $\lambda = \hbar/mc$ is the Compton length of the particle. The integration constant is chosen so as to ensure the identification of $\varrho = \psi\psi^\dagger$ with a probability density for the particle.

As shown by Zastawniak [77] and as can be easily recovered from the definition (129), the quadratic invariant of special motion-relativity, $v^\mu v_\mu = 1$, is naturally generalized as

$$\mathcal{V}^\mu\mathcal{V}^\dagger_\mu = 1, \tag{138}$$

where $\mathcal{V}^\dagger_\mu$ is the complex conjugate of $\mathcal{V}_\mu$. This ensures the covariance (i.e. the invariance of the form of equations) of the theory at this level.

6.1.2 Quadratic invariant, Leibniz rule and complex velocity operator

However, it has been recalled by Pissondes [73, 74] that the square of the complex four-velocity is no longer equal to unity, since it is now complex. It can be derived directly from (136) after accounting for the Klein-Gordon equation. One obtains the generalized energy (or quadratic invariant) equation:

$$\mathcal{V}_\mu\mathcal{V}^\mu + i\lambda\partial_\mu\mathcal{V}^\mu = 1. \tag{139}$$

Now taking the gradient of this equation, one obtains:

$$\partial_\alpha(\mathcal{V}_\mu\mathcal{V}^\mu + i\lambda\partial_\mu\mathcal{V}^\mu) = 0 \Rightarrow \left(\mathcal{V}^\mu + \frac{1}{2}i\lambda\partial^\mu\right)\partial_\alpha\mathcal{V}_\mu = 0, \tag{140}$$

which is equivalent to Eq. (135) in the case of free motion, since in the absence of external field $\partial_\alpha\mathcal{V}_\mu = \partial_\mu\mathcal{V}_\alpha$.

Clearly, the new form of the quadratic invariant comes only under 'weak covariance'. Pissondes has therefore addressed the problem of implementing the strong covariance (i.e., of keeping the free, Galilean form of the equations of physics even in the new, more complicated situation) at all levels of the description. The additional terms in the various equations find their origin in the very definition of the 'quantum-covariant' total derivative operator. Indeed, it contains derivatives of first order (namely, $\partial/\partial t + \mathcal{V}\nabla$), but also derivatives of second order ($-i\mathcal{D}\Delta$). Therefore, when one is led to compute quantities like $\acute{d}(fg)/dt = 0$ the Leibniz rule to use becomes a linear combination of the first order and second order Leibniz rules. There is no problem provided one always come back to the definition of the covariant total derivative. (Some inconsistency would appear if one, in contradiction with this definition, wanted to use only the first order Leibniz rule $d(fg) = f dg + g df$). Indeed, one finds:

$$\frac{\acute{d}}{ds}(fg) = f\,\frac{\acute{d}\,g}{ds} + g\,\frac{\acute{d}\,f}{ds} + i\lambda\,\partial^\mu f\,\partial_\mu g. \tag{141}$$

Pissondes attempted to find a formal tool in terms of which the form of the first order Leibniz rule would be preserved. He introduced the following 'symmetric product':

$$f \circ \frac{\acute{d}\, g}{ds} = f\, \frac{\acute{d}\, g}{ds} + i\, \frac{\lambda}{2}\, \partial^\mu f\, \partial_\mu g, \tag{142}$$

and he showed that, using this product, the covariance can be fully implemented. In particular, one recovers the form of the derivative of a product, $\acute{d}\,(fg) = f \circ \acute{d}\, g + g \circ \acute{d}\, f$, and the standard decomposition in terms of partial derivatives, $\acute{d}\, f = \partial_\mu f \circ \acute{d}\, x^\mu$.

However, one of the problem with this tool is that it depends on two functions f and g. We shall therefore use another equivalent tool, which has the advantage to depend only on one function. We define a complex velocity operator:

$$\widehat{\mathcal{V}_\mu} = \mathcal{V}^\mu + i\, \frac{\lambda}{2}\, \partial^\mu, \tag{143}$$

so that the covariant derivative can now be written in terms of an operator product that keeps the standard, first order form:

$$\frac{\acute{d}}{ds} = \widehat{\mathcal{V}^\mu}\, \partial_\mu\ . \tag{144}$$

More generally, one defines the operator:

$$\widehat{\frac{\acute{d}\, g}{ds}} = \frac{\acute{d}\, g}{ds} + i \frac{\lambda}{2}\, \partial^\mu g\, \partial_\mu \tag{145}$$

which has the advantage to be defined only in terms of g. The covariant derivative of a product now writes

$$\frac{\acute{d}\,(fg)}{ds} = \widehat{\frac{\acute{d}\, f}{ds}}\, g + \widehat{\frac{\acute{d}\, g}{ds}}\, f \tag{146}$$

i.e., one recovers the form of the first order Leibniz rule. Since $\widehat{f} g \neq \widehat{g} f$, one is led to define a symmetrized product, following Pissondes [74]. One defines $\dot{f} = \acute{d}\, f/ds$, then

$$\dot{f} \otimes \dot{g} = \widehat{\dot{f}} \dot{g} + \widehat{\dot{g}} \dot{f} - \dot{f} \dot{g}. \tag{147}$$

This product is now commutative, $\dot{f} \otimes \dot{g} = \dot{g} \otimes \dot{f}$, and in its terms the standard expression for the square of the velocity is recovered, namely,

$$\mathcal{V}^\mu \otimes \mathcal{V}_\mu = 1. \tag{148}$$

The introduction of such a tool, that may appear formal in the case of free motion, becomes particularly useful in the presence of an electromagnetic field: this point will be further developed in Sect. 7. We shall show that the introduction of a new level of complexity in the description of a relativistic fractal space-time, namely, the account of resolutions that become functions of coordinates, leads to a new geometric theory of gauge fields, in particular of the U(1) electromagnetic field. We find that the complex velocity is given in this case by:

$$\mathcal{V}^\mu = i\lambda\, D^\mu \ln\psi = i\lambda\, \partial^\mu \ln\psi - \frac{e}{mc^2} A^\mu, \tag{149}$$

where A^μ is a field of dilations of internal resolutions that can be identified with an electromagnetic field.

Inserting this expression in Eq. (140) yields the standard Klein-Gordon equation with electromagnetic field, $[i\hbar\partial_\mu - (e/c)A_\mu][i\hbar\partial^\mu - (e/c)A^\mu]\psi = m^2c^2\psi$ [74].

6.1.3 Application to gravitation in the motion-relativistic case

Let us consider the motion of a free particle in a curved and fractal space-*time*. One can define a motion-covariant and quantum-covariant derivative that combine the general-relativistic covariant derivative (which describes the effects of curvature) and the scale-relativistic quantum-covariant derivative (which describes the effects of fractality), namely,

$$\frac{\bar{D}A^\mu}{ds} = \left(\frac{\partial}{\partial s} + \mathcal{V}^\nu \partial_\nu + i\frac{\lambda}{2}\partial^\nu \partial_\nu\right) A^\mu + \Gamma^\mu_{\rho\nu} \mathcal{V}^\rho A^\nu. \tag{150}$$

The equation of motion of a free particle can now be written as a geodesics equation by using this covariant derivative. However, one should take care that the combination of the two covariant derivatives imply the appearance of a new term in the geodesics equation [71, 26, 73]. This is easily established by starting from the quadratic invariant, [74] $\mathcal{V}_\mu \mathcal{V}^\mu + i\lambda\partial_\mu \mathcal{V}^\mu = 1$, which becomes in the general-relativistic case:

$$\mathcal{V}_\mu \mathcal{V}^\mu + i\lambda\, D_\mu \mathcal{V}^\mu = 1, \tag{151}$$

where we now have $\mathcal{V}_\mu \mathcal{V}^\mu = g_{\mu\nu} \mathcal{V}^\mu \mathcal{V}^\nu$, D_μ being Einstein's covariant derivative. The equations of motion are obtained by differentiating this relation. One obtains [73]:

$$\frac{\acute{d}}{ds}\mathcal{V}^\mu + \Gamma^\mu_{\nu\rho} \mathcal{V}^\nu \mathcal{V}^\rho - i\frac{\lambda}{2}\, R^\mu_\nu\, \mathcal{V}^\nu = 0. \tag{152}$$

This equation can be integrated in terms of ψ to yield a generalized "Einstein-Klein-Gordon" equation of motion:

$$\lambda^2 \left(g_{\mu\nu}\, \partial^\mu \partial^\nu \psi + \partial_\nu (\ln\sqrt{-g})\, \partial^\nu \psi\right) = -1, \tag{153}$$

where g is the metrics determinant. A detailed study of this equation, although interesting, is outside the scope of the present contribution. It may have physical applications, e.g., for the description of the close environment of black holes.

6.2 Dirac Equation

6.2.1 Reflection symmetry breaking of spatial differential element

One of the main result of the scale-relativity theory is its ability to provide a physical origin for the complex nature of the wave function in quantum mechanics. Indeed, we have seen that in its framework, it is a direct consequence of the non-differentiable geometry of space-time, which involves a symmetry breaking of the reflection invariance $dt \leftrightarrow -dt$, and therefore a two-valuedness of the classical velocity vector.

Going to motion-relativistic quantum mechanics amounts to introduce not only a fractal space, but a fractal space-time. The invariant parameter becomes in this case the proper time s instead of the time t. As a consequence the complex nature of the four-dimensional wave function in the Klein-Gordon equation comes from the discrete symmetry breaking $ds \leftrightarrow -ds$.

However, this is not the last word of the new structures implied by the non-differentiability. The total derivative of a physical quantity also involves partial derivatives with respect to the space variables, $\partial/\partial x^\mu$. Once again, from the very definition of derivatives, the discrete symmetry under the reflection $dx^\mu \leftrightarrow -dx^\mu$

should also broken at a more profound level of description. Therefore, we expect the possible appearance of a new two-valuedness of the generalized velocity.

At this level one should also account for parity violation. Finally, we have suggested that the three discrete symmetry breakings

$$ds \leftrightarrow -ds \qquad dx^\mu \leftrightarrow -dx^\mu \qquad x^\mu \leftrightarrow -x^\mu$$

can be accounted for by the introduction of a bi-quaternionic velocity. It has been subsequently shown by Célérier [66, 24] that one can derive in this way the Dirac equation, namely as an integral of a geodesics equation: this demonstration is summarized in what follows. In other words, this means that this new two-valuedness is at the origin of the bi-spinor nature of the electron wave function.

6.2.2 Spinors as bi-quaternionic wave-function

Since $\mathcal{V}^\mu$ is now bi-quaternionic, the Lagrange function is also bi-quaternionic and, therefore, the same is true of the action. Moreover, it has been shown [66] that, for s-stationary processes, the bi-quaternionic generalisation of the quantum-covariant derivative keeps the same form as in the complex number case, namely,

$$\frac{\acute{d}}{ds} = \mathcal{V}^\nu \partial_\nu + i\frac{\lambda}{2}\partial^\nu \partial_\nu. \tag{154}$$

A generalized equivalence principle, as well as a strong covariance principle, allows us to write the equation of motion under a free-motion form, i.e., under the form of a differential geodesics equation

$$\frac{\acute{d}\mathcal{V}_\mu}{ds} = 0, \tag{155}$$

where $\mathcal{V}_\mu$ is the bi-quaternionic four-velocity, e.g., the covariant counterpart of $\mathcal{V}^\mu$.

The elementary variation of the action, considered as a functional of the coordinates, keeps the usual form

$$\delta\mathcal{S} = -mc\,\mathcal{V}_\mu\,\delta x^\mu. \tag{156}$$

We thus obtain the bi-quaternionic four-momentum, as

$$\mathcal{P}_\mu = mc\mathcal{V}_\mu = -\partial_\mu \mathcal{S}. \tag{157}$$

We are now able to introduce the wave function. We define it as a re-expression of the bi-quaternionic action by

$$\psi^{-1}\partial_\mu\psi = \frac{i}{cS_0}\partial_\mu\mathcal{S}, \tag{158}$$

using, in the left-hand side, the quaternionic product. The bi-quaternionic four-velocity is derived from Eq. (157), as

$$\mathcal{V}_\mu = i\frac{S_0}{m}\psi^{-1}\partial_\mu\psi. \tag{159}$$

This is the bi-quaternionic generalization of the definition used in the Schrödinger case: $\psi = e^{iS/S_0}$. It is worth stressing here that we could choose, for the definition of the wave function in Eq. (158), a commutated expression in the left-hand side, i.e., $(\partial_\mu \psi)\psi^{-1}$ instead of $\psi^{-1}\partial_\mu \psi$. But with this reversed choice, owing to the non-commutativity of the quaternionic product, we could not obtain the motion equation as a vanishing four-gradient, as in Eq. (165). Therefore, we retain the above simplest choice, as yielding an equation which can be integrated.

Note that this non-commutativity is at a more profound level than that already involved on the quantum operatorial tool by the fractal and non-differentiable description [74]. It is now at the level of the fractal space-time itself, which therefore fundamentally comes under Connes's noncommutative geometry [81, 82]. Moreover, this non-commutativity might be considered as a key for a future understanding of the parity and CP violation, which will not be developed here.

Finally, the isomorphism which can be established between the quaternionic and spinorial algebrae through the multiplication rules applying to the Pauli spin matrices allows us to identify the wave function ψ to a Dirac bispinor. Indeed, spinors and quaternions are both a representation of the SL(2,C). This identification is reinforced by the result [66, 24] that follows, according to which the geodesics equation written in terms of bi-quaternions is naturally integrated under the form of the Dirac equation.

6.2.3 Free-particle bi-quaternionic Klein-Gordon equation

The equation of motion, Eq. (155), writes

$$\left(\mathcal{V}^\nu \partial_\nu + i\frac{\lambda}{2}\partial^\nu \partial_\nu\right)\mathcal{V}_\mu = 0. \tag{160}$$

We replace $\mathcal{V}_\mu$, (respectively $\mathcal{V}^\nu$), by their expressions given in Eq. (159) and obtain

$$i\frac{S_0}{m}\left(i\frac{S_0}{m}\psi^{-1}\partial^\nu \psi \partial_\nu + i\frac{\lambda}{2}\partial^\nu \partial_\nu\right)\left(\psi^{-1}\partial_\mu \psi\right) = 0. \tag{161}$$

The choice $\mathcal{S}_0 = m\lambda$ allows us to simplify this equation and we get

$$\psi^{-1}\partial^\nu \psi \; \partial_\nu(\psi^{-1}\partial_\mu \psi) + \frac{1}{2}\partial^\nu \partial_\nu(\psi^{-1}\partial_\mu \psi) = 0. \tag{162}$$

The definition of the inverse of a quaternion

$$\psi\psi^{-1} = \psi^{-1}\psi = 1, \tag{163}$$

implies that ψ and ψ^{-1} commute. But this is not necessarily the case for ψ and $\partial_\mu \psi^{-1}$ nor for ψ^{-1} and $\partial_\mu \psi$ and their contravariant counterparts. However, when we derive Eq. (163) with respect to the coordinates, we obtain

$$\begin{aligned} \psi \; \partial_\mu \psi^{-1} &= -(\partial_\mu \psi)\psi^{-1}, \\ \psi^{-1}\partial_\mu \psi &= -(\partial_\mu \psi^{-1})\psi, \end{aligned} \tag{164}$$

and identical formulae for the contravariant analogues.

Developing Eq. (162), using Eqs. (164) and the property $\partial^\nu \partial_\nu \partial_\mu = \partial_\mu \partial^\nu \partial_\nu$, we obtain, after some calculations,

$$\partial_\mu[(\partial^\nu \partial_\nu \psi)\psi^{-1}] = 0. \tag{165}$$

We integrate this four-gradient as

$$(\partial^\nu \partial_\nu \psi)\psi^{-1} + C = 0, \tag{166}$$

of which we take the right product by ψ to obtain

$$\partial^\nu \partial_\nu \psi + C\psi = 0. \tag{167}$$

We therefore recognize the Klein-Gordon equation for a free particle with a mass m, after the identification $C = m^2c^2/\hbar^2 = 1/\lambda^2$. But in this equation ψ is now a biquaternion, i.e. a Dirac bispinor.

6.2.4 Derivation of the Dirac equation

We now use a long-known property of the quaternionic formalism, which allows to obtain the Dirac equation for a free particle as a mere square root of the Klein-Gordon operator (see Refs. in [66, 24]).

We first develop the Klein-Gordon equation as

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{m^2c^2}{\hbar^2}\psi. \tag{168}$$

Thanks to the property of the quaternionic and complex imaginary units $e_1^2 = e_2^2 = e_3^2 = i^2 = -1$, we can write Eq. (168) under the form

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = e_3^2\frac{\partial^2\psi}{\partial x^2}e_2^2 + ie_1^2\frac{\partial^2\psi}{\partial y^2}i + e_3^2\frac{\partial^2\psi}{\partial z^2}e_1^2 + i^2\frac{m^2c^2}{\hbar^2}e_3^2\psi e_3^2. \tag{169}$$

We now take advantage of the anticommutative property of the quaternionic units ($e_ie_j = -e_je_i$ for $i \neq j$) to add to the right-hand side of Eq. (169) six vanishing couples of terms which we rearrange as

$$\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\psi}{\partial t}\right) &= e_3\frac{\partial}{\partial x}\left(e_3\frac{\partial\psi}{\partial x}e_2 + e_1\frac{\partial\psi}{\partial y}i + e_3\frac{\partial\psi}{\partial z}e_1 - i\frac{mc}{\hbar}e_3\psi e_3\right)e_2 \\
&+ e_1\frac{\partial}{\partial y}\left(e_3\frac{\partial\psi}{\partial x}e_2 + e_1\frac{\partial\psi}{\partial y}i + e_3\frac{\partial\psi}{\partial z}e_1 - i\frac{mc}{\hbar}e_3\psi e_3\right)i \\
&+ e_3\frac{\partial}{\partial z}\left(e_3\frac{\partial\psi}{\partial x}e_2 + e_1\frac{\partial\psi}{\partial y}i + e_3\frac{\partial\psi}{\partial z}e_1 - i\frac{mc}{\hbar}e_3\psi e_3\right)e_1 \\
&- i\frac{mc}{\hbar}e_3\left(e_3\frac{\partial\psi}{\partial x}e_2 + e_1\frac{\partial\psi}{\partial y}i + e_3\frac{\partial\psi}{\partial z}e_1 - i\frac{mc}{\hbar}e_3\psi e_3\right)e_3.
\end{aligned} \tag{170}$$

We see that Eq. (170) is obtained by applying two times to the bi-quaternionic wavefunction ψ the operator

$$\frac{1}{c}\frac{\partial}{\partial t} = e_3\frac{\partial}{\partial x}e_2 + e_1\frac{\partial}{\partial y}i + e_3\frac{\partial}{\partial z}e_1 - i\frac{mc}{\hbar}e_3(\quad)e_3. \tag{171}$$

The three first Conway matrices $e_3(\quad)e_2$, $e_1(\quad)i$ and $e_3(\quad)e_1$ [80], appearing in the right-hand side of Eq. (171), can be written in the compact form $-\alpha^k$, with

$$\alpha^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix},$$

the σ_k's being the three Pauli matrices, while the fourth Conway matrix

$$e_3(\quad)e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

can be recognized as the Dirac β matrix. We can therefore write Eq. (171) as the non-covariant Dirac equation for a free fermion

$$\frac{1}{c}\frac{\partial\psi}{\partial t} = -\alpha^k \frac{\partial\psi}{\partial x^k} - i\frac{mc}{\hbar}\beta\psi. \tag{172}$$

The covariant form, in the Dirac representation, can be recovered by applying $ie_3(\quad)e_3$ to Eq. (172).

6.2.5 Pauli equation

Finally it is easy to derive the Pauli equation, since it is known that it can be obtained as a non-relativistic approximation of the Dirac equation [85]. Two of the components of the Dirac bi-spinor become negligible when $v << c$, so that they become Pauli spinors (i.e., in our representation the bi-quaternions are reduced to quaternions) and the Dirac equation is transformed in a Schödinger equation for these spinors with a magnetic dipole additional term. Such an equation is but the Pauli equation. Therefore the Pauli equation is understood in the scale-relativity framework as a manifestation of the fractality of space (but not time), while the symmetry breaking of space differential elements is nevertheless at work:

	fractal space ($ds = dt$)	fractal space-time
$ds \leftrightarrow -ds$	Schrödinger	Klein-Gordon
$dx \leftrightarrow -dx$	Pauli	Dirac

7 Gauge theories and scale relativity

7.1 Introduction

Let us now review another important field of application of the fractal space-time / scale-relativity theory. In the previous sections, we have (i) developed the scale laws, internal to a given space-time point, that can be constructed from first physical principles; (ii) studied the consequences on motion of the simplest of these laws.

In that description the resolution variables $\ln(\lambda/\varepsilon)$ can take all the values of the scale-space, but, as a first step, they do not themselves vary in function of other variables. Then we have considered the new situation of 'scale dynamics', in which 'scale-accelerations' are defined, so that the resolutions may vary with the djinn (variable scale-dimension).

We shall now consider the case when the ε variables become functions of the coordinates, $\varepsilon = \varepsilon(x, y, z, t)$. This means that the resolutions become themselves a field. Such a case can be described as a coupling between motion and scales, but it also comes under a future 'general scale-relativitistic' description in which scale and motion will be treated on the same footing. As we shall now recall, this approach provides us with a new interpretation of gauge transformations and therefore with a geometric interpretation of the nature of gauge fields [25, 23, 83].

In the present physical theory, one still does not really understand the nature of the electric charge and of the electromagnetic field. As recalled by Landau ([84], Chap. 16), in the classical theory the very existence of the charge e and of the electromagnetic 4-potential A_μ are ultimately derived from experimental data. Moreover, the form of the action for a particle in an electromagnetic field cannot be chosen only from general considerations, and it is therefore merely postulated. Said another way, contrarily to general relativity in which the equation of trajectories (i.e., the fundamental equation of dynamics) is self-imposed as a geodesics equation, in today's theory of electromagnetism the Lorentz force must be added to Maxwell's equations. Quantum field theories have improved the situation thanks to the link between the nature of charges and gauge invariance, but we still lack a fundamental understanding of the nature of gauge transformations.

We shall now review the proposals of the scale-relativity approach made in order to solve these problems, that involves a new interpretation of gauge transformations, then we shall recall some of its possible consequences.

7.2 Nature of the electromagnetic field (classical theory)

7.2.1 Electromagnetic potential as dilation field

The theory of scale relativity allows one to get new insights about the nature of the electromagnetic field, of the electric charge, and about the physical meaning of gauge invariance. Consider an electron (or any other charged particle). In scale relativity, we identify the particle with a family of fractal trajectories, described as the geodesics of a nondifferentiable space-time. These trajectories are characterized by internal (fractal) structures.

Now consider anyone of these structures, lying at some (relative) resolution ε smaller that the Compton length of the particle (i.e. such that $\varepsilon < \lambda_c$) for a given relative position of the particle. In a displacement of the particle, the relativity of scales implies that the resolution at which this given structure appears in the new position will a priori be different from the initial one. Indeed, if the whole internal fractal structure of the electron was rigidly fixed, this would mean an absolute character of the scale-space, which would bein contradiction with the principle of the relativity of scales.

Therefore we expect the occurrence of dilatations of resolutions induced by translations, which read:

$$q\frac{\delta\varepsilon}{\varepsilon} = -A_\mu \delta x^\mu. \tag{173}$$

In this expression, the elementary dilation is written as $\delta\varepsilon/\varepsilon = \delta \ln(\varepsilon/\lambda)$: this is justified by the Gell-Mann-Levy method, from which the dilation operator is found to take the form $\tilde{D} = \varepsilon\partial/\partial\varepsilon = \partial/\partial \ln \varepsilon$ (see Sect. 4.1.1). Since the elementary displacement in space-time δx^μ is a four-vector and since $\delta\varepsilon/\varepsilon$ is a scalar, one must introduce a four-vector A_μ in order to ensure covariance. The constant q measures the amplitude of the scale-motion coupling; it will be subsequently identified with the active electric charge that intervenes in the potential. This form ensures that the dimensionality of A_μ be CL^{-1}, where C is the electric charge unit (e.g., $\varphi = q/r$ for a Coulomb potential).

This behaviour can be expressed in terms of the introduction of a scale-covariant derivative. Namely, the total effect is now the sum of an inertial effect and of a geometric effect described by A_μ:

$$q\, \partial_\mu \ln(\lambda/\varepsilon) = q\, D_\mu \ln(\lambda/\varepsilon) + A_\mu. \tag{174}$$

(Note the correction of sign with respect to previous papers). This method is analogous to Einstein's contruction of generalized relativity of motion, in which the Christoffel components $\Gamma^{\mu}_{\nu\varrho}$ can be introduced directly from the mere principle of relativity of motion (see e.g. ref. [84]).

7.2.2 Nature of gauge invariance

Let us go one with the dilation field A_μ. If one wants such a "field" to be physical, it must be defined whatever the initial scale from which we started. Therefore, starting from another relative scale $\varepsilon' = \varrho\varepsilon$, where the scale ratio ϱ may be any function of coordinates $\varrho = \varrho(x, y, z, t)$, we get (considering only Galilean scale-relativity for the moment):

$$q\,\frac{\delta\varepsilon'}{\varepsilon'} = -A'_\mu\,\delta x^\mu, \tag{175}$$

so that we obtain:

$$A'_\mu = A_\mu + q\,\partial_\mu \ln\varrho(x, y, z, t). \tag{176}$$

Therefore the 4-vector A_μ depends on the relative "state of scale", or "scale velocity", $\ln\varrho = \ln(\varepsilon/\varepsilon')$.

We have suggested [25, 23] to identify A_μ with an electromagnetic 4-potential and Eq. (176) with the gauge invariance relation for the electromagnetic field, that writes in the standard way:

$$A'_\mu = A_\mu + q\,\partial_\mu\chi(x, y, z, t), \tag{177}$$

where χ is usually considered as an arbitrary function of coordinates devoid of physical meaning. This is no longer the case here, since it is now identified with a scale ratio $\chi = \ln\varrho$ between internal structures of the electron geodesics (at scales smaller than its Compton length). Our interpretation of the nature of the gauge function is compatible with its inobservability. Indeed, such a scale ratio is impossible to measure explicitly, since it would mean to make two measurements of two different relative scales smaller than the electron Compton length. But the very first measurement with resolution ε would change the state of the electron: namely, just after the measurement, its de Broglie length would become of order $\lambda_{\rm dB} \approx \varepsilon$ (see e.g. [4]), so that the second scale ε' would not be measured on the same electron. Therefore the ratio $\varrho = \varepsilon'/\varepsilon$ is destined to remain a virtual quantity. However, as we shall see in what follows, even whether it can not be directly measured, it has indirect consequences, so that the knowledge of its nature finally plays an important role: it allows one to demonstrate the quantization of the electron charge and to relate its value to that of its mass.

7.2.3 Electromagnetic field, electric charge and Lorentz force

Let us now show that the subsequent developments of the properties of this dilation field support its interpretation in terms of electromagnetic potentials.

If one considers translation along two different coordinates (or, in an equivalent way, displacement on a closed loop), one may write a commutator relation (once again, in analogy with the definition of the Riemann tensor in Einstein's general relativity):

$$q\,(\partial_\mu D_\nu - \partial_\nu D_\mu)\ln(\lambda/\varepsilon) = -(\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{178}$$

This relation defines a tensor field:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{179}$$

which, contrarily to A_μ, is independent of the initial relative scale (i.e., of the gauge). One recognizes in $F_{\mu\nu}$ the expression for the electromagnetic field. The first group of Maxwell equations directly derives from this expression, namely,

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{180}$$

In this interpretation, the property of gauge invariance recovers its initial status of scale invariance, in accordance with Weyl's initial ideas [86]. However, equation (176) may represent a progress compared with these early attempts and with the status of gauge invariance in today's physics. Indeed the gauge function, which has, up to now, been considered as arbitrary and devoid of physical meaning, is now identified with the logarithm of internal resolutions. As we shall recall in what follows, this interpretation has physical consequences concerning the quantization of the electric charge and its value.

Let us now derive the equation of motion of a charge in an electromagnetic field. Consider the action S for the electron. In the framework of a space-time theory based on a relativity principle, as it is the case here, it should be given directly by the length invariant s, i.e., $dS = -mc\, ds$. This relation ensures that the stationary action principle $\delta \int dS = 0$ becomes identical with a geodesics (Fermat) principle $\delta \int ds = 0$. Now the fractality of the geodesical curves to which the electron is identified means that, while S is an invariant with respect to space-time changes of the coordinate system, it is however a function of the scale variable, $S = S(\ln \rho)$, at scales smaller than λ.

Therefore its differential reads:

$$dS = \frac{\partial S}{\partial \ln \rho}\, d \ln \rho = \frac{\partial S}{\partial \ln \rho}\, (D \ln \rho + \frac{1}{q} A_\mu dx^\mu), \tag{181}$$

so that we obtain:

$$\partial_\mu S = D_\mu S + \frac{\partial S}{q\, \partial \ln \rho}\, A_\mu. \tag{182}$$

The first term of the product actually provides us with a definition for the 'passive' charge:

$$\frac{e}{c} = -\frac{\partial S}{q\, \partial \ln \rho} \tag{183}$$

This is an important point worth to be emphasized, since it will play an important role for the forthcoming generalizations to non-Abelian gauge theories. In the standard theory, the charge is set from experiment, then it is shown to be related to gauge transformations, while the gauge functions are considered to be arbitrary and devoid of physical meaning. In the scale-relativity approach, the charges are built from the symmetries of the scale space. One indeed recognizes in Eq. 183 the standard expression that relates, though the derivative of the action, a conservative quantity to the symmetry of a fundamental variable (here the internal relative resolution), following Noether's theorem.

Finally, the known form of the coupling term in the action is now demonstrated, while it was merely postulated in the standard theory:

$$S_{\rm pf} = \int -\frac{e}{c} A_\mu dx^\mu. \tag{184}$$

We can now write the total action under the form (recall that the field term is self-imposed to be the square of the electromagnetic tensor, see e.g. [84]):

$$S = S_{\rm p} + S_{\rm pf} + S_{\rm f} = -\int mc\, ds - \int \frac{e}{c} A_\mu dx^\mu - \frac{1}{16\pi c} \int F_{\mu\nu} F^{\mu\nu} d\Omega. \tag{185}$$

The variational principle applied on the two first terms of this action finally yields the searched motion equation as a geodesics equation (since tha action is now proportioanl to the length-invariant) and therefore the known expression for the Lorentz force acting on charge e:

$$mc\,\frac{du_\mu}{ds} = \frac{e}{c} F_{\mu\nu} u^\nu. \tag{186}$$

The variational principle applied on the two last terms (after their generalization to the current of several charges) yields Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} j^\nu. \tag{187}$$

In conclusion of this section, the progress here, respectively to the standard classical electromagnetic theory, is that, instead of being independently constructed, the Lorentz force and the Maxwell equations are derived in the scale relativity theory as being both manifestations of the fractal geometry of space-time. Moreover, a new physical meaning can be given to the electric charge and to gauge transformations.

We shall now consider its consequences for the quantum theory of electrodynamics.

7.3 Scale-relativistic quantum electrodynamics

7.3.1 Analysis of the problem

It is well-known that the quantum theory of electromagnetism and of the electron has added a new and essential stone in our understanding of the nature of charge. Indeed, in its framework, gauge invariance becomes deeply related to phase invariance of the wave function. The electric charge conservation is therefore directly related to the gauge symmetry. However, despite the huge progress that such a success has been (in particular, the extension of the approach to non-Abelian gauge theories has allowed to incorporate the weak and strong field into the same scheme) the lack of a fundamental understanding of the nature of gauge transformations has up to now prevented from reaching the final goal of gauge theories: namely, understand why charge is quantized and, as a consequence, theoretically calculate its quantized value.

Let us indeed consider the wave function of an electron. It writes:

$$\psi = \psi_0 \, \exp\left\{ \frac{i}{\hbar}(px - Et + \sigma\varphi + e\chi) \right\} \tag{188}$$

Its phase contains the usual products of fundamental quantities (space position, time, angle) and of their conjugate quantities (momentum, energy, angular momentum). They are related through Noether's theorem. Namely, the conjugate variables are the conservative quantities that originate from the space-time symmetries. This means that our knowledge of what are the energy, the momentum and the angular momentum and of their physical properties is founded on our knowledge of the nature of space, time and its transformations (translations and rotations).

This is true already in the classical theory, but there is something more in the quantum theory. In its framework, the conservative quantities are quantized when the basic variables are bounded. Concerning energy-momentum, this means that it is quantized only in some specific circumstances (e.g., bound states in atoms for which $r > 0$ in spherical coordinates). The case of the angular momentum is instructive: its differences are quantized in an universal way in units of $\hbar$ because angles differences can not exceed 2π.

In comparison, the last term in the phase of Eq. (188) keeps a special status in today's standard theory. The gauge function χ remains arbitrary, while it is clear from a comparison with the other terms that the meaning of charge e and the reason for its universal quantization can be obtained only from understanding the physical meaning of χ and why it is universally limited, since it is the quantity conjugate to the charge. As we shall now see, the identification of χ with the resolution scale factor $\ln \varrho$ that we have developped in the previous sections in the classical framework, can be transported to the quantum theory and allows one to suggest solutions to these problems in the special scale-relativity framework.

7.3.2 QED covariant derivative

Let us first show how one can recover the standard QED quantum derivative in the scale-relativity approach. Let us consider again the generalized action introduced in the previous section, which depends on motion and on scale variables. In the scale-relativistic quantum description, the 4-velocity is now complex (see Sec. 2.4), so that the action writes, $\mathcal{S} = \mathcal{S}(x^\mu, \mathcal{V}^\mu, \ln \varrho)$. Recall that the complex action gives the fundamental meaning of the wave function, namely, ψ is defined as:

$$\psi = e^{i\mathcal{S}/\hbar}. \tag{189}$$

The decomposition performed in the framework of the classical theory still holds and now becomes:

$$d\mathcal{S} = -mc\mathcal{V}_\mu dx^\mu - \frac{e}{c} A_\mu dx^\mu. \tag{190}$$

This leads us to a QED-covariant expression for the velocity:

$$\mathcal{V}_\mu = i\lambda\, D_\mu(\ln \psi) = i\lambda\, \partial_\mu(\ln \psi) - \frac{e}{mc^2} A_\mu, \tag{191}$$

where $\lambda = \hbar/mc$ is the Compton length of the electron.

We recognize in this derivative the standard QED-covariant derivative operator acting on the wave function ψ:

$$-i\hbar D_\mu = -i\hbar\partial_\mu + \frac{e}{c}\, A_\mu, \tag{192}$$

since we can write Eq. (191) as $mc\mathcal{V}_\mu\psi = [i\hbar\partial_\mu - (e/c)A_\mu]\ \psi$.

We have therefore reached an understanding from first principles of the nature and origin of the QED covariant derivative, while it was merely set as a rule devoid of geometric meaning in the standard quantum field theory.

This covariant derivative is exactly the previous one introduced in the classical framework. Indeed, the classical covariant derivative was written (for $q = e$), $D_\mu = \partial_\mu - (1/e)A_\mu$ acting on ϱ, while $\psi = \psi_0\ \exp[(-i/\hbar)(e^2/c) \ln \varrho]$. We therefore recover expression (192) acting on ψ.

7.3.3 Klein-Gordon equation in the presence of an electromagnetic field

We can now combine the two main effects of the fractality of space-time, namely, the induced effects that lead to quantum laws and the effects of coordinate-dependent resolutions that lead to the appearance of an electromagnetic field. The account of both covariant derivatives allows one to derive from a geodesics equation the relativistic wave equation for a scalar particle in the presence of an electromagnetic field. Following Pissondes [74], let us start from the fully covariant form of the quadratic invariant, as established in Sec. 6:

$$\mathcal{V}^\mu \otimes \mathcal{V}_\mu = (\mathcal{V}^\mu + i\lambda\, \partial^\mu)\mathcal{V}_\mu = 1. \tag{193}$$

By taking its gradient, we obtain (in terms of the operator $\widehat{\mathcal{V}^\mu} = \mathcal{V}^\mu + i(\lambda/2)\partial^\mu$ introduced in Sec. 6.1.2

$$\widehat{\mathcal{V}^\mu}\, \partial_\alpha \mathcal{V}_\mu = 0. \tag{194}$$

This equation is no longer equivalent to $\widehat{\mathcal{V}^\mu}\partial_\mu\mathcal{V}_\alpha = 0$ in the presence of an electromagnetic field, since in this case

$$\partial_\alpha \mathcal{V}_\mu = \partial_\mu \mathcal{V}_\alpha + \frac{e}{mc^2} F_{\mu\alpha}. \tag{195}$$

Therefore Eq. 194 becomes

$$\widehat{\mathcal{V}^\mu}\left(\partial_\mu \mathcal{V}_\alpha + \frac{e}{mc^2} F_{\mu\alpha}\right) = 0, \tag{196}$$

and we finally recover the form of the Lorentz equation of electrodynamics [74]:

$$mc\frac{d\!\!\!/}{ds}\mathcal{V}_\alpha = \widehat{\mathcal{V}^\mu}\, F_{\alpha\mu}. \tag{197}$$

This equation is equivalent to that found by Pissondes [74], but now the electromagnetic field itself is built from the fractal geometry, instead of being simply added by applying the standard (but previously misunderstood) QED rules. The Klein-Gordon equation with electromagnetic field is easily obtained by replacing in Eq. (193) the complex velocity by its expression in function of ψ (Eq. 191):

$$\left(i\hbar\partial_\mu - \frac{e}{c}A_\mu\right)\left(i\hbar\partial^\mu - \frac{e}{c}A^\mu\right)\psi = m^2c^2\psi. \tag{198}$$

7.3.4 Nature of the electric charge (quantum theory)

In a gauge transformation $A'_\mu = A_\mu + e\partial_\mu\chi$ the wave function of an electron of charge e becomes:

$$\psi' = \psi\, \exp\left\{\frac{-i}{\hbar} \times \frac{e}{c} \times e\chi\right\}. \tag{199}$$

We have reinterpreted in the previous sections the gauge transformation as a scale transformation of resolution, $\varepsilon \to \varepsilon'$, yielding an identification of the gauge function with a scale ratio, $\chi = \ln\varrho = \ln(\varepsilon/\varepsilon')$, which is a function of space-time coordinates. In such an interpretation, the specific property that characterizes a charged particle is the explicit scale-dependence on resolution of its action, then of its wave function. The net result is that the electron wave function writes

$$\psi' = \psi \exp\left\{-i\frac{e^2}{\hbar c}\ln\varrho\right\}. \tag{200}$$

Since, by definition (in the system of units where the permittivity of vacuum is 1),

$$e^2 = 4\pi\alpha\hbar c, \tag{201}$$

where α is the fine structure constant, equation (200) becomes

$$\psi' = \psi \, e^{-i4\pi\alpha \ln \varrho}. \tag{202}$$

This result supports the previous solution brought to the problem of the nature of the electric charge in the classical theory. Indeed, considering now the wave function of the electron as an explicitly resolution-dependent function, we can write the scale differential equation of which it is solution as:

$$i\hbar \frac{\partial \psi}{\partial \left(\frac{e}{c} \ln \varrho \right)} = e\psi. \tag{203}$$

We recognize in $\tilde{D} = i(\hbar c/e)\partial/\partial \ln \varrho$ a dilatation operator similar to that introduced in Section 3. Equation (203) can then be read as an eigenvalue equation issued from an extension of the correspondence principle (but here, demonstrated),

$$\tilde{D}\psi = e\psi. \tag{204}$$

This is the quantum expression of the above classical suggestion, according to which the electric charge is understood as the conservative quantity that comes from the new scale symmetry, namely, from the uniformity of the resolution variable $\ln \varepsilon$.

7.3.5 Charge quantization and mass-charge relations

While the results of the scale relativity theory described in the previous sections mainly deal with a new interpretation of the nature of the electromagnetic field, of the electric charge and of gauge invariance, we now arrive at the principal consequences of this approach: as we shall see, it allows one to establish the universality of the quantization of charges (for any gauge field) and to theoretically predict the existence of fundamental relations between mass scales and coupling constants.

In the previous section, we have recalled our suggestion [25, 23] to elucidate the nature of the electric charge as being the eigenvalue of the dilation operator corresponding to resolution transformations. We have written the wave function of a charged particle under the form Eq. (203).

Let us now consider in more detail the nature of the scale factor $\ln \varrho$ in this expression. This factor describes the ratio of two relative resolution scales ε and ε' that correspond to structures of the fractal geodesical trajectories that we identify with the electron. However the electron is not structured at all scales, but only at scales smaller than its Compton length $\lambda = \hbar/m_e c$. We can therefore take this upper limit as one of the two scales and write:

$$\psi' = \exp\left\{ -i \, 4\pi\alpha \, \ln\left(\frac{\lambda}{\varepsilon}\right) \right\} \, \psi. \tag{205}$$

In the case of Galilean scale-relativity, such a relation leads to no new result, since ε can go to zero, so that $\ln(\lambda/\varepsilon)$ is unlimited. But in the framework of special scale-relativity, scale laws take a log-Lorentzian form below the scale λ

(see Section 2). The Planck length l_P becomes a minimal, unreachable scale, invariant under dilations, so that $\ln(\lambda/\varepsilon)$ becomes limited by $\mathbb{C} = \ln(\lambda/l_P)$. This implies a quantization of the charge which amounts to the relation $4\pi\alpha\mathbb{C} = 2k\pi$, i.e.:

$$\alpha\,\mathbb{C} = \frac{1}{2}k, \tag{206}$$

where k is integer. Since $\mathbb{C} = \ln(\lambda/l_p)$ and is equal to $\ln(m_P/m_e)$ for the electron (where m_P is the Planck mass and m_e the elctron mass), equation (206) amounts to a general relation between mass scales and coupling constants..

In order to explicitly apply such a relation to the electron, we must account for the fact that we expect the electric charge to be only a residual of a more general, high energy electroweak coupling in the framework of Grand Unified theories. From the U(1) and SU(2) couplings, one can define an effective electromagnetic coupling as:

$$\alpha_0^{-1} = \frac{3}{8}\alpha_2^{-1} + \frac{5}{8}\alpha_1^{-1}. \tag{207}$$

It is such that $\alpha_0 = \alpha_1 = \alpha_2$ at unification scale and it is related to the fine structure constant at Z scale by the relation $\alpha = 3\alpha_0/8$. This means that, because the weak gauge bosons acquire mass through the Higgs mechanism, the interaction becomes transported at low energy only by the residual null mass photon. As a consequence the amplitude of the electromagnetic force abruptly falls at the WZ scale. Therefore, we have suggested that it is the coupling α_0 instead of α which must be used in Eq. (206) for relating the electron mass to its charge.

Finally, disregarding as a first step threshold effects (that occur at the Compton scale of the electron), we get a mass-charge relation for the electron [25, 23]:

$$\ln\frac{m_P}{m_e} = \frac{3}{8}\alpha^{-1}. \tag{208}$$

The existence of such a relation between the mass and the charge of the electron is supported by the experimental data. Indeed, using the known experimental values, the two members of this equation agree to 0.2 %: $\mathbb{C}_e = \ln(m_P/m_e) = 51.528(1)$ while $(3/8)\alpha^{-1} = 51.388$. The agreement is made even better if one accounts from the fact that the measured fine structure constant (at Bohr scale) differs from the limit of its asymptotic behavior (that includes radiative corrections). One finds that the asymptotic inverse running coupling at the scale where the asymptotic running mass reaches the observed mass m_e is $\alpha_0^{-1}\{r(m = m_e)\} = 51.521$, which lies within 10^{-4} of the value of $\mathbb{C}_e$ (see Fig. 18).

8 'Third quantization': quantum mechanics in scale-space

8.1 Motivation

Let us now consider a new tentative development of the scale relativity theory. Recall that this theory is founded on the giving up of the hypothesis of differentiability of space-time coordinates. We reached the conclusion that the problem of dealing with non-derivable coordinates could be circumvented by replacing them by fractal functions of the resolutions. These functions are defined in a space of resolutions, or 'scale-space'. The advantage of this approach is that it sends the

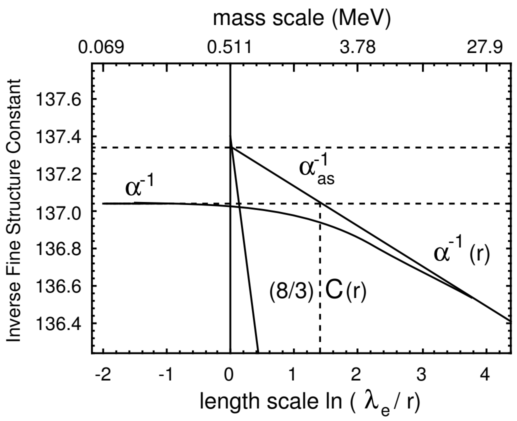

Figure 18: Observed convergence at the electron mass-scale (i.e. at the Compton length of the electron) of the asymptotic running electromagnetic inverse coupling $\alpha^{-1}(r)$ and of the running scale-relativity constant $8\mathbb{C}(r)/3 = 8\ln[m_P/m(r)]$. Such a convergence is theoretically expected in the framework of the scale-relativistic interpretation of gauge transformations, that yields a mass-charge relation for the electron, that reads $8\alpha_e \ln(m_P/m_e) = 1$. The final low energy fine structure constant differs from its asymptotic value by a small theshhold effect (see e.g. [85], [4] Sec.6.2)

problem of non-derivability to infinity in the scale space ($\ln(\lambda/\varepsilon) \to \infty$). But, in such a framework, standard physics should be completed by scale laws allowing to determine the physically relevant functions of resolution. We have suggested that these fundamental scale laws be written in terms of differential equations (which amounts to define a differential fractal 'generator'). Then the effects induced by these internal scale laws on the dynamics can be studied: we have found that the simplest possible scale laws that are consistent (i) with the principle of scale relativity and (ii) with the standard laws of motion and displacements, lead to a quantum-like mechanics in space-time.

However, the choice to write the transformation laws of the scale space in terms of standard differential equations [involving $\partial/\partial\ln\varepsilon, \partial^2/(\partial\ln\varepsilon)^2$ as a first step, then $\partial/\partial\,\delta$, etc... as a second step], even though it allows non-differentiability in standard space-time, implicitly assumes differentiability in the scale space. This is once again a mere hypothesis that can be again given up.

We may therefore use the method that has been built for dealing with non-differentiability in space-time and explore a new level of structures that may be its manifestation. As we shall now see, this results in the obtention of scale laws that take quantum-like forms instead of a classical ones. Now, these new proposals should be considered as tentative in view of their novelty. Their self-consistency and their ability to describe real systems remain to be established. Moreover, the complementary problem of constructing the motion laws that are induced by such quantum scale laws is left open to future studies.

8.2 Schrödinger equation in scale-space

Recall that, for the construction of classical scale differential equations, we have mainly considered two representations: (i) the logarithms of resolution are fundamental variables; (ii) the main new variable is the djinn and the resolutions are deduced as derivatives.

These two possibilities are also to be considered for the new present attempt to construct quantum scale-laws. The first one, that we shall only briefly study here, consists of introducing a scale-wave function $\psi[\ln\varepsilon(x,t),x,t]$. In the simplified case where it depends only on the time variable, one may write a Schrödinger equation acting in scale-space:

$$\mathcal{D}_\varepsilon^2 \frac{\partial^2 \psi}{(\partial \ln \varepsilon)^2} + i\,\mathcal{D}_\varepsilon \frac{\partial \psi}{\partial t} - \frac{1}{2}\phi\psi = 0. \tag{209}$$

This is the quantum equivalent of the classical stationary wave equation giving rise to a log-periodic behavior (Section 4.3.1). It is also related to the scale-relativity re-interpretation of gauge invariance (Section 7), in which the resolutions become 'fields' depending on space and time variables, so that the wave function becomes a function of the $\ln\varepsilon$. However only the phase was affected, while now the modulus of the scale-wave function depends on the resolution scale. This means that the solutions of such an equation give the probability of presence of a structure in the scale-space, and that time-dependent solutions describe the propagation in the 'zoom' dimension of quantum waves.

8.3 Schrödinger equation in terms of the djinn

Let us consider now the second representation in which the 'djinn' (variable scale dimension) has become the primary variable. Start with the general Euler-Lagrange form given to scale laws in Sec. 4.3.2 after introduction of the djinn δ,

$$\frac{d}{d\delta}\frac{\partial \tilde{L}}{\partial \mathbb{V}} = \frac{\partial \tilde{L}}{\partial \ln \mathcal{L}}\ . \tag{210}$$

where we recall that $\mathcal{L}$ is a fractal coordinate, δ is the djinn that generalizes to a variable fifth dimension the scale dimension, $\tilde{L}$ is the scale Lagrange function and $\mathbb{V} = \ln(\lambda/\varepsilon)$ is the 'scale-velocity'.

It becomes in the Newtonian case

$$\frac{d^2 \ln \mathcal{L}}{d\delta^2} = -\frac{\partial \Phi_S}{\partial \ln \mathcal{L}}. \tag{211}$$

Since the scale-space is now assumed to be itself non-differentiable and fractal, the various elements of the new description can be used in this case, namely:

(i) Infinity of trajectories, leading to introduce a scale-velocity field $\mathbb{V} = \mathbb{V}(\ln\mathcal{L}(\delta),\delta)$;

(ii) Decomposition of the derivative of the fractal coordinate in terms of a 'classical part' and a 'fractal part', and introduction of its two-valuedness because of the symmetry breaking of the reflection invariance under the exchange ($d\delta \leftrightarrow -d\delta$);

(iv) Introduction of a complex scale-velocity $\tilde{\mathcal{V}}$ based on this two-valuedness;

(v) Construction of a new total covariant derivative with respect to the djinn, that reads:

$$\frac{\acute{d}}{d\delta} = \frac{\partial}{\partial \delta} + \tilde{\mathcal{V}}\,\frac{\partial}{\partial \ln \mathcal{L}} - i\,\mathcal{D}_s\,\frac{\partial^2}{(\partial \ln \mathcal{L})^2}\ . \tag{212}$$

(vi) Introduction of a wave function as a re-expression of the action (which is now complex), $\Psi_s(\ln\mathcal{L}) = \exp(i\mathcal{S}_s/2\mathcal{D}_s)$;

(vii) Transformation and integration of the above Newtonian scale-dynamics equation under the form of a Schrödinger equation now acting on scale variables:

$$\mathcal{D}_s^2 \frac{\partial^2 \Psi_s}{(\partial \ln \mathcal{L})^2} + i\,\mathcal{D}_s \frac{\partial \Psi_s}{\partial \delta} - \frac{1}{2}\Phi_s \Psi_s = 0. \tag{213}$$

8.4 Complexergy

In order to understand the meaning of this new Schrödinger equation, let us review the various levels of evolution of the concept of physical fractals adapted to a geometric description of a non-derivable space-time.

The first level in the definition of fractals is Mandelbrot's concept of 'fractal objects' [2].

The second step has consisted to jump from the concept of fractal objects to scale-relativistic fractals. Namely, the scales at which the fractal structures appear are no longer defined in an absolute way: only scale ratios do have a physical meaning, not absolute scales.

The third step, that is achieved in the new interpretation of gauge transformations recalled hereabove, considers fractal structures (still defined in a relative way) that are no longer static. Namely, the scale ratios between structures become a field that may vary from place to place and with time.

The final level (in the present state of the theory) is given by the solutions of the above scale-Schrödinger equation. The Fourier transform of these solutions will provide probability amplitudes for the possible values of the logarithms of scale ratios, $\Psi_s(\ln \varrho)$. Then $|\Psi_s|^2(\ln \varrho)$ gives the probability density of these values. Depending on the scale-field and on the boundary conditions (in the scale-space), peaks of probability density will be obtained, this meaning that some specific scale ratios become more probable than others. Therefore, such solutions now describe quantum probabilistic fractal structures. The statement about these fractals is no longer that they own given structures at some (relative) scales, but that there is a given probability for two structures to be related by a given scale ratio.

Concerning the direct solutions of the scale-Schrödinger equation, they provide probability densities for the position on the fractal coordinate (or fractal length) $\ln \mathcal{L}$. This means that, instead of having a unique and determined $\mathcal{L}(\ln \varepsilon)$ dependence (e.g., the length of the Britain coast), an infinite family of possible behaviors is defined, which self-organize in such a way that some values of $\ln \mathcal{L}$ become more probable than others.

A more complete understanding of the meaning of this new description can be reached by considering the case of a scale-harmonic oscillator potential well. This is the quantum equivalent of the scale force considered previously, but now in the attractive case. The stationary Schrödinger equation reads in this case:

$$2\mathcal{D}_s^2 \frac{\partial^2 \Psi_s}{(\partial \ln \mathcal{L})^2} + \left[I\!\!E - \frac{1}{2}\omega^2 (\ln \mathcal{L})^2 \right] \Psi_s = 0. \tag{214}$$

The stationarity of this equation means that it does no longer depend on the djinn δ. (Recall that the djinn is to scale laws what time is to motion laws, i.e., it can be identified with a 'scale-time', while the resolutions are 'scale-velocities').

A new important quantity, denoted here $I\!\!E$, appears in this equation. It is the conservative quantity which, according to Noether's theorem, must emerge from the uniformity of the new djinn variable (or fifth dimension). It is defined, in terms of the scale-Lagrange function $\tilde{L}$ and of the resolution $\mathbb{V} = \ln(\lambda/\varepsilon)$, as:

$$\mathbb{E} = \mathbb{V}\frac{\partial \tilde{L}}{\partial \mathbb{V}} - \tilde{L}. \tag{215}$$

This new fundamental prime integral had already been introduced in Refs. [16, 4], but its physical meaning remained unclear.

As we shall now see, the behavior of the above equation suggests an interpretation for this conservative quantity and allows one to link it to the complexity of the system under consideration. For this reason, and because it is linked to the djinn in the same way as energy is linked to the time, we have suggested to call this new fundamental quantity 'complexergy'.

Indeed, let us consider the momentum solutions $a[\ln(\lambda/\varepsilon)]$ of the above scale-Schrödinger equation. Recall that the main variable is now $\ln \mathcal{L}$ and that the scale-momentum is the resolution, $\ln \rho = \ln(\lambda/\varepsilon) = d \ln \mathcal{L}/d\delta$ (since we take here a scale-mass $\mu = 1$). The squared modulus of the wave function yields the probability density of the possible values of resolution ratios:

$$|a_n\,(\ln \rho)\,|^2 = \frac{1}{2^n n! \sqrt{2\pi \mathcal{D}_s \omega}}\; e^{-(\ln \rho)^2/2\mathcal{D}_s\omega}\; H_n^2\left(\frac{\ln \rho}{\sqrt{2\mathcal{D}_s\omega}}\right), \tag{216}$$

where the H_n's are the Hermite polynomials (see Fig. 19).

The complexergy is quantized, in terms of the quantum number n, according to the well-known relation for the harmonic oscillator:

$$\mathbb{E}_n = 2\mathcal{D}_s\,\omega\,\left(n + \frac{1}{2}\right). \tag{217}$$

As can be seen in Fig. 19, the solution of minimal complexergy shows a unique peak in the probability distribution of the $\ln(\lambda/\varepsilon)$ values. This can be interpreted as describing a system characterized by a single, more probable relative scale. Now, when the complexergy increases, the number of probability peaks $(n+1)$ increases. Since these peaks are nearly regularly distributed in terms of $\ln \varepsilon$ (i.e., probabilistic log-periodicity), it can be interpreted as describing a system characterized by a hierarchy of imbricated levels of organization. Such a hierarchy of organization levels is one of the criterions that define complexity. Therefore increasing complexergy corresponds to increasing complexity, which justifies the chosen name for the new conservative quantity.

More generally, one can remark that the djinn is universally limited from below ($\delta > 0$), which implies that the complexergy is universally quantized. The same is true for the energy of systems which are described by the above scale-Schrödinger equation. In the case where $\ln(\mathcal{L}/\mathcal{L}_0) = \ln(\mathcal{T}/\mathcal{T}_0)$ is mainly a time variable (as for example in motion-relativistic high energy physics), the associated conservative quantity is $\mathcal{E} = \ln(E/E_0)$ (see [4], p.242). Because of the fractal-nonfractal transition, $\ln(\mathcal{T}/\mathcal{T}_0) > 0$ is also limited from below, so that we expect the energy to be generally quantized, but now in exponential form. In other words, it describes a hierarchy of energies.

8.5 Applications

8.5.1 Elementary particle physics

A natural domain of possible application of these new concepts is the physics of elementary particles. Indeed, there is an experimentally observed hierarchy of elementary particles, that are organized in terms of three known families, with mass increasing with the family quantum number. For example, there is a (e, μ, τ) universality among leptons, namely these three particles have exactly the same properties except for their mass and family number. However, there is, in the present standard model, no understanding of the nature of the families and no prediction of the values of the masses.

Hence, the experimental masses of charged leptons and of the 'current' quark masses are ([91]:

- $m_e = 0.510998902(21)$ MeV; $m_\mu = 105.658357(5)$ MeV; $m_\tau = 1776.99(28)$ MeV;
- $m_u = 0.003$ GeV; $m_c = 1.25$ GeV; $m_t = 174$ GeV;
- $m_d = 0.006$ GeV; $m_s = 0.125$ GeV; $m_b = 4.2$ GeV.

Basing ourselves on the above definition of complexergy and on this mass hierarchy, we suggest that the existence of particle families are a manifestation of increasing complexergy, i.e., that the family quantum number is nothing but a complexergy quantum number. This would explain why the electron, muon and tau numbers are conserved in particle collisions, since such a fondamental conservative quantity (like energy) can be neither created nor destroyed.

We have shown in Sec. 7 that the scale-relativistic re-interpretation of gauge transformations allowed one to suggest a relation between the mass of the electron and its electric charge (in terms of the fine structure constant). This result is compatible with the mass of the electron mainly being of electromagnetic origin. More generally, the observed mass hierarchy between (neutrinos, charged leptons, quarks) also goes in this direction, suggesting that their masses are respectively of (weak, electroweak and electroweak+strong) origin.

Although a full treatment of the problem must await a more advanced level of development of the theory, that would mix the 'third quantization' description with the gauge field one, some remarkable structures of the particle mass hierarchy already support such a view:

(i) The above values of quark and lepton masses are clearly organized in a hierarchical way. This suggests that their understanding is indeed to be searched in terms of structures of the scale-space, for example as manifestation of internal structures of iterated fractals [4].

(ii) For example, concerning the quarks, we have suggested that QCD was linked to a 3D harmonic oscillator scale-potential (which should be the source of the SU(3) gauge symmetry, understood here as a scale-dynamical symmetry). In such a framework, the energy ratios are expected to be quantized as $\ln E \propto (3+2n)$. It may therefore be significant in this regard that the s/d mass ratio, which is far more precisely known that the individual masses since it can be directly determined from the pion and kaon masses, is found to be $m_s/m_d = 20.1$ [91], to be compared with $e^3 = 20.086$, which is the fundamental ($n = 0$) level predicted by the above formula.

However, it is clear that one should wait for a more complete development of such a new quantum mechanics in scale-space before deciding whether such an approach is meaningful.

8.5.2 Biology: nature of first evolutionary leaps

Another tentative application of the complexergy concept concerns biology. In the CGN fractal model of the tree of life [40, 41, 42], we have voluntarily limited ourselves to an analysis of only the chronology of events (see Fig. 4), independently of the nature of the major evolutionary leaps. We have now at our disposal a tool that allows us to reconsider the question. As well the question of the origin of life as that of species evolution can be posed in a renewed way.

We indeed suggest that life evolution proceeds in terms of increasing quantized complexergy. This would account for the existence of punctuated evolution [93], and for the log-periodic behavior of the leap dates, which can now be interpreted in terms of probability density of the events, $P = \psi\psi^{\dagger} \propto \sin^2[\omega \ln(T - T_c)]$. Moreover, one may contemplate the possibility of an understanding of the nature of the events, even though in a rough way as a first step.

Indeed, one expects the formation of a structure at the fundamental level (lowest complexergy) characterized by one length-scale (Fig. 19). Moreover, the most probable value for this scale of formation is predicted to be the 'middle' of the scale-space (in the free case, it is determined by the boundary conditions). The universal boundary conditions are the Planck-length l_P in the microscopic domain and the cosmic scale $\mathbb{L} = \Lambda^{-1/2}$ given by the cosmological constant Λ in the macroscopic domain [4, 23, 35]. From the predicted and now observed value of the cosmological constant, one finds $\mathbb{L}/l_{\mathbb{P}} = 5.3 \times 10^{60}$, so that the mid scale is at $2.3 \times 10^{30}\, l_{\mathbb{P}} = 40\ \mu$m. A quite similar result is obtained from the scale boundaries of living systems (0.5 Angströms - 30 m). This scale of 40 μm is indeed a typical scale of living cells. Moreover, the first 'prokaryot' cells appeared about three Gyrs ago had only one hierarchy level (no nucleus).

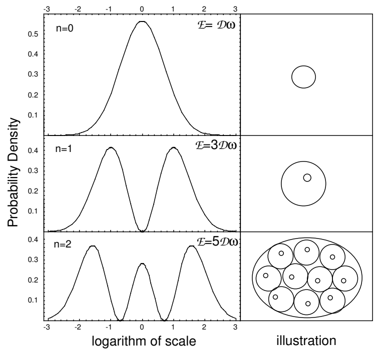

Figure 19: Solutions of increasing complexergy of the scale-Schrödinger equation for an harmonic oscillator scale-potential. These solutions can be interpreted as describing systems characterized by an increasing number of hierarchical levels, as illustrated in the right hand side of the figure. For example, living systems such as procaryots, eucaryots and simple multicellular organisms have respectively one (cell size), two (nucleus and cell) and three (nucleus, cell and organism) characteristic scales.

In this framework, a further increase of complexergy can occur only in a quantized way. The second level describes a system with two levels of organization, in agreement with the second step of evolution leading to eukaryots about 1.7 Gyrs ago (see upper Fig. 4). One expects (in this very simplified model), that the scale of nuclei be smaller than the scale of prokaryots, itself smaller than the scale of eucaryots: this is indeed what is observed.

The following expected major evolutionary leap is a three organization level system, in agreement with the apparition of multicellular forms (animals, plants and fungi) about 1 Gyr ago (third event in upper Fig. 4). It is also predicted that the multicellular stage can be built only from eukaryots, in agreement with what is observed. Namely, the cells of multicellulars do have nuclei; more generally, evolved organisms keep in their internal structure the organization levels of the preceeding stages.

The following major leaps correspond to more complicated structures then functions (supporting structures such as exoskeletons, tetrapody, homeothermy, viviparity), but they are still characterized by fundamental changes in the number of organization levels. The above model (based on spherical symmetry) is clearly too simple to account for these events. But the theoretical biology approach outlined here for the first time is still in the infancy: future attempts of description using the scale-relativity methods will have the possibility to take into account more complicated symmetries, boundary conditions and constraints, so that the field seems to be wide open to investigation.

9 Conclusion and prospect

We have attempted, in the present review paper, to give an extended discussion of the various developments of the theory of scale-relativity, including some new proposals concerning in particular the quantization in the scale-space and tentative applications to the sciences of life.

The aim of this theory is to describe space-time as a continuous manifold (either derivable or not) that would be constrained by the principle of relativity (of motion and of scale). Such an attempt is a natural extension of general relativity, since the two-times differentiable continuous manifolds of Einstein's theory, that are constrained b the principle of relativity of motion, are particular sub-cases of the new geometry to be built.

Now, giving up the differentiability hypothesis involves an extremely large number of new possible structures to be investigated and described. In view of the immensity of the task, we have chosen to proceed by adding self-imposed structures in a progressive way, using presently known physics as a guide. Such an approach is rendered possible by the result according to which small-scale structures issued from non-differentiability are smoothed out beyond some transitions at large scale. Moreover, these transitions have profound physical meaning, since they are themselves linked to fundamental mass scales.

This means that the program that consists of developing a full scale-relativistic physics is still in its infancy. Much work remains to be done, in order (i) to describe the effect on motion laws of the various levels of scale laws that have been considered, and of their generalizations still to come (general scale-relativity); (ii) to take into account the various new symmetries, as well continuous as discrete, of the new variables that must be introduced for the full description of a fractal space-time, and of the conservative quantities constructed from them (including their quantum counterparts which are expected to provide us with an explanation of various still misunderstood quantum numbers in elementary particle physics).

Let us conclude by a final remark: one of the main interest of the new approach is that, being based on the universality of fractal geometry already unveiled by Mandelbrot, it allows one to go beyond the frontiers between sciences. In particular, it opens the hope of a future refoundation on first principles of sciences of life and of some human sciences.

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