Riemann's Vision of a New Approach to Geometry

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The General Concept of Manifold and Herbart's "Serial Forms"

How can a mathematician outline a fundamentally new vision of a mathematical discipline? He might turn to the philosophy of mathematics and speak *about* mathematics, i.e. on a metalevel, reflecting his own and other mathematicians' work. Or he might try to sketch the architecture of the new mathematical discipline in question. In the latter case he has to introduce concepts, constructions, and theorems as the central technical building blocks of a mathematical theory. Usually he can draw upon a whole network of results of other scientists, which brings his view closer to tradition and attenuates the novelty of his views. Thus, if an epistemological break is intended, at least some elements of the first, more philosophical approach have to be taken up. The occasion of sharp epistemological turns are rare in the history of mathematics. Riemann's contribution to geometry is a most prominent example.

As is well known, Riemann organized his approach to geometry around the new concept of *manifold (Mannigfaltigkeit)* which for obvious reasons he could not define in a mathematical technical sense. He therefore did it in a semi-philosophical way, drawing consciously and cautiously upon hints by C. F. Gauss who had spoken geometrically about complex numbers (Gauss 1831) and J. F. Herbart who had argued for the use of geometrical imagery in all kind of concept formation, his so-called *serial forms (Reihenformen)*. Vaguely speaking a continuous serial form is produced in the imagination when a class of mental images, or presentations (*Vorstellungen*), undergoes what Herbart called a *graded fusion (abgestufte Verschmelzung)*, i.e. a mental fusion which does not destroy the individual presentations but glues them together, with the result that continuous transition from one to another becomes possible (Herbart 1825, 193*).[1] Riemann did not bother much about the specific ontological theorization which Herbart gave of this prima facie psychological process of concept formation in what Herbart called *synechology (Synechologie)*.

[1] An asterix in a Herbart citation indicates that Riemann read and excerpted the corresponding passage during his studies of Herbartian philosophy. For more details see [Scholz 1982a].

Riemann rather preferred to allude only vaguely to this Herbartian conception (1854, 273). He presupposed the existence of concepts, mathematical or not, which may arise as the result of a "graded fusion" into serial forms.[2] He took up the result and opened it to mathematical consideration, thus forming his concept of ***multiply extended magnitude (mehrfach ausgedehnte Grösse)*** or ***manifold.***

To rephrase his approch briefly: In contrast to a set theoretic approach, Riemann presupposed a concept taken from any field of investigation, thus pre-existing in some epistemic sense in both levels of traditional logic, intension (defined by properties in terms of the field specified) and extension (endowed with a well-determined range of instantiation). For Riemann, the extensional aspect of the concept became of prime importance, with little more scrutiny into the related foundational questions. Leaving some sparse – even if in the light of the later development of set theory important – remarks on finite or discrete manifolds aside, Riemann proceeded immediately to those situations where the particular instances of the concept admit *continuous transitions.* That was to be understood in an intuitive sense, as the concept of continuity came to be mathematically analyzed only after the formal definitions of real numbers had appeared and set theoretic ideas were being formulated, that is not before the 1870/80s.[3] Riemann's approach was somehow parallel to the introduction of Herbart's serial forms; but Riemann specified the idea further by a local successive reconstruction in a quasi-cinematical sense, by 1-parameter, 2-parameter, ..., $(n-1)$-parameter, and finally n-parameter variation of the determination of instances of the concept. In these cases he admitted the obvious, but drastically generalized, geometric terminology of *point* for a particular instance of the general concept (manifold).

Another aspect of this local reconstruction is, according to Riemann, the possibility of introducing n-function systems, the values of which separate (locally) points in the manifold. This leads to coordinate systems which make the manifold accessible to mathematical constructions and further investigations. Riemann even hinted at the possibility of infinite dimensional manifolds at this place.[4] This approach is similar to the modern introduction of topological, differential, Riemannian, etc. manifolds; the role of the topological space, however, being taken in a vague sense by a Herbartian-type of "serial form", backed by mathematical intuition.

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[2] Riemann emphasized, however, that, according to his opinion and in contrast to Herbart's, such continuous *serial forms* (Herbart) or *extended magnitudes* (Riemann) are much more frequent in "higher mathematics" than in other parts of knowledge.

[3] Cf. [Johnson 1979/1981, 1987], [Moore 1989].

[4] "By an n-time repetition of this process, the determination of position in an n-fold extended manifold is reduced to n numerical determinations, and therefore the determination of position in a given manifold is reduced, whenever this is possible, to a finite number of numerical determinations. There are, however, also manifolds in which the fixing of position requires not a finite number, but either an infinite sequence or a continuous manifold of numerical measurements. Such manifolds form e.g. the possibilities for a function in a given region, the possible shapes of a solid figure, etc." (Riemann 1854, 276 – English translation [Spivak 1970]).

Differential Geometry

Already in his inaugural lecture (1854) Riemann pointed clearly to a basic distinction among mathematical investigations which are possible in manifolds, between those "independent of measurement" (analysis situs) and those assuming metrical structures (differential geometry). Even though Riemann hinted in his talk only very briefly at the first branch of this division (*analysis situs*), restricting himself in this respect to the local reconstruction and the choice of coordinates in $\mathbb{R}^n$, it was already quite clear to him that the topological theory of manifolds was a challenging and promising mathematical subject of its own right. I shall come back to this point in the next section of this article.

The main subject of Riemann's lecture, however, was metrical geometry of manifolds, which he introduced as a profound generalization of Gauss' differential geometry of surfaces. It is true that Gauss had prepared the way for Riemann in the best possible way, by working out the intrinsic nature of the metrical geometry of surfaces in his *Disquisitiones generales circa superficies curvas* (1828);[5] and Riemann referred to it quite clearly (1854, 276). Nevertheless Gauss' concept was severely restricted by sticking to the conceptual framework of euclidean space. His surfaces were always embedded in euclidean 3-space, even if their metrical investigation led to intrinsic aspects. The autonomy of the geometrical object (here: surface) was only intended implicitly; Gauss did not, and could not in his framework, state it explicitly. Thus Gauss did not dare to formulate this autonomy conceptually, because he preferred to accept the delimitations of the "euclidean philosophy" of geometry, at least in his published writings. This restriction allowed generalizations of geometric thought as analogy, imagery, or metaphor in other mathematical contexts only.

That changed completely with Riemann. His concept of manifold was formed exactly to transform imagery and metaphor into strictly mathematical concepts of a generalized geometric framework, thus liberating geometrical thought from the euclidean straightjacket. He introduced the concept of a metrical manifold (*Mannigfaltigkeit mit Maßbestimmung*) in the well-known way by the selection of a positive determinate quadratic differential form,

$$ds^2 = \sum_{i,j} g_{ij}\, dx_i dx_j \qquad (1 \leq i,j \leq n)\,,$$

which enabled him to transfer essential constructions of Gauss' theory of surfaces to the generalized geometry of manifolds.

Most important in this respect was the transfer of the curvature concept to manifolds. In his inaugural lecture Riemann did so by introducing the *sectional curvature* of an infinitesimal surface element, even if only by a description of the procedure for deriving it without giving an explicit general formula. This allowed him to speak about *manifolds of constant curvature* as a connecting link between the general differential geometric theory of manifolds and the theory of physical space.

He mentioned two main results:

[5] In particular he elaborated the central role of the *Theorema egregium* (the intrinsic determination of the curvature of the surface) and the *Theorem aureum* (angle sum of geodetic triangles).

1. Manifolds of constant curvature are exactly those in which free mobility of rigid figures is possible.
2. In a manifold of constant curvature α it is always possible to choose local coordinates such that the metric is given by

$$ds^2 = \frac{\alpha \sum_i dx_i^2}{(1 + \frac{\alpha}{4} \sum_i x_i^2)^2} \quad .$$

I mention Riemann's later work (1861) on heat distribution in a homogeneous body only in passing. There Riemann introduced the famous 4-index symbol in second derivatives of the metric as a criterion of flatness of a metrical (Riemannian) manifold, which was later identified as *curvature tensor.*[6]

Steps towards a Topological Theory of Manifolds

I now want to come back to Riemann's basic distinction between studies of "manifolds with metrics" and of manifolds "independent of measurement" or *analysis situs.* Already in his well known doctoral dissertation on complex function theory (1851) Riemann had dealt with questions of analysis situs. There he had introduced *Riemann surfaces* for multi-valued analytic functions in a complex region and had begun to study the topology of compact oriented surfaces with boundaries in detail. He had used *dissection of the surfaces* into simply connected components, resulting in a complete classification of these by the *number of boundary components* and the *order of connectivity* (*Ordnung des Zusammenhangs*) m. The latter had been introduced by him as alternating sum of the number e of cross cuts and the number f of simply connected components

$$m = e - f = -\chi(F), [\quad \chi(F) \quad \text{the Euler characteristic of F}].$$

Of course, one of Riemann's central arguments in this passage was that this number is independent of the specific choices of the dissection process (1851, 11f.).

Three years after his inaugural lecture, in his great *memoir on abelian functions* Riemann used another approach to the characterization of the the topology of surfaces, building on *boundary relations* of systems of closed curves inside the surface (Riemann 1857). This time he studied closed oriented surfaces exclusively and looked for systems of closed curves, which form a complete boundary of a part of the surface. In this connection he introduced an appropriate equivalence concept of curve systems and showed that the maximum number of closed curves which do not form a complete boundary is independent of the specific choice of curves and gives a good classification of the topological type of the surface. As he could show that in the case of closed (Riemann) surfaces the maximum number of boundary independent closed curves (cycles) is even, he was led to the well known *numerical invariant* $2p$ for these surfaces (p was later called *genus* by Clebsch). And of course he showed how the genus characterization translated to the "order of connectivity" m of (1851): Punctuate the closed surface [Euler characteristic $\chi(F) = 2 - 2p \longrightarrow \chi(F') = 1 - 2p$], so you get a surface with boundary to which application of

[6] A recent investigation of this work is given in [Farwell/Knee 1990].

the dissection method shows that $2p$ cross cuts can be used to dissect the surface into one simply connected piece (1857, 93ff.). So in the terminology of (1851) the order of connectivity of the punctured surface is $m = 2p - 1$ [which is indeed $-\chi(F')$].

So, if one adds Riemann's insights of (1851) and (1857) to his inaugural lecture, one sees that he did, in the twodimensional case, take the first step to a topological theory of manifolds, containing the two complementary aspects of

- dissection into simply connected cells and counting the Euler characteristic of the cell complex,
- homological characterization of closed surfaces, counting the first Betti number.

All this has been discussed from different points of view in the historical literature, as has the fact that Riemann started to think about a generalization of his analysis situs methods to higher dimensional manifolds [Bollinger 1972, Pont 1974; Scholz 1980 etc.]. He left a *Nachlass*-fragment on this topic, where he experimented with dissection and bordism ideas in higher dimensional manifolds (*Werke* 479–482).

I should only add that an investigation of the fragmentary manuscript itself (Riemann *Nachlass* **16**, 44^r, 46^{r-v}, 49^{r-v}) gives some circumstancial evidence that Riemann worked on these ideas in the time between his doctoral dissertation and his inaugural lecture, not about or even later than 1857 [Scholz 1982b]. So we can read the fragments on analysis situs as a (hidden) background to Riemann's short and in itself rather vague reference to the topology of manifolds in the lecture of 1854.

First Glances at Other Geometrical Structures

The next point I want to discuss is the surprising fine and differentiated approach to geometric thinking that was opened up by Riemann on the basis of his manifold concept. This view on geometry was in line with the most far-reaching and deep-going changes of geometric thought during the turn towards "modern mathematics" of the late 19th and the early 20th century. These changes concern both semantics and the internal structure of geometry. From the point of view of semantics the most striking feature of 19th century development is the turn away of geometric theories from predominantly or even exclusive reference to physical space (even if perhaps understood in a philosophical *a priori* disguise). On the other hand, a whole range of new reference fields arose inside mathematical knowledge itself, in particular analysis, algebra, and arithmetic. As a companion to and a result of this development the geometrical theories became more abstract and more diverse. And yet they were to be kept together by central organizing ideas.

For Riemann the latter function was taken over by his manifold concept which admitted different enrichments with structural ideas derived from the contextual situations. Again it was his function theoretic work, where he developed most clearly (albeit restricted to the real two- or complex one-dimensional case) some basic ideas for the study of manifolds with structures going beyond those he talked about in his 1854 lecture. Most important, from this point of view, are his investigations of *abelian functions* (1857), which contain fundamentally new ideas on surfaces (or curves, depending on the standpoint) from the *complex analytic* and/or complex *birational* point of view.

Here is not the place to discuss these questions in detail,[7] but I have to mention at this point two general and fundamental insights of Riemann. The first is his analysis of the *meromorphic structure* of a compact surface of genus p. Riemann characterized abelian integrals of the second kind by a set of independent conditions (pole behaviour and real part of the periods) using the later disputed Dirichlet principle. In a second step he derived an estimation of the number of linearly independent meromorphic functions on the surface with simple poles in m chosen points as

$$\mu \geq m - p + 1 .$$

The result was later sharpened by his student Roch to the *Riemann- Roch theorem:*

$$\mu = m - p + \tau + 1 ,$$

(with τ = number of linearly independent abelian differentials of the first kind with zeroes at the m given points). *This was the first result of modern geometry to establish a deep-rooted connection between the topology of a manifold and a more refined structure, here the complex analytic one on a complex compact curve.*

The next point to mention in this context is Riemann's insight that the meromorphic functions on a curve can in fact be expressed as *rational functions* in two of them, say z and t, which, read as inhomogeneous coordinates in in $P(2, \mathbb{C})$, let the curve be represented algebraically as

$$F(z,t) = 0, \qquad F \in \mathbb{C}[z,t].$$

This makes it possible, as Riemann stated clearly, to study any compact complex curve from a purely algebraic birational point of view. In particular, the change of representing coordinates (z,t) to (z',t') is given by rational transformations in both directions. Thus Riemann indicated the way towards a purely algebraic structure linked to his manifold concept. The elaboration of these ideas would, of course, later lead to an adaptation of the underlying concept of manifold to the algebraic-geometric context and a transformation into different types of algebraic varieties. This is a story far away from Riemann's days and closer to the present than most of the other points mentioned here.[8]

Foundations of Geometry

Coming back to the inaugural lecture, it has to be said that Riemann gave only slight indications that the manifold concept could be developed further from the point of view of analytic or even purely algebraic structures.[9] The title of his talk indicated another line of investigation, namely the foundations of geometry. There is no doubt that, after

[7] See e.g. [Dieudonné 1974, 42ff.; Gray 1989, 361ff.; Scholz 1980, 68ff.]

[8] See [Dieudonné 1974] for a first historiographic overview.

[9] Riemann mentioned in passing, however, that "...Such investigations [of analysis situs of manifolds, E.S.] have become a necessity for several parts of mathematics, e.g., for the treatment of many-valued analytic functions, and the dearth of such studies is one of the principal reasons why the celebrated theorem of Abel and the contributions of Lagrange, Pfaff and Jacobi to the general theory of differential equations have remained unfruitful for so long." (1854, 274)

his inaugural lecture became accessible to the wider scientific public with its publication in 1867 (*Göttinger Abhandlungen*), the early reception of Riemann's geometric ideas was eminently important and influential in the debate about the character and interpretation of ***non-euclidean geometry.*** That is true, in particular, for Riemann's influence on Beltrami (to be seen by the latter's conceptual progress between his two 1868 papers on non-euclidean geometry), on Helmholtz (even if also Helmholtz's ideas on free mobility as the central "fact" lying at the base of geometry were developed before he had read Riemann) and on Clifford, to give just the three most outstanding examples.

Indeed Riemann's approach, and in particular his discussion of manifolds of constant curvature, can easily be read in the context of the investigations of Bolyai and Lobachevskii, because Riemann outlined a sophisticated conceptual framework for a possible and satisfying interpretation of non-euclidean geometry. Surprisingly ***there are no indications whatsoever that Riemann knew more than superficially of Bolyai's and Lobachevskii's work and maybe even not at all.*** Consequently he did not bother about the intimate potential connection between his and their considerations. I shall try to give the main arguments for this thesis which I have discussed more in detail elsewhere [Scholz 1982b].

It may even be surprising that in Riemann's inaugural lecture the only "modern reformer of geometry" cited by name was Legendre. That fits the observation that Riemann never in his talk (or elsewhere) mentioned the axiom of parallels – not even as a sideremark comparable to that referring to the topological theory of manifolds and its role in complex function theory. That must be startling if one tries to see a conscious reference to non-euclidean geometry in Riemann's title, *Hypotheses which lie at the basis of geometry.* This is the more so, as the last two sections of the second part of the talk (on differential geometry in manifolds) deal with manifolds of constant curvature, so that a sideremark on the different behaviour of parallels, dependent on the curvature, would have had an obvious place and context.

It becomes even clearer that Riemann never bothered about the foundational questions of geometry in the logical sense, when one takes into account a passage in his *Nachlass,* which was apparently written in the years 1852/1853, i.e. some time, although not much, before his inaugural lecture. In this pre-1854 fragment Riemann experimented with the idea of a manifold and dealt with the relationship of the manifold concept to foundational questions of geometry.[10] In particular he pointed out that a treatment of geometry from the manifold point of view would make superfluous all the specifically geometric axioms of Euclid and provide the possibility of reducing the necessary axioms to those "which hold for quantities in general...". As the only example of what could be proved in this framework, Riemann cited Euclid's axiom 9 which states that, given two points, there exists only one straight line incident with them. Again the parallel problem is not even mentioned.

In what follows Riemann stated quite frankly why he was content with this simple example and why he did not see a reason to go further into these foundational questions:

> "But even if it is of interest to grasp the possibility of this mode of treatment of geometry, the execution of the latter would be extremely fruitless, as by this means we

[10]Riemann *Nachlass* (map **16**, folio 40^{r-v}) published in [Scholz 1982b, 228–230] with a correction by E. Neuenschwander.

> would not find new theorems, and what seems simple and clear in the presentation in space, would thereby get involved and difficult." (Riemann Nachlass, **16**, 40^r)

This is clear testimony that Riemann did not show much interest in detailed studies of the logical foundations of geometry, precisely because he presumed them to be fruitless from the point of view of new theorems. This position is completely understandable from his point of view, *but it cannot be upheld if one is familiar with the works of Bolyai and/or Lobachevskii.* Their studies of absolute geometry and of horocycle geometry in the non-euclidean case [Gray 1979], to name just two examples, is too obviously incompatible with such a strict verdict of fruitlessness.

On the other hand, Riemann gave a clue why, in his opinion, the study of manifolds *really* mattered. In direct continuation of the quotation just given he went on:

> "Therefore one has taken everywhere the opposite road, and each time one encounters manifolds of several dimensions in geometry [mathematics? E.S.], as in the doctrine of definite integrals in the theory of imaginery quantities, one takes spatial intuition as an aid. It is well known how one gets thus a real overview over the subject and how only thus are precisely the essential points emphasized."

This quotation agrees completely with the main line of thought in Riemann's inaugural lecture, as far as manifolds inside mathematics are concerned. If Riemann had obtained a deeper knowledge of the more recent foundational studies of Bolyai or Lobachevskii during the time which elapsed between the formulation of these remarks and his inaugural lecture, he would have had sufficient reason to mention these new and unexpected aspects. That, however, is not the case.

Manifold Concept and Physical Space

We have to conclude that Riemann's "hypotheses which lie at the basis of geometry" have connotations different from the ones given in the late 1860s and 1870s when the debate on non-euclidean geometry ran high. One of these other connotations has already been mentioned. Riemann very consciously began the introduction of (non-metaphoric) geometric language into other mathematical fields (complex function theory, differential equations etc.) and needed therefore new fundamental concepts and "hypotheses" of geometry. This was the purely mathematical aspect of his enterprise, which was kept in second place in his 1854-talk.

The *main goal* of Riemann's inaugural lecture, on the other hand, was a *reformulation of the conceptual foundations of physical geometry.* This is made completely clear by the talk's architecture and the selection of main topics which culminate in a proposal of how the manifold concept could be used to analyze more deeply the properties of physical space. This goal also illuminates the reasons why Riemann choose manifolds of constant curvature as the only class of examples for a more detailed treatment in his second, differential geometric, part.

Riemann's intention in the last part of his lecture was to outline methodologically how the manifold concept could be used to improve the comprehension of physical space.

Already at the beginning of this discussion he stated that for such an application of the new concept it is essential to know something about conditions "which are sufficient and necessary for determining the metric relations of Space" (taking for granted that physical space can be analyzed as a manifold with Riemannian differential geometric metric). manifolds of constant curvature became important in this context, because in this class of examples such conditions are easily characterized.

- If free movement of rigid bodies is assumed, curvature is constant, and determination of the angle sum in one triangle uniquely determines curvature and metric of the whole manifold.
- If the angle sum of all triangles equals two right angles, all sectional curvatures are zero and space is euclidean (at least locally).
- If neither is the case the determination of the metric is completely open.

After a short excursion into the questions of unboundedness and infiniteness of space, which Riemann himself classified as "idle questions (müssige Fragen) for the explanation of nature" (1854, 285), he came back to the metrical relations of space. He thought them of much greater importance, as "... to discover causal connections one pursues phenomena into the spatially small, just as far as the microscope permits." And "... (u)pon the exactness with which we pursue phenomena into the infinitely small, does our knowledge of their causal connections essentially depend." (*ibid.*)

Here again he discussed the existing empirical evidence from the point of view of a theoretical alternative. The main observation referred to (without mentioning names) was Bessel's recent astronomical measurements (in 1838) of fixed star parallaxes, which had the highest technical standards of the time and had given positive values for some (nearer) stars but zero for most of them. That was good evidence for the existence of large triangles of astronomical scale with angle sum π (if one excludes the case of positive curvature, what Riemann apparently did without further notice). Riemann now drew the widely known conclusion:

- If one assumes free mobility of rigid bodies, then space is euclidean with the best precision available at the time.
- If, however, free mobility does not hold, then only "the total curvature of every measurable portion of Space is not perceptibly different from zero", leaving open the possibility of drastic changes of curvature in the small, which cancel out if integrated over larger regions (1854, 285).

Riemann closed this passage with the warning that one should not take euclidean geometry for granted, however convincing it may seem at the time. His argument in this connection even casts some doubts on the general applicability of his own metrical concept and shows that also his new concepts cannot be considered as a new type of (neo-)Kantian *a priori*. He noted that "... the empirical notions on which the metric determinations of Space are based, the concept of a solid body and that of a light ray, lose their validity in the infinitely small." Therefore one should always be open to a revision of the fundamental concepts of (physical) geometry "... as soon as it permits a simpler way of explaining phenomena." (*ibid.*) This is, from hindsight, a particularly striking remark, as the change to a Lorentzian or semi-Riemannian metric in special and general relativity was a revision of fundamental concepts of this type and – to comment this remark of Riemann even more anachronistically – something similar is being looked for today in the ongoing search for a quantum structure of space(-time).

All this shows clearly, how Riemann wanted to proceed in the elaboration of the "hypotheses which lie at the basis of (physical) geometry". These have to be rethought and perhaps revised time and again, with each fundamentally new piece of evidence about the physical tools of metrical measurement. He finished with a short remark on what was the role of mathematics in this process:

> "Investigations like the one just made, which begin from general concepts, can serve only to insure that this work is not hindered by too restrictive concepts, and that progress in comprehending the connection of things is not obstructed by traditional prejudices." (1854, 286)

This remark leads to our last point, Riemann's philosophical definition of the task of mathematics in the cognition of physical reality.

Riemann's Epistemology of Mathematics

Riemann obviously was neither an empiricist nor a Kantian. He could, in contrast to e.g. Gauss, go beyond the restrictive limits of euclidean geometry so easily, because he had worked out a good understanding of post-Kantian German dialectical philosophy, in particular by his extensive and detailed studies of the philosophy of Johann Friedrich Herbart (1776 – 1841). In fact, Herbart had defended a sideline of philosophy, which was essentially realistic in its methodology and ontology without losing its commitments to dialectics in its epistemology. This brought it closer to the lines of thought of scientists than the mainstream of German idealist philosophy of the time. I cannot go here into much detail,[11] but I want to outline some philosophical aspects underlying Riemann's mathematical work, which were heavily influenced by Herbart and surely of a greater overall importance for his work than just the vague reference to Herbart's "serial forms".

First of all the Herbartian background gave Riemann a post-Kantian view of epistemology. Herbart, in contrast to the idealist dialecticians of the time, saw the role of dialectical development mainly in concept formation and in the methodology of knowledge. He was no dialectician as far as ontology is concerned. We know from Riemann's *Nachlass* that the latter studied exactly intensely, and essentially with agreement, those parts which were constitutive for Herbart's epistemology.[12] That formed the background for Riemann's own developmental dialectical position with respect to epistemology, stated explicitly e.g. in the philosophical fragments published by Weber in Riemann's *Werke* (521–525).

As a consequence, Riemann did not share the restrictive Kantian view of a uniquely determined structure of synthetic knowledge *a priori*, of which mathematics, according

[11] See for this subject [Scholz 1982a].

[12] Riemann *Nachlaß* (**16**, 59^v, 64^r, 141^r) – see [Scholz 1982a] – and Riemann's selfdescription: "The author [Riemann, E.S.] is a Herbartian in psychology and epistemology [...]; in most cases he cannot agree, however, with Herbart's natural philosophy and the metaphysical disciplines (ontology and synechology) referring to it." (Riemann *Werke*, 508)

to Kant, forms an essential part. For Riemann there was no place for a purely *a priori* deduction of transcendental forms of cognition.[13] But on the other hand he also did not give in to the pitfalls of empiricism. Theoretical knowledge, in particular mathematical theory, insofar as it constitutes a conceptual framework for scientific knowledge, plays, according to Riemann, a role of what I want to call a ***relative*** or ***dialectical a priori*** with respect to empirical knowledge.

- This knowledge is *a priori*, because it is never to be derived by induction, generalization, or even straightforward idealization from experience. It is constituted by a deliberate conceptual creation and serves as a theoretical system of reference for empirical investigations and thus plays a formative role for the cognition of the empirical world.
- On the other hand this knowledge is *relative* and *dialectical*. Its structure is not uniquely determined, i.e. there is place for theoretical choices in the process of generation of the concepts, and these choices are done in consideration of the available empirical evidence. Just as little is it stable in time; it is subject to changes during the historical process of refinement of knowledge. *Refinement* (Riemann's term) may be read as a pragmatic expression for a type of conceptual change which overcomes the old structure without destroying completely the latter's validity. It thus shares the characteristic features of dialectical negation (*Aufhebung*), even if presented in less elaborate language.

Both aspects were already inherent in Herbart's epistemology, but they were formulated by Riemann in his epistemological fragments as his own position.[14] Mathematics plays, according to Riemann, an essentially critical role. It has to ensure that cognition of reality "... is not hindered by too restrictive concepts, and that progress in comprehending the connection of things is not obstructed by traditional prejudices..." (full citation above). It goes without saying that for Riemann the critical function of mathematical investigations is not restricted to undermining the validity of the old concepts, but it also establishes new and wider ones.

This point of view enabled Riemann to conceive such a fundamental revision of the conceptual framework of physical space as given in his inaugural lecture. We know from Gauss that the latter had thought about the necessity of going beyond the Kantian-Euclidean standpoint, but had never dared to come out with such a position in the scientific public. Riemann knew of the changes which had taken place on the philosophical terrain during the early century and used these as a positive reference system for his own proposals with respect to physical geometry.

There remains one more point to add. Riemann's clear conceptual orientation, which led him to single out central concepts in the different mathematical fields he worked in

[13]The anti Kantian tendency of Riemann's 1854 talk is discussed more in detail by [Nowak 1989].

[14]"...By the concepts through which we conceive nature, not only are our perceptions complemented in each moment, but also future perceptions are singled out as necessary, or, insofar as the conceptual system is not complete enough for that purpose, determined as probable ..." And a little later: "... The conceptual systems which underlie them now [the exact sciences, E.S.], have been formed by gradual change of older conceptual systems, and the reasons which resulted in new modes of explanation, can be reduced to contradictions or improbabilites, which turned up in the older modes of explanations." (Riemann *Werke*, 521)

(manifolds in geometry, Riemann surface in function theory, Riemann integral in convergence theory of trigonometric series etc.) was also much in line with Herbart's concept of *philosophical studies of the sciences*. Herbart, as one of Germany's educational philosophers in the first decades of 19th century, had seen a close connection between philosophy, philosophical studies of the sciences and a type of social reform which was fostered by the state but carried on by a class of (scientifically) educated men. In order to fulfil such a function the sciences should not be pursued in a predominantly technical style, but studied in what he called *philosophical spirit*. This lead Herbart to postulate a continually renewed search for and elaboration of central concepts in the different scientific disciplines. In fact the sciences ought to organize themselves around *central concepts (Hauptbegriffe)* (Herbart 1807). Philosophy proper should then work out the connections and dissolve possible contradictions between the central concepts of the different scientific fields. By such a systematic but open communication between philosophical studies of science and philosophy proper, science and philosophy would be able to fulfil their *Bildungsauftrag*, their educational goal. Riemann had studied intensely and excerpted these comments of Herbart, as we know from his *Nachlass* [Scholz 1982a, 424ff.]. That seems to have given him a sort of mirror for the self-reflection on the task and method of mathematics.

Riemann's mathematical work is penetrated by such a deep conceptual orientation that it cannot be better characterized than as "philosophical study of mathematics" in the Herbartian sense. And it even throws some light on the relationship between mathematics and physical sciences, as seen by Riemann, when one substitutes mathematics for philosophy in the Herbartian communication network of the sciences and philosophy. In fact, Riemann did in his inaugural lecture, what philosophers should do, according to Herbart, with respect to the scientific disciplines. He investigated the central concept of manifold, to be found in different mathematical and physical sciences, in order to clarify the connections between its different specifications, dissolve possible contradictions and elaborate it, in order to insure the possibility of further progress of scientific knowledge.

So again we find a deep convergence of ideas between Herbart and Riemann on this methodological level, referring to the most general description of the task of mathematical research. This convergence leads us to the claim that Riemann, in his mathematical research, took up orientations for scientific investigations, which had been worked out by Herbart, among others, on a philosophical level and which signify a broader social and cultural influence on the sciences and mathematics in the first half of the nineteenth century.

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