CONJUGATE HARMONIC FUNCTIONS IN SEVERAL VARIABLES

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1. Introduction and background

The purpose of this note is to report on some recent progress in the theory of conjugate harmonic functions in several variables.

We begin by sketching the requisite background.

In the Euclidian n-dimensional space, E_n, we consider an n-tuple of functions $u_1, u_2, \dots, u_n$ which satisfies the equations

$$\left.\begin{aligned} \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} &= 0, \\ \text{and} \qquad \frac{\partial u_k}{\partial x_j} &= \frac{\partial u_j}{\partial x_k}. \end{aligned}\right\} \tag{1}$$

Clearly (1) is locally equivalent with the statement that the vector $u = (u_1, \dots, u_n)$ is the gradient of a harmonic function H, that is $u = \nabla H$. That such a system might be a fruitful n-dimensional analogue of the usual notion of an analytic function (the case $n=2$) was suggested by various writers. This seems to be borne out particularly when one considers (1) in connection with problems of $n-1$ dimensional Fourier analysis.

For this purpose we distinguish one of the variables, say x_1, and consider the half space E_n^+ of points where $x_1 > 0$ and $(x_2, \dots, x_n)$ are arbitrary. The subspace E_{n-1} of points $(x_2, \dots, x_n)$ may be identified with the boundary hyper-plane $x_1 = 0$ of E_n^+.

Now let $f(x_2, \dots, x_n)$ be an arbitrary (say L^2) function on E_{n-1}, and let $u(x_1, \dots, x_n)$ be its Poisson integral. That is, u is harmonic in E_n^+ and takes on the boundary value f when $x_1 = 0$. Starting from u it is easy to pass to an n-tuple of functions in E_n^+ which satisfy (1), with $u_1 = u$, and so that $u_2, \dots, u_n$ are also Poisson integrals of their (L^2) boundary values. Finally set $f_k(x_2, \dots, x_n) = u_k(0, x_2, \dots, x_n)$, $k = 2, \dots, n$.

In this way we see that starting from an "arbitrary" function on E_{n-1} we can pass to a system of functions satisfying (1); this construction also leads to $n-1$ linear transformations of conjugacy

$$R_k : f \to f_k \quad (k = 2, \dots, n). \tag{2}$$

The Riesz transforms, R_k, are the natural generalizations of the Hilbert transform. Their significance can most easily be understood in terms of the Fourier transform. Thus in these terms R_k is a multiplier transformation corresponding to the factor $ix_k(x_2^2 + x_3^2 \dots x_n^2)^{-\frac{1}{2}}$; (see Horvath [3]).

A further significance of the R_k is that they are the basic building blocks for the "singular integral" transformations which have found wide application in partial differential equations; (see Calderon and Zygmund [1]). We add here an important inequality satisfied by the Riesz transforms:

$$\|R_k f\|_p \leqslant A_p \|f\|_p, \; 1 < p < \infty.$$

Another significant consequence of the notion (1) of conjugacy is the fact that $|u|^p = (u_1^2 + ... u_n^2)^{p/2}$ is sub-harmonic for $p \geqslant (n-2)/(n-1)$. This, together with harmonic majorization, allows us to develop a theory of H^p spaces extending the classical notion of the Hardy spaces (when $n=2$). The basic result when $p=1$ is the following extension of a well-known theorem of F. and M. Riesz. If $d\mu$ is a finite measure on E_{n-1} and its $n-1$ Riesz transforms are also finite measures, then all n measures are absolutely continuous. For these results see Stein and Weiss [6].

Finally, it is possible to extend results of Plessner and Privalov dealing with boundary behavior of analytic functions to n-tuples satisfying (1). We may state the result as follows: Let E be a subset of E_{n-1} of positive measure. If for every $(\bar{x}_2, ..., \bar{x}_n) \in E$, $\lim u_1$ exists as the point $(x_1, ..., x_n)$ converges to $(0, \bar{x}_2, ..., \bar{x}_n)$ non-tangentially, then the same is true for $u_2, ..., u_n$ for almost all points in E; the converse statement also holds; (see Stein [5]).

We shall now describe two new directions of investigation in this subject. The first reveals intimate connection with various classical expansions. The second is closely related to the theory of representations of the rotation group in n-variables.

2. Classical expansions

Our first point of departure is the consideration of harmonic functions which are radial in $n-1$ of the variables. This, as is well known, leads to problems analagous to Fourier analysis but for Bessel functions of order $(n-3)/2$, and by analogy to Bessel functions of real order $\geqslant -\frac{1}{2}$.(1) A similar situation holds for ultraspherical (Gegenbauer) expansions, and other classical expansions. Let us consider the matter in more detail. We set $x = x_1$ and $y = (x_2^2 + ... x_n^2)^{\frac{1}{2}}$. Consider the system (1) in the half-space E_n^+, and let $u_1(x_1, ..., x_n) = u(x, y)$ be radial in the variables $x_2, ..., x_n$; suppose that it is the Poisson integral of an $f(x_2, ..., x_n) = f(y)$ defined on E_{n-1}. Then $u(x,y) = \partial H/\partial x$ where $H(x,y)$ is harmonic as a function of $x_1, ..., x_n$. Thus H satisfies the equation

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{n-2}{y}\frac{\partial H}{\partial y} = 0. \tag{3}$$

As one might guess, there are now no longer $n-1$ distinct conjugates to $u(x,y)$, but only one. In fact

$$u_k(x,y) = \frac{x_k}{y}\frac{\partial H}{\partial y} \quad (k = 2, \dots, n).$$

Thus set $v(x,y) = \partial H/\partial y = (u_2^2 + ..., u_n^2)^{\frac{1}{2}}$. Then u and v together satisfy the singular "Cauchy-Riemann equations"

$$\begin{aligned} u_y - v_x &= 0, \\ u_x + v_y + \frac{n-2}{y} v &= 0. \end{aligned} \tag{4}$$

(1) This is also the point of view of axially symmetric potential theory, see e.g. Weinstein [7].

Now replace the constant $n-2$ appearing in (3) and (4) by the positive constant 2λ. Then it is natural to expand $u(x,y)$ (as a function of y) as a Hankel integral involving the Bessel function of order $\lambda-\frac{1}{2}$; and its conjugate $v(x,y)$ in terms of the Bessel function of order $\lambda+\frac{1}{2}$. One might expect a variety of results for this type of conjugacy similar to those discussed in the preceding section; this turns out to be the case. We shall not go into further details, however, concerning the problem of expansions in Bessel functions; instead we shall discuss at some length the analagous problem involving ultraspherical expansions. In fact, the results that can be obtained for Bessel expansions are very similar to the results we shall now state for the ultraspherical case.

The problem for ultraspherical expansions arises, in the first place, by considering functions which are harmonic in the interior of the unit sphere in n-dimensions in terms of their boundary values on the surface of the sphere; and then, by restricting consideration to those boundary functions which are "radial" ("spherical" in the modern terminology) that is, depend only on the distance from the "north pole". The notion of conjugacy, (4), turns out to be the one which links the normal to the tangential derivatives of such harmonic functions. More particularly this leads to ultraspherical expansions of type $(n-2)/2$, and to conjugate expansions of type $n/2$. By analogy we pass to expansions of type λ, and to their conjugate expansions of type $\lambda+1$, $\lambda\geqslant 0$.

The ultraspherical polynomials of type λ, P_n^λ, are defined by

$$\sum_{n=0}^{\infty} r^n P_n^\lambda(\cos\theta) = (1-2r\cos\theta+r^2)^{-\lambda} \quad (\lambda>0).$$

Then P_n^λ are orthogonal with respect to the measure $(\sin\theta)^{2\lambda}d\theta$ on $(0,\pi)$. It is with respect to this measure that the L^p spaces are then defined.

Suppose now that $f(\theta)$ is an arbitrary function on $(0,\pi)$, and let it have the expansion $f(\theta)\sim\sum a_n P_n^\lambda$ $(\cos\theta)$. We then define its "Poisson integral" $f(r,\theta)$ by

$$f(r,\theta)=\sum_{n=0}^{\infty} a_n r^n P_n^\lambda(\cos\theta) \quad (0\leqslant r<1), \tag{5}$$

and set $f(r,\theta)=u(x,y)$ in cartesian coordinates. Then u satisfies the singular Laplace equation (3) (with $n-2=2\lambda$), in the upper semidisc $x^2+y^2<1$, $y>0$. If we look for its conjugate function $v(x,y)$ given by (4) (again $n-2=2\lambda$), then we obtain

$$v(x,y)=\tilde{f}(r,\theta)=-2\lambda\sum_{n=1}^{\infty}\frac{a_n}{n+2\lambda}r^n\sin\theta P_{n-1}^{\lambda+1}(\cos\theta). \tag{6}$$

Finally, this leads us to consider the natural conjugacy mapping $f(\theta)\to\tilde{f}(\theta)$, where

$$\tilde{f}(\theta)\sim -2\lambda\sum_{n=1}^{\infty}\frac{a_n}{n+2\lambda}\sin\theta P_{n-1}^{\lambda+1}(\cos\theta).$$

Some of the results obtained may be listed as follows:

(*a*) The inequality $\|\tilde{f}\|_p\leqslant A_p\|f\|_p$, $1<p<\infty$, which is a generalization of the classical conjugacy inequality of M. Riesz.

(*b*) A maximum principle for those functions u which are "subharmonic" in the sense that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2\lambda}{y}\frac{\partial u}{\partial y}\geqslant 0.$$

The regions considered may contain the singular line $y=0$. Because of this singularity the usual Hopf maximum principle is not applicable; but the situation can be remedied by limiting oneself to functions even in the y variable.

(*c*) If u and v satisfy (4) (with $n-2=2\lambda$) then $(u^2+v^2)^{p/2}$ is "subharmonic" if $p\geqslant(2\lambda)/(2\lambda+1)$.

(*d*) From (*b*) and (*c*) one can obtain a theory of H^p spaces, for ultraspherical expansions, whenever $p\geqslant(2\lambda)/(2\lambda+1)$. In fact if $F(re^{i\theta})=f(r,\theta)+i\tilde{f}(r,\theta)$, (see equations 5 and 6), and $\sup_{r<1}\int_0^\pi |F(re^{i\theta})|^p \sin^{2\lambda}\theta d\theta<\infty$, $p>(2\lambda)/(2\lambda+1)$, then $\lim_{r\to 1} F(re^{i\theta})$ exists almost everywhere, in the L^p norm, and dominatedly. Incidentally, (*a*) may be viewed as describing the H^p spaces, when $p>1$.

(*e*) The results of the H^p theory, when $p=1$, may be restated as follows. Suppose $d\mu_1$ and $d\mu_2$ are two measures on $(0,\pi)$ which are finite, in the sense that $\int_0^\pi \sin^{2\lambda}\theta\,|d\mu_i(\theta)|<\infty$. Let

$$d\mu_1\sim\sum a_n P_n^\lambda(\cos\theta)$$

and

$$d\mu_2\sim 2\lambda\sum\frac{a_n}{n+2\lambda}\sin\theta P_{n-1}^{\lambda+1}(\cos\theta).$$

Then both $d\mu_1$ and $d\mu_2$ are absolutely continuous.

The results sketched in this section, and others of this type, were obtained jointly with B. Muckenhoupt.

3. Representations of the rotation group

The second general area we shall deal with concerns the study of other systems, akin to (1), which are in a natural sense generalizations of the Cauchy-Riemann equations.

The results are closely connected to the theory of representations of the rotation group in n variables. Here we shall limit ourselves to the case of three variables, where the problems raised can be solved explicitly. While the situation for any number of variables is more complex, a similar outcome for the general case is indicated.

We begin by giving examples of some of the systems, in addition to (1), which we shall study in a general setting. Let us use the notation

$$\partial=\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2},\quad \bar{\partial}=\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_1},\quad \partial_3=\frac{\partial}{\partial x_3}.$$

A system considered by Moisil and Theodoresco (see [4]) consists of two *complex*-valued functions u and v satisfying the equations

$$\begin{aligned}\bar{\partial}u+\partial_3 v&=0,\\ \partial v-\partial_3 u&=0.\end{aligned}\tag{7}$$

It may be of interest to point out that (7) is merely the electron equation of Dirac in the case without external force, with zero mass, and independent of time.

It should also be observed that passing to the real and imaginary components of u and v, (7) gives four equations in these four components. If we had taken one of these components to be zero (e.g. the imaginary part of u), then we recover essentially the gradient system (1), when $n=3$. Thus it is natural to raise the same problems about system (7) as we raised in the case of system (1). In particular we may ask; for what $p>0$, is $(|u|^2+|v|^2)^{p/2}$ sub-harmonic, when u and v satisfy (7). The answer is $p \geq \frac{1}{2}$; this is identical with the result for system (1), when $n=3$. Thus, by what was said above this conclusion actually extends our previous result for (1), when $n=3$.

Another variant of (1) occurs when we take a given harmonic function H and form its gradient of order l, l a positive integer. Thus $u=\nabla^l H$, and u is actually a tensor of rank l whose components consist of the various lth. derivatives of H. Such tensors u can also be characterized by a system of equations like (1). If $|u|$ denotes the usual Euclidean norm of this tensor, we may again ask when $|u|^p$ is sub-harmonic. It was shown by Calderon and Zygmund (1) that $|u|^p$ is sub-harmonic when $p \geq 1/(l+1)$. We should note that the larger the system becomes in terms of the number of components and relations between them, the stronger the result for sub-harmonicity that obtains.

From the examples cited above the following two general problems arise: (*a*) to characterize all possible natural generalizations of the Cauchy-Riemann equations; (*b*) to find the best exponent p so that the analogue of $|u|^p$ is sub-harmonic, in each given case.

To study problem (*a*) we have to consider the invariance properties of the examples cited. These systems are trivially invariant under the group of translations. They are distinguished, however, by their transformation properties under rotations. In fact, if the underlying space is transformed according to the group of rotations, each system is naturally transformed according to an appropriate representation of this group. (2) In the case of (1) this representation is the principal one—which assigns to each rotation itself. This is most easily seen by writing $u=\nabla H$. In the case of the gradient of order l, $u=\nabla^l H$, the representation is the one induced on spherical harmonics of degree l. Alternatively this representation, which is irreducible, can be described as acting under symmetric tensors of rank l, all of whose traces vanish. Finally, a study of example (7) shows that we must include the two-valued as well as the single-valued representations of the rotation group. In fact let R_3 denote the 3-dimensional rotation group, and U its "spinor" group; i.e. U is the two-dimensional group of unitary matrices of determinant one. The group R_3 is not simply-connected, but has U as its simply connected covering group. Thus U has only single-valued representation, and these are in one-one correspondence with the totality of single-valued or double-valued representations of R_3. The system (7) then corresponds to the principal representation of U, which is the usual spinor representation of R_3.

To state our general results we shall need some further notation. Let r

(1) This result is unpublished.

(2) For a survey of the theory of representations of the 3-dimensional rotation group see Gel'fand and Shapiro [2].

denote a positive integer, and let $l = r/2$. Thus $l = 0, \frac{1}{2}, 1, \ldots$ For each such l, let V_l denote the (complex unitary) vector space of symmetric tensors of rank r whose indices take on the values 1 or 2. Since the group U consists of 2×2 matrices which are unitary, there is a natural representation of U on V_l; and thus there is a natural (possibly two-valued) irreducible representation of R_3 on V_l. Let $\varrho \to R^l(\varrho)$ denote this representation, $\varrho \in R_3$. In fact, $R^l(\varrho)$ is single-valued if l is integral, and double-valued otherwise. This scheme describes all the irreducible representations of R_3.

We return to our first problem. To each l we shall associate a system, S_l, generalizing (1). Let $u(x)$ be a function on E_3 whose values are in V_l; the differential equations determining the system are, in components,

$$(S_l) \qquad \begin{cases} \bar{\partial} u_{1 i_2 i_3 \ldots i_r} + \partial_3 u_{2 i_2 i_3 \ldots i_r} = 0, \\ \partial u_{2 i_2 i_3 \ldots i_r} - \partial_3 u_{1 i_2 i_3 \ldots i_r} = 0. \end{cases}$$

It may be shown that the system S_l transforms according to the representation R^l in the following sense. If u satisfies (S_l), then $v(x) = R^l(\varrho)[u(\varrho^{-1}(x))]$ satisfies (S_l) also, for each $\varrho \in R_3$.

Therefore we see, in answer to the first problem raised, how to assign in a natural way a system to each irreducible representation of R_3. It is desirable to describe the system S_l in a more intrinsic fashion, which will also indicate its n-dimensional generalization.

We had $u(x) \in V_l$; therefore its gradient $\nabla u(x)$ belongs to $V_l \otimes V_1$. ∇u transforms according to the representation $R^l \otimes R^1$, ($\otimes$ denotes the tensor product). However the tensor product representation is not irreducible; it reduces into a direct sum of irreducible ones. Let $V_{l,1}$ denote the irreducible sub-space of $V_l \otimes V_1$ containing the *highest* weight vector. Then $R^l \otimes R^1$ restricted to $V_{l,1}$ is equivalent with R^{l+1}. Incidentally, this gives us the so-called Cartan composition of R^l with R^1.

The content of equations (S_l) is exactly the statement that $\nabla u(x)$ belongs to the sub-space $V_{l,1}$ of $V_l \otimes V_1$.

It is to be observed that when $l = \frac{1}{2}$, (S_l) reduces immediately to the system (7). When $l = 1$, then after a simple change of basis, (S_l) becomes the system (1). Similarly for any integral l this system can be shown to be locally characterized as the gradient of order l of a single harmonic function.

We come now to the second question raised earlier. For what p is $|u|^p$ sub-harmonic(¹) when u satisfies the system (S_l)?

The result is as follows: When l is integral, then $|u|^p$ is subharmonic if $p \geq 1/(l+1)$. When l is half-integral, then $|u|^p$ is sub-harmonic when $p \geq 1/(l + \frac{3}{2})$. Both these conclusions are best possible. The interest of this result is that it allows us to extend the theory of H^p spaces to the systems (S_l) for the above mentioned range of p's.

The results in this section were obtained in collaboration with G. Weiss.

(¹) Recall that V_e is a unitary vector space, and hence the norm $|u|$ is defined on it.

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